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RETRACTS OF ABELIAN LATTICE ORDERED GROUPS

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Retracts of partially ordered sets were investigated in [1], [2], [3] and [4]. In particular, in [4] they were applied for defining the notion of a variety of partially ordered sets (namely, a nonempty class of partially ordered sets is said to be a variety if it is closed with respect to retracts and direct products).

The present paper deals with retracts of direct products of abelian lattice ordered groups.

Assume that an abelian lattice ordered group H is an internal direct product of its l-subgroups A and B. Let G be a retract of H. It will be shown that G need not be, in general, an internal direct product of a retract of A and a retract of B. Nevertheless, it will be proved that there exist a retract R_1 of A and a retract R_2 of B such that G is isomorphic to the direct product of R_1 and R_2.

The above result will be applied for investigating retracts of lattice ordered groups which can be represented as direct products of a finite number of linearly ordered groups. Further, complete retract mappings of a complete lattice ordered group are studied.

Retract varieties of abelian lattice ordered groups (defined analogously as varieties of partially ordered sets [4]) will be investigated in a subsequent paper by using the results established here.

PRELIMINARIES

We recall that a nonempty subset Q of a partially ordered set P is said to be a *retract of P* if there is an isotone mapping f of P onto Q such that $f(q) = q$ for each element q of Q.

Let us remark that each nonempty subset of a partially ordered set is viewed as being partially ordered by the inherited relation of the partial order.

Let H be an abelian lattice ordered group and let G be an l-subgroup of H. If there exists a homomorphism f of H onto G such that $f(g) = g$ for each element g of G, then G will be said to be a *retract of H*. Also, the mapingp f with the just mentioned properties will be called a *retract mapping of H onto G*.

We shall apply the following notation which concerns direct products and lexicographic products.

Assume that A and B are l-subgroups of H such that the following conditions are satisfied:

(i) for each $h \in H$ there exist uniquely determined elements $h(A) \in A$ and $h(B) \in B$ such that $h = h(A) + h(B)$;

(ii) for each $h, h' \in H$ and each $t \in \{+, \wedge, \vee\}$ we have

$$(hth')(A) = h(A)\, t\, h'(A),$$

and similarly for B.

Under these assumptions H is said to be an *internal direct product of A and B*; we shall express this fact by writing $H = (\mathrm{i})\, A \times B$.

If $A_1, A_2, \ldots, A_n$ are l-subgroups of H, then the relation

$$H = (\mathrm{i})\, A_1 \times A_2 \times \ldots \times A_n$$

is defined analogously as in the case of two internal direct factors.

If A_1 and B_1 are lattice ordered groups, then their direct product $A_1 \times B_1$ is defined in the usual way. Let $H = (\mathrm{i})\, A \times B$. The mapping $\varphi\colon H \to A \times B$ defined by $\varphi(h) = (h(A), h(B))$ for each $h \in H$ is an isomorphism of H onto $A \times B$.

Let

$$H = (\mathrm{i})\, A \times B \tag{1}$$

be valid and let $X \subseteq H$. Then we put

$$X(A) = \{x(A)\colon x \in X\};$$

$X(B)$ is defined analogously.

Now assume that H is linearly ordered. Next, let A and B be l-subgroups of H such that the condition (i) above is satisfied and that, moreover, the following conditions hold:

(iii) for each $h, h' \in H$ we have $(h + h')(A) = h(A) + h'(A)$, and similarly for B;

(iv) for each $h \in H$ we have $h \geqq 0$ if and only if either $h(A) > 0$, or $h(A) = 0$ and $h(B) \geqq 0$.

Under these assumptions H will be said to be an *internal lexicographic product of A and B*, and we write $H = (\mathrm{i})\, A \circ B$.

If A_1 and B_1 are linearly ordered groups, then we can define their (external) lexicographic product $A_1 \circ B_1$ (cf., e.g., [5]). If

$$H = (\mathrm{i})\, A \circ B \tag{2}$$

holds, then the mapping $\varphi\colon H \to A \circ B$ defined by $\varphi(h) = (h(A), h(B))$ for each $h \in H$ is an isomorphism of H onto $A \circ B$.

If (2) holds and $X \subseteq H$, then $X(A)$ and $X(B)$ are defined analogously as in the case of the direct product.

The following two lemmas are easy to verify.

1.1. Lemma. *Let (1) be valid and let X be an l-subgroup of H. Then the following conditions are equivalent:*

(i) $X(A) \subseteq X$ *and* $X(B) \subseteq X$;

(ii) $X = (\mathrm{i})\, X(A) \times X(B)$.

Moreover, if there are l-subgroups A_1, B_1 *of* A *and* B, *respectively, such that* $X = (\mathrm{i})\, A_1 \times B_1$, *then* $A_1 = X(A)$ *and* $B_1 = X(B)$.

1.2. Lemma. *Let* H *be linearly ordered and let* (2) *hold. Let* X *be an l-subgroup of* H. *Then the assertion of* 1.1 *remains valid if the symbol* $\times$ *is replaced by* $\circ$.

1.3. Lemma. *Let* $f: H \to G$ *be a retract mapping and let* H_1 *be the kernel of* f. *Then the group* H *is a direct sum of its subgroups* G *and* