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LOOP COHOMOLOGY

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1. INTRODUCTION

Associated with every algebraic category there is a cohomology theory. In the category of groups cohomology theory is well-established, and one of the interpretations of the higher cohomology groups of a group leads to the considerations of extensions in the category of loops ([6]). In general very few algebraic tools are available in the category of loops and so there is a possibility that the cohomology theory will lead to interesting algebraic insights into categories of varieties of loops. This is the point of departure of this paper. In it the basis is laid for the study of cohomology theory for any variety in the category $\boldsymbol{L}$ of loops and the theory is worked out for $\boldsymbol{L}$ itself. Not surprisingly, the cohomological functors in this case are trivial in the sense that $\boldsymbol{L}^1$ is an extension functor (corresponding to H^2 in group theory) and $\boldsymbol{L}^n = 0$ for $n > 1$. However, from the point of view of homological algebra the situation is less trivial in that $\boldsymbol{L}$ affords an example of a *balanced* category, i.e. the cohomology theory vanishes on injective modules. This is a rather rare but important property. As is well-known, the category of associative loops, i.e. groups, is also balanced and it was thought interesting to investigate the category of commutative loops, which turns out not to be balanced. One of the interesting facets of the work is that the concept of module in the category of a variety of loops is in general much less trivial than that in the category of groups, and in fact the investigation of modules in the categories of Moufang and Bol loops, which was initiated by contact with the work in this paper, has already given insight into the structure of such loops ([13], [15]).

The appropriate definition of a loop module is given in § 2. We remark that relative to $\boldsymbol{L}$ the multiplicative structure of the loop is brought into play only insofar as it determines the identity element, but for other varieties the multiplication becomes more important. In § 3 the group $\mathrm{Ext}(Q, A)$ of extensions of a Q-module A by a loop Q is calculated. In § 4 the cohomology groups $\boldsymbol{L}^n(Q, A)$ are defined and it is proved that they have the properties mentioned above. A brief description of the 'standard' cohomology theory of an algebraic category is given in § 5, via Rinehart's approach. It is shown that the cohomology groups defined in § 4 are indeed the standard cohomology groups. In § 6 it is shown that $\boldsymbol{L}$ is balanced and that the category of commutative loops is not.

A *quasi group* is a set Q with a multiplication defined in such a way that, if $ab = c$, any two of a, b and c determines the third. A *loop* is a quasi-group with identity 1. Thus a group is an associative loop. We use the dot notation to avoid excessive brackets; for example $a \,.\, bc = a(bc)$. An *abelian* loop is required to be associative as well as commutative; in this case composition may be denoted by $+$. The *kernel* of a loop homomorphism has the natural meaning as in group theory. The set of non-identity elements of a loop Q is denoted by $Q^{\#}$.

An *algebraic* category is a category whose objects are sets endowed with operations of given arity, subject to identity relations, and whose morphisms are the set-theoretic maps that preserve these operations. For example, $\boldsymbol{L}$ is an algebraic category with one nullary operation given by the identity element and three binary operations. If $ab = c$ the first of these operations takes (a, b) to c, the second takes (a, c) to b, and the third takes (b, c) to a. These operations are subject to the obvious relations. A *variety* in an algebraic category is an algebraic category obtained from the original one by imposing further relations. Thus the category of groups is a variety in the category of loops. An algebraic category has free objects, characterised by the usual universal property, and fibre products. For example, in $(S, *)$, the category of pointed sets, every object is free. The fibre product, or pull-back, $C_1 \times_C C_2$ of two homomorphisms γ_i: $C_i \to C$, $i = 1, 2$, is the sub-object $\{(c_1, c_2)$: $\gamma_1 c_1 = \gamma_2 c_2\}$ of $C_1 \times C_2$. If C is a fixed object in an algebraic category $\boldsymbol{C}$ the category $\boldsymbol{C}/C$ is defined as follows. An object of $\boldsymbol{C}/C$ is an object in $\boldsymbol{C}$ supplied with a homomorphism into C (formally an object of $\boldsymbol{C}/C$ is a homomorphism in $\boldsymbol{C}$ with codomain C) and if γ_i: $C_i \to C$, $i = 1, 2$, are objects of $\boldsymbol{C}/C$ then a morphism from γ_2 to γ_1 in $\boldsymbol{C}/C$ is a homomorphism Θ: $C_2 \to C_1$ such that $\gamma_2 = \gamma_1 \Theta$. In general, $\boldsymbol{C}/C$ is not equivalent to an algebraic category. Finally we mention some 'forgetful' functors. First there is the functor V: $\boldsymbol{L} \to (S, *)$ taking the loop Q to the underlying set of Q, with special element 1. Then if Q is a fixed loop there is a forgetful functor from $\boldsymbol{L}/Q$ to $(S, *)$ taking γ: $H \to Q$ to $V(H)$.

There have been previous treatments of extensions of loops, see [1], [4], [6].

II. LOOP MODULES

If Q is a loop requirements of two kinds must be met in defining a Q-module. First we require a Q-module A to have sufficient structure for the split extension $Q \sqsubset A$ of A by Q to be defined. The loop $Q \sqsubset A$ has underlying set $Q \times A$, and identity $(1, 0)$, and the projection Π on the first variable is a homorphism with kernel isomorphic, as abelian group, to A. It must also satisfy the following two conditions.

(2.1) The map $q \to (q, 0)$ is a splitting of Π .

(2.2) For all $q \in Q$ and $a \in A$, $(q, a) = (q, 0)(1, a)$.

In order to define a Q-module A a structure map $\mu: A \times A \times Q \times Q \to A$ must be defined so that, writing $(a_1, a_2; q_1, q_2)$ for $\mu(a_1, a_2; q_1, q_2)$, multiplication in $Q \sqsubset A$ is given by

(2.3) $$(q_1, a_1)(q_2, a_2) = (q_1 q_2, (a_1, a_2; q_1, q_2))$$

It is easy to see that μ defines a Q-module if and only if the following four conditions are satisfied.

For all fixed $q_1, q_2 \in Q$, a quasi-group structure is defined on A by

(2.4) $$(a_1, a_2) \to (a_1, a_2; q_1, q_2) .$$

For all

(2.5) $$q_1, q_2 \in Q\,, \quad (0, 0; q_1, q_2) = 0 .$$

For all

(2.6) $$a \in A \quad \text{and} \quad q \in Q\,, \quad (0, a; 1, q) = (a, 0; q, 1) = (0, a; q, 1) .$$

For all

(2.7) $$a_1, a_2 \in A\,, \quad (a_1, a_2; 1, 1) = a_1 + a_2 .$$

One further condition on a loop module arises as follows. If H is any loop supplied with homomorphism γ into Q a *derivation* from H to A is a map $\delta: H \to A$ such that $h \to (\gamma h, \delta h)$ is a homomorphism of H into $Q \sqsubset A$, i.e. such that, for all $h_1, h_2 \in H$,

(2.8) $$\delta(h_1 h_2) = (\delta h_1, \delta h_2; \gamma h_1, \gamma h_2) .$$

For every fixed H and γ we require the derivations of H into A to form an abelian group under addition induced from A. This will be the case if, for all $a_1, a_2, b_1, b_2 \in A$ and $q_1, q_2 \in Q$,

(2.9) $$(a_1 + a_2,\ b_1 + b_2; q_1, q_2) = (a_1, b_1; q_1, q_2) + (a_2, b_2; q_1, q_2) .$$

For then, if δ_1, δ_2 are derivations with $\delta_i h_1 = a_i$, $\delta_i h_2 = b_i$, $i = 1, 2$, one has

$$(\delta_1 + \delta_2)(h_1 h_2) = \delta_1(h_1 h_2) + \delta_2(h_1 h_2) = (a_1, b_1; \gamma h_1, \gamma h_2) +$$
$$+ (a_2, b_2; \gamma h_1, \gamma h_2) = (a_1 + a_2,\ b_1 + b_2; \gamma h_1, \gamma h_2) =$$
$$= ((\delta_1 + \delta_2) h_1,\ (\delta_1 + \delta_2) h; \gamma h_1, \gamma h_2) .$$

So, $\delta_1 + \delta_2$ is a derivation. Conversely, if the derivations of H into A are to form an abelian group for every H and γ, then condition (2.9) is necessary. For if H is free on $\{h_1, h_2\}$ then a_1, a_2, b_1 and b_2 can be chosen arbitrarily.

Note that (2.9) implies (2.5) and, in conjunction with (2.6), implies (2.7). Thus a *Q-module* (strictly a *Q-loop-module*) A is defined to be an abelian group with structure map μ satisfying (2.4), (2.6) and (2.9). Another consequence of (2.9) is that, for all $a, b \in A$ and $q_1, q_2 \in Q$,

(2.10) $$(a, b; q_1, q_2) = (a, 0; q_1, q_2) + (0, b; q_1, q_2) .$$

So, for all $q_1, q_2 \in Q$, we define maps $L(q_1, q_2)$ and $R(q_1, q_2)$ from A into itself by

$$(2.11)\qquad \begin{aligned} aL(q_1, q_2) &= (a, 0; q_1, q_2)\,, \\ aR(q_1, q_2) &= (0, a; q_1, q_2)\,. \end{aligned}$$

Note that (2.3) now becomes $(q_1, a_1)(q_2, a_2) = (q_1 q_2, a_1 L(q_1, q_2) + a_2 R(q_1, q_2))$. In view of (2.10) R and L determine the module structure of A. By (2.4) and (2.9) $R(q_1, q_2)$ and $L(q_1, q_2)$ are automorphisms, of A, and (2.6) implies that, for all $q \in Q$,

$$(2.12)\qquad L(q, 1) = R(q, 1) = R(1, q) = 1_A\,.$$

Conversely, given automorphisms $L(q_1, q_2)$ and $R(q_1, q_2)$ of A for all $q_1, q_2 \in A$, satisfying (2.12), and using (2.10) and (2.11) to define a suitable structure on A it is easy to check (2.4), (2.6) and (2.9), so that A becomes a Q-module. In view of this remark we define the *enveloping algebra* $E(Q)$ of Q to be the integral group ring of the free group on symbols

$$\{\mathscr{L}(q_1, q_2) \colon q_1 \in Q,\ q_2 \in Q^{\#}\} \cup \{\mathscr{R}(q_1, q_2) \colon q_1, q_2 \in Q^{\#}\}\,,$$

and define $\mathscr{L}(q, 1) = \mathscr{R}(q, 1) = \mathscr{R}(1, q) = 1$ for all $q \in Q$. So a Q-module may equivalently be regarded as a right $E(Q)$-module. In particular, it makes sense to speak of an injective Q-module. Note that $E(Q)$ depends on the identity element of Q, but not on its multiplicative structure. Technically the functor E factors through pointed sets.

Let $J(Q)$, the *representing module* for Q, be the right $E(Q)$-module on generators $\{\bar{q} \colon q \in Q\}$ with defining relations

$$(2.13)\qquad \begin{aligned} \overline{q_1 q_2} &= (\overline{q_1}, \overline{q_2}; q_1, q_2)\,, \quad \text{i.e.} \\ \overline{q_1 q_2} &= \overline{q_1}\, L(q_1, q_2) + \overline{q_2}\, R(q_1, q_2)\,, \end{aligned}$$

for all $q_1, q_2 \in Q$. Putting $q_1 = q_2 = 1$ gives $\bar{1} = 0$. Clearly $J(Q)$ represents derivations in that, taking $\gamma = 1_Q$,

$$\operatorname{Der}(Q, A) = \operatorname{Hom}_{E(Q)}(J(Q), A)\,.$$

More generally, given any $\gamma \colon H \to Q$,

$$\operatorname{Der}(H, A) = \operatorname{Hom}_{E(H)}(J(H), A)\,,$$

where A is an H-module via γ.

The constructions in this paragraph can be paralleled in any algebraic category. It is easy to check that our definition of Q-module is precisely as required to ensure that the split extension $Q \sqsubset A$ is an abelian object in the category $\boldsymbol{L}/Q$ of loops over Q. In [7] Lawvere proves that if $\boldsymbol{C}$ is any algebraic category and $C \in \boldsymbol{C}$ then the category of abelian objects in $\boldsymbol{C}/C$ is equivalent to the category of modules over some ring $E(C)$ and that derivations are represented by some $E(C)$-module. For example if $\boldsymbol{C}$ is the category of groups, respectively abelian groups, respectively Lie algebras over some field, then $E(C)$ is the integral group ring of C, respectively the ring of integers, respectively the enveloping algebra of C.

Since the category of groups is a variety in the category of loops, the integral group ring $\mathbb{Z}(G)$ of a group G is a homomorphic image of the enveloping algebra $E(G)$ of G, and since the augmentation ideal $I(G)$ represents derivations for groups, $I(G) = J(G) \otimes_{E(G)} \mathbb{Z}(G)$. Given a $\mathbb{Z}(G)$-module A, in the (group) split extension of A by G one has $(g_1, a_1)(g_2, a_2) = (g_1 g_2, g_2^{-1} a_1 g_2 a_2)$ for all $g_1, g_2 \in G$ and $a_1, a_2 \in A$. Since this must also be the loop split extension $L(g_1, g_2)$ is conjugation by g_2 and $R(g_1, g_2)$ is the identity map. Thus the natural map $E(G) \to \mathbb{Z}(G)$ is given by $L(g_1, g_2) \to g_2$, $R(g_1, g_2) \to 1$. Note however that, if Q is a loop, $J(Q)$ cannot in general be embedded as a submodule in $E(Q)$, or in any free $E(Q)$-module. For suppose that $q \in Q$ is of order 2. Then by (2.13) $\bar{q}(R(q, q) + L(q, q)) = 0$. But $\bar{q} \neq 0$ since a derivation need not take q to 0, and $L(q, q) + R(q, q)$ is not a zero divisor in $E(Q)$ since the integral group ring of a free group is a domain, see [14]. The isomorphism of $J(G) \otimes_{E(G)} \mathbb{Z}(G)$ onto $I(G)$ is given by $\bar{g} \otimes 1 \to 1 - g$.

III. EXTENSIONS

The first problem is to decide what is to be meant by an extension of a loop Q by a module A. In the first place an extension must be an exact sequence $0 \to A \to \to H \to^{\pi} Q \to 1$, i.e. a surjection of H onto Q with kernel A. It remains to consider the module structure. For all $a_1, a_2 \in A$ and $h_1, h_2 \in H$ define $(a_1, a_2; h_1, h_2)$ by

(3.1) $$h_1 a_1 \,.\, h_2 a_2 = h_1 h_2 \,.\, (a_1, a_2; h_1, h_2)\,.$$

So $(a_1, a_2; h_1, h_2) \in A$, and the second and last requirement is that

(3.2) $$(a_1, a_2; h_1, h_2) = (a_1, a_2; \pi h_1, \pi h_2)\,,$$

where the right-hand side is defined by the given Q-module structure on A. For example the split extension is a module extension.

In the case of groups every surjection with abelian kernel is a module extension for the appropriate action of the quotient group on the kernel. In the case of loops the corresponding result is false in that given a short exact sequence as above, $(a_1, a_2; h_1, h_2)$, as defined in (3.1), may not be determined by $a_1, a_2, \pi h_1$ and πh_2, so that (3.2) is not satisfied for any Q-module structure on A.

3.1. Lemma. *Let $0 \to A \to^{i} H \to^{\pi} Q \to 1$ be an exact sequence of loops, where A is abelian. Use i to identify A with its image in H. Then the sequence is a module extension for a suitable Q-module structure on A if and only if, for all $h \in H$ and $a, b \in A$,*

(3.3) $$ha \,.\, b = h \,.\, ab\,,$$

and, for all $a_1, a_2, b_1, b_2 \in A$ and $h_1, h_2 \in H$, with the notation of (3.1),

(3.4) $$(a_1 + a_2, b_1 + b_2; h_1, h_2) = (a_1, b_1; h_1, h_2) + (a_2, b_2; h_1, h_2)$$

Proof. Assume that the sequence is a module extension. Then (3.4) follows from (2.9) and (3.2). Also, $ha \,.\, b = ha \,.\, 1b = h(a, b; h, 1) = h(a, b; \pi h, 1) = h \,.\, ab$ since $L(\pi h, 1) = R(\pi h, 1) = 1_A$.

Conversely, assume that (3.3) and (3.4) hold. Let $a_i, b_i \in A$ and $h_i \in H$, $i = 1, 2$ Then

$$\begin{aligned}(h_1 b_1 \,.\, h_2 b_2)(a_1, a_2; h_1 b_1, h_2 b_2) &= (h_1 b_1 \,.\, a_1)(h_2 b_2 \,.\, a_2) = \\ &= (h_1 \,.\, b_1 a_1)(h_2 \,.\, b_2 a_2) = h_1 h_2 \,.\, (b_1 + a_1, b_2 + a_2; h_1, h_2) = \\ &= (h_1 h_2 \,.\, (b_1, b_2; h_1, h_2))(a_1, a_2; h_1, h_2) = \\ &= (h_1 b_1 \,.\, h_2 b_2)(a_1, a_2; h_1, h_2)\,.\end{aligned}$$

Thus (3.2) can be said to define $(a_1, a_2; \pi h_1, \pi h_2)$ unambiguously. Then (2.4), (2.6) and (2.9) follow, so A becomes a Q-module and we have a module extension.

Let A be a Q-module and Ext (Q, A) denote the set of module extensions of A by Q, modulo the usual equivalence as in group theory; that is, two extensions are equivalent if the middle terms are isomorphic via an isomorphism inducing the identity on A and Q. There is a natural addition on Ext (Q, A) given by the Baer sum. Given two module extensions $0 \to A \to H \to Q \to 1$, $i = 1, 2$, the Baer sum is defined as follows. Take the fibre product $H_1 \times_Q H_2$; this has a natural homomorphism onto Q with kernel $A \times A$. Dividing out by $\{(a, -a);\ a \in A\}$ gives rise to a loop H with a surjection onto Q with kernel A. This is a module extension, and is the Baer sum of the given extensions. Thus Ext (Q, A) becomes an abelian group and is covariant in A, contravariant in Q, in the usual way. The zero element of Ext (Q, A) corresponds to the split extension. Clearly the extension $0 \to A \to H \to^{\pi} Q \to 1$ is equivalent to the split extension if and only if π has a one-sided inverse. Thus if Q is free every extension splits and Ext $(Q, A) = 0$.

We now calculate Ext (Q, A) in terms of cocycles. To construct a module extension $0 \to A \to^{i} H \to^{\pi} Q \to 1$ as above, form H on the underlying set $Q \times A$, with $\pi(q, a) = q$, $ia = (1, a)$, and $(q, a) = (q, 0)(1, a)$. Let $(q_1, 0)(q_2, 0) = (q_1 q_2, f(q_1, q_2))$ in H, where $f\colon Q \times Q \to A$. Since $(1, 0)$ is to be the identity element of H we have, for all $q \in Q$,

$$f(1, q) = f(q, 1) = 0\,. \tag{3.5}$$

Any map satisfying (3.5) will be said to be a cocycle. Then, in H,

$$\begin{aligned}(q_1, a_1)(q_2, a_2) &= (q_1, 0)(1, a_1) \,.\, (q_2, 0)(1, a_2) = \\ &= (q_1, 0)(q_2, 0) \,.\, (a_1, a_2; q_1, q_2) \text{ by } (3.2) \\ &= (q_1 q_2, f(q_1, q_2))(a_1, a_2; q_1, q_2) = (q_1 q_2, f(q_1, q_2) + (a_1, a_2; q_1, q_2))\end{aligned}$$

by (3.3). So f determines H, and clearly any cocycle f determines a module extension in this way.

Now let the cocycle $f_i\colon Q \times Q \to A$ define the extension H_i, for $i = 1, 2$. These extensions are equivalent if and only if there is a map $\omega: Q \to A$, with $\omega(1) = 0$, such that $(q, a) \to (q, a + \omega q)$ is an isomorphism of H_1 onto H_2. This is easily seen to be equivalent to

$$(3.6) \qquad f_2(q_1, q_2) - f_1(q_1, q_2) = \omega(q_1 q_2) - (\omega q_1, \omega q_2; q_1, q_2).$$

A map $g: Q \times Q \to A$ given by $g(q_1, q_2) = \omega(q_1 q_2) - (\omega q_1, \omega q_2; q_1, q_2)$ for some $\omega: Q \to A$ with $\omega(1) = 0$ is a *coboundary*. In view of (2.6) a coboundary is a cocycle. Clearly the coboundaries form an abelian group under addition induced via A, and (3.6) shows that $\mathrm{Ext}(Q, A)$ is bijective with the group of cocycles $f: Q \times Q \to A$ modulo the coboundaries, the bijection being natural in both variables, and hence an isomorphism.

IV. LOOP COHOMOLOGY

For any loop Q we make the following definitions:

$B_{-1}(Q) = J(Q)$;

$B_0(Q)$ is the free $E(Q)$-module on $\{[q]: q \in Q^{\#}\}$, put $[1] = 0$;

$B_1(Q)$ is the free $E(Q)$ module on $\{[q_1, q_2]: q_1, q_2 \in Q^{\#}\}$, put $[q, 1] = [1, q] = 0$;

$B_n(Q) = \{0\}$ for $n > 1$.

$d_1: B_1(Q) \to B_0(Q)$ and $d_0: B_0(Q) \to J(Q)$ are Q-module homomorphisms given by $d_1[q_1, q_2] = q_1 q_2 - ([q_1], [q_2]; q_1, q_2)$ and $d_0[q] = \bar{q}$. One checks immediately that these are well defined, i. e. that $d_1[q, 1] = d_1[1, q] = d_0[1] = 0$ for all $q \in Q$.

The definition of $J(Q)$ shows that

$$(4.1) \qquad B_1(Q) \to^{d_1} B_0(Q) \to^{d_0} J(Q) \to 0$$

is exact. We shall see in § 6 that d_1 is injective, a fact which may make the definition of B_n for $n > 1$ more natural. If A is a Q-module the *loop cohomology* groups $L^*(Q, A)$ are defined to be the cohomology of the sequence

$$0 \to \mathrm{Hom}_Q(B_0(Q), A) \to^{d_1^*} \mathrm{Hom}_Q(B_1(Q), A) \to 0 \to 0 \to \dots,$$

where $d_1^* = \mathrm{Hom}_Q(d_1, A)$, so that

$$L^0(Q, A) = \mathrm{Hom}_Q(J(Q), A) \simeq \mathrm{Der}(Q, A),$$

$$L^1(Q, A) = \mathrm{Hom}_Q(B_1(Q), A)/\mathrm{im}\, d_1^*, \quad \text{and}$$

$$L^n(Q, A) = 0 \quad \text{for} \quad n > 1.$$

Now $\mathrm{Hom}_Q(B_1(Q), A)$ is naturally isomorphic to the group of cocycles, and a coboundary is just an element in the image of d_1^*, where 'cocycle' and 'coboundary' are as defined in § 3. Thus

$$L^1(Q, A) \simeq \mathrm{Ext}(Q, A).$$

We finish this section with two simple remarks needed in the next section to prove that the cohomology groups defined here coincide with standard cohomology.

4.1. Proposition. *If Q is a free loop and A is a Q-module then $\mathbf{L}^n(Q, A) = 0$ for all $n > 0$.*

Proof. Only the case $n = 1$ needs to be considered, and we have already seen that $L^1(Q, A) \simeq \mathrm{Ext}(Q, A) = 0$ whenever Q is free.

If H is a loop supplied with a homomorphism $\gamma: H \to Q$ the Q-module A becomes an H-module so that $\mathrm{Hom}_{E(H)}(B_i(H), A)$ is defined for $i = 0, 1$, and $\mathrm{Hom}_{E(-)}(B_i(-), A)$ can be regarded as a functor from $\mathbf{L}/Q$ to Ab^{op}.

4.2. Proposition. *The functors* $\mathrm{Hom}_{E(-)}(B_i(-), A)$, $i = 0, 1$ *factor through pointed sets. More precisely if* $V: \mathbf{L}/Q \to (\mathbf{S}, *)$ *is the forgetful functor there are functors* $T_i: (\mathbf{S}, *) \to Ab^{\mathrm{op}}$, $i = 0, 1$, *such that*

$$\mathrm{Hom}_{E(-)}(B_i(-), A) \simeq T_i V.$$

Proof. If $(S, *)$ is a pointed set define $S \wedge S$ to be the pointed set obtained from $S \times S$ by identifying all elements of the form $(s, *)$ or $(*, s)$, and taking this to be the special element. Let $|A|$ denote the underlying pointed set of A; i.e. the underlying set of A with special element 0. Then define

$$T_0(S) = (\mathbf{S}, *)\,(S, |A|)\,,$$

i.e. the set-theoretic maps of S to A taking $*$ to 0, and

$$T_1(S) = (\mathbf{S}, *)\,(S \wedge S, |A|)\,.$$

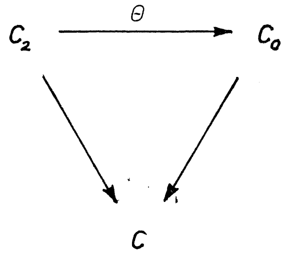
Clearly T_0 and T_1 define functors in the usual way satisfying the required conditions.

V. STANDARD COHOMOLOGY

In this section a brief account of standard cohomology will be given, followed by a proof that the loop cohomology defined in section 4 is the standard cohomology of loops. If $\mathscr{C}$ is a suitable category, $\mathbf{A}$ is an abelian category, and $T: \mathscr{C} \to \mathbf{A}$ is a functor, the problem is to define the derived functors of T. It should be mentioned in passing that it is no longer necessary to assume that $\mathbf{A}$ is abelian; this is an important development as it allows algebraic K-theory and Baer invariants to appear as derived functors. We shall restrict attention to the case when $\mathbf{A}$ is abelian and is either an algebraic category or the category $\mathbf{C}/C$ for some algebraic category $\mathbf{C}$ and object C of $\mathbf{C}$. Henceforth $\mathbf{A}$ and $\mathscr{C}$ shall denote such categories. If C is an object in any algebraic category $\mathbf{C}$ there is a canonical definition of *C-module* and of a *derivation* into a *C-module A*, as mentioned at the end of section 2. Then derivations define a functor $\mathrm{Der}(-, A): \mathbf{C}/C \to Ab^{\mathrm{op}}$ whose derived functors are the *standard cohomology* functors $\mathbf{C}^*(-, A)$.

The derived functors of a functor $T: \mathscr{C} \to \mathbf{A}$ can be defined in several equivalent ways. For example one can use triples, see Barr and Beck [3]; or simplicial resolutions, see Tierney and Vogel [17] or André [2] § 7; or André's definition in [2] § 1 in terms of models. We shall use yet another approach due to Rinehart [16].

A *surjection* in $\mathscr{C}$ is defined as follows. If $\mathscr{C}$ is an algebraic category this is the usual definition. If $\mathscr{C}$ is $\boldsymbol{C}/C$ for some algebraic category $\boldsymbol{C}$ and object C of $\boldsymbol{C}$ a morphism



in $\boldsymbol{C}/C$ is a *surjection* if θ is. Similarly we use the familiar definition of free object in $\boldsymbol{C}$, and define $F \to C$ in $\boldsymbol{C}/C$ to be free if F is. Note that $\mathscr{C}$ has fibre products. If $\mathscr{C}$ is algebraic this is the usual construction. If $\mathscr{C}$ is $\boldsymbol{C}/C$ the fibre product of $C_1 \to C$ and $C_2 \to C$ over $C_0 \to C$ is just the fibre product $C_1 \times_{C_0} C_2$ in $\boldsymbol{C}$ supplied with the composite $C_1 \times C_2 \to C_0 \to C$.

Let T: $\mathscr{C} \to \boldsymbol{A}$ be a functor such that, whenever ϱ: B_0 is a surjection in

$$T(B_1 \times_{B_0} B_1) \to^{\alpha} T(B_1) \to^{\beta} T(B_0) \to 0$$

is exact, where $\alpha = T(\pi_1) - T(\pi_0)$ and $\beta = T(\varrho)$, π_1 and π_0: $B_1 \times_{B_0} B_1 \to B_0$ are the projections. Then T is said to be *right exact*. This formulation of right exactness generalises the usual definition of right exactness between abelian categories. It can be shown that Der$(-, A)$, as above, is right exact for any algebraic category $\boldsymbol{C}$.

Let $\mathbb{R}(\mathscr{C}, \boldsymbol{A})$ denote the category of right exact functors from $\mathscr{C}$ to $\boldsymbol{A}$, regarded as a full subcategory of the category $(\mathscr{C}, \boldsymbol{A})$ of all functors from $\mathscr{C}$ to $\boldsymbol{A}$. (There is a set-theoretic difficulty here which can be removed by replacing with a suitable small category, e.g. by the use of Grothendieck universes.) Then the inclusion functor I: $\mathbb{R}(\mathscr{C}, \boldsymbol{A}) \to (\mathscr{C}, \boldsymbol{A})$ is an additive right exact functor between abelian categories with derived functors $S_n I$: $\mathbb{R}(\mathscr{C}, \boldsymbol{A})$ as in the standard theory of abelian categories.

We shall need three simple facts. The first is standard, and the others are to be found, for example, in [16].

1. The sequence $T_2 \to T_1 \to T_0$ in the abelian category $(\mathscr{C}, \boldsymbol{A})$ is exact if and only if, for all objects B of $\mathscr{C}$, $T_2(B) \to T_1(B) \to T_0(B)$ is exact. This remains true if $\mathscr{C}$ is replaced with any (small) category.
2. Let $\mathscr{F}$ denote the full subcategory of free objects in $\mathscr{C}$. Then restriction induces an equivalence between $\mathbb{R}(\mathscr{C}, \boldsymbol{A})$ and $(\mathscr{F}, \boldsymbol{A})$. Thus, in view of 1, a sequence $T_2 \to T_1 \to T_0$ of functors is exact (in $\mathbb{R}(\mathscr{C}, \boldsymbol{A})$ if and only if $T_2(F) \to T_1(F) \to$ $\to T_0(F)$ is exact in $\boldsymbol{A}$ for all free F in $\mathscr{C}$.
3. If D satisfies the conditions imposed on $\mathscr{C}$; e.g. if $\mathscr{C}$ is an algebraic category;

and if $V: \mathscr{C} \to D$ preserves pull-backs, and takes free objects to free objects and surjections to surjections; and if $T: D \to A$ is right exact, then TV is right exact and $S_n(TV) = S_n(T)V$ for all $n \geqq 0$.

Let Q be a loop, A be a Q-module, and G_i denote the functor $\mathrm{Hom}_{E(-)}(B_i(-), A)$ for $i = 1, 0$ as in section 4.

5.1. Proposition. *The sequence* $0 \to G_1 \to^{\delta_1} G_0 \to^{\delta_0} \mathrm{Der}(-, A) \to 0$, *where* δ_1 *and* δ_0 *are induced by* d_1 *and* d_0 (cf. § 4), *is an exact sequence in* $\mathbb{R}(\boldsymbol{L}/Q, \boldsymbol{AB}^{\mathrm{op}})$. *Also,* $S_n(G_i) = 0$ *for* $i = 0, 1$ *and* $n > 0$.

Proof. If T is any functor from $(\boldsymbol{S}, *)$ to an abelian category $\boldsymbol{A}$ then T is right exact and $S_nT = 0$ for $n > 0$; for every object in $(\boldsymbol{S}, *)$ is free, so $\mathbb{R}((S, *), \boldsymbol{A}) = ((S, *), \boldsymbol{A})$. Now as in Proposition 4.2, $G_i = T_iV$, where $V: \boldsymbol{L}/Q \to (\boldsymbol{S}, *)$ is the forgetful functor. Clearly V satisfies the conditions of 3 above, and T_i is right exact, so G_i is right exact, and $S_nG_i = (S_nT_i)V = 0$ for $n > 0$. Exactness of the sequence is equivalent to the exactness, in Ab, of $0 \to \mathrm{Der}(F, A) \to G_0(F) \to G_1(F) \to 0$, for all free F in $\boldsymbol{L}/Q$. Exactness at $G_1(F)$ is equivalent to Proposition 4.1, and exactness at the other points follows from the exactness of sequence (4.1), and does not rely on the freeness of F.

5.2. Theorem. *If* Q *is a loop and* A *is a* Q*-module, then* $\boldsymbol{L}^n(Q, A)$, *as defined in section* 4, *is the standard nth loop cohomology group of* Q *with coefficients in* A, *for all* $n \geqq 0$.

Proof. Using Proposition 5.1, and 1 above, classical homological algebra shows that the standard loop cohomology $\boldsymbol{L}^n(Q\,.\,A)$, as defined in this section, is the nth cohomology group of the sequence $\ldots \to 0 \to 0 \to G_1(Q) \to G_0(Q) \to 0$ in Ab^{op}. But this is the definition of $\boldsymbol{L}^n(Q, A)$ in section 4.

VI. CLASSICAL COHOMOLOGY

There is an alternative method of defining cohomology in an algebraic category $\boldsymbol{C}$ which does not in general agree with the standard cohomology of the last section. That is, if C is an object in $\boldsymbol{C}$, A is a C-module, and $E(C)$ and $J(C)$ are defined analogously to the definitions given in section 2 for $\boldsymbol{L}$, we define the *classical cohomology* groups $\boldsymbol{C}^n\langle C, A\rangle$ of C with coefficients in A to be $\mathrm{Ext}^n_{E(C)}(J(C), A)$. If $\boldsymbol{C}^n(C, A) \simeq \boldsymbol{C}\langle C, A\rangle$ for all C and A then $\boldsymbol{C}$ is said to be *balanced*. For example the categories of groups, Lie algebras over a fixed field, and associative algebras over a fixed field, are balanced. Thus if $\boldsymbol{C}$ is the category of groups then $\boldsymbol{C}^n(C, A) \simeq \boldsymbol{C}\langle G, A\rangle \simeq H^{n+1}(G, A) \simeq \mathrm{Ext}^n_{\boldsymbol{Z}(G)}(I(G), A)$ for $n > 0$, and $\boldsymbol{C}^0(G, A) \simeq \boldsymbol{C}^0\langle G, A\rangle \simeq \mathrm{Der}(G, A)$. These isomorphisms are all obvious (hence the name 'classical cohomology group') except for the isomorphism between $\boldsymbol{C}^n(G, A)$ and $\boldsymbol{C}^n\langle G, A\rangle$. This can be proved in exactly the same way that the cohomology groups $\boldsymbol{L}^n(G, A)$ of section 4 were proved in the last section to be isomorphic to the standard cohomology groups; one uses the standard homogeneous resolution to obtain an acyclic resolution of $\mathrm{Der}(G, A)$ by right exact functors that factor through pointed sets. A similar argument can be used for Lie algebras or associative algebras. We have not yet proved that the category of loops is balanced since we have not proved that d_1 in (4.1) is a monomorphism; this is clearly equivalent to the statement that

$$\boldsymbol{L}^n(Q, A) \simeq \mathrm{Ext}^n_{E(Q)}(J(Q), A) \quad \text{for all} \quad n \geqq 0 .$$

For the case where $\boldsymbol{C}$ is a variety of groups, see, for example, [8], [9], [10] and [11]. In [8] and [11] it is shown that various varieties are not balanced; for example the only balanced variety of groups containing all metabelian groups is the variety of all groups. In [12] it is shown that locally finite variety of groups $\boldsymbol{V}$ is balanced if and only if every p-group in $\boldsymbol{V}$ is abelian. The proof used is very different from that used in this paper, since no standard resolution is produced. In [8] the group-theoretic significance of the standard cohomology groups in a variety of groups is discussed. A similar investigation into the cohomology of varieties of loops might be of interest.

An interpretation of standard cohomology in any algebraic category is given by Duskin in [5].

6.1. Proposition. *The following conditions are equivalent:*

(a) *the category of loops is balanced;*
(b) *for every loop Q, d_1 in (4.1) is an injection;*
(c) *for every injective Q-module I, $\boldsymbol{L}^1(Q, I) = 0$;*
(d) *for every module extension of a Q-module A by a loop Q there is an injection of A into a Q-module B such that the induced extension of B by Q splits.*

Proof. We have already seen that (a) is equivalent to (b). Clearly (b) si equivalent to (c) and (c) implies (d). Finally, (d) implies (c); for let $\xi \in \boldsymbol{L}^1(Q, I)$, where I is injective. Then by (d) there is a Q-module B and an injection $i: I \to B$ such that $i^*(\xi) = 0$. But i, and hence i^*, is a split injection; so $\xi = 0$ as required.

6.2. Theorem. *The category of loops is balanced.*

Proof. We use criterion (d) of Proposition 6.1; it does not seem easy to check (b) directly. Let $0 \to A \to L \to^{\pi} Q \to 1$ be a module extension. Pick elements $\{t_q : q \in Q\}$ such that $\pi(t_q) = q$ for all q, and $t_1 = 1$. Then for all $q, r \in Q$, $t_q t_r = t_{qr} f(q, r)$, where f is a cocycle as in section 3. Let $B = A \oplus M$ as abelian group, where M is a free abelian group on $\{m_q : q \in Q^{\#}\}$; put $m_1 = 0$. We turn B into a Q-module as follows. The action of Q on A is already determined. The action of L and R on M are defined as follows.

$$m_r R(q, r) = \quad m_r - m_q \quad \text{if} \quad r \neq q \quad \text{and} \quad q \neq 1 \neq r ,$$

$$m_t R(q, r) = m_t \quad \text{if} \quad t \neq r \quad \text{or} \quad r = q \quad \text{or} \quad q = 1 \quad \text{or} \quad r = 1 ,$$

$$m_q L(q, r) = \quad m_q - m_r + m_{qr} - f(q, r) \quad \text{if} \quad r \neq q \quad \text{and} \quad q \neq 1 \neq r,$$

$$m_q L(q, r) = -m_q + m_{qr} - f(q, r) \quad \text{if} \quad r = q \quad \text{and} \quad q \neq 1\,,$$
$$m_t L(q, r) = \quad m_t \quad \text{if} \quad t \neq q \quad \text{or} \quad q = 1 \quad \text{or} \quad r = 1\,.$$

For all q and r, $R(q, r)$ and $L(q, r)$ clearly extend to automorphisms of B.

It remains to check that the induced extension E of B by Q splits. Define $\{s_q: q \in Q\} \subset E$ by $s_q = t_q m_q$, so $s_1 = 1$. Then for all $q, r \in Q$,

$$s_q s_r = t_q m_q \,.\, t_r m_r = t_q t_r \,.\, (m_q, m_r; q, r) =$$
$$= t_{qr} f(q, r) \,.\, (m_q L(q, r) + m_r R(q, r)) =$$
$$= t_{qr} f(q, r) \,.\, (m_{qr} - f(q, r)) = t_{qr} m_{qr} = s_{qr}\,.$$

So E splits as required.

If the category considered is that of commutative loops then the corresponding definitions of commutative loop modules and commutative loop cohomology functors are trivial translations of the above. If Q is a commutative loop and A is a Q-commutative loop module then the only extra condition on the R's and L's which define A is that $L(x, y) = R(y, x)$ for all x, y in Q. A commutative cocycle f satisfies $f(x, y) = f(y, x)$ for all x, y in Q. Proposition 6.1 holds if the obvious changes are made: all loops are commutative and all Q-modules are Q-commutative loop modules and the cohomology functors are those for commutative loops.

6.3. Theorem. *The category of commutative loops is not balanced.*

Proof. It is sufficient to give an example of a commutative loop Q and an extension of a Q-commutative loop module A by Q which does not satisfy (d) of 6.1 (after the changes mentioned above). Let Q be the Klein 4 group $= \{1, q, r, qr\}$. Let A be the cyclic group of order 4 generated by z, with trivial Q-action i.e. $L(x, y) = 1$ for all x, y in Q. Define $f: Q \times Q \to A$ by $f(q, r) = f(r, q) = z$, and $f(x, y) = 0$ for $(x, y) \neq (q, r)$. Let H_1 be the resulting extension of A by Q. Suppose there is an injection of A into the Q-commutative loop module B such that the induced extension splits. We may consider H_1 to be embedded in H_2. If $t: Q \to H_1$ is a set theoretic splitting of H_1 which defines f, i.e. $t_x t_y = t_{xy} f(x, y)$ for x, y in Q, and $s: Q \to H_2$ is a loop theoretic splitting of H_2, i.e. $s_x s_y = s_{xy}$ for x, y in Q then $s_x = t_x m_x$ where $x \in Q$, $m_x \in B$ and we may assume $m_1 = 0$. Since if $x \in Q$, $s_x \,.\, s_x = s_{x^2} = 0$, we have $0 = t_x m_x . t_x m_x = t_x t_x . 2m_x = t_{x^2} + f(x, x) + 2m_x = 2m_x$. Now $s_q s_q = t_q m_q . t_r m_r = t_{qr} f(q, r) + m_q + m_r$. Hence $m_{qr} = f(q, r) + m_q + m_r$. This is a contradiction since $2 f(q, r) \neq 0$.

We note that if Q is a commutative loop with no element x such that $x \,.\, x = 1$ then Q satisfies condition (d). In particular this is true if Q has finite odd order.

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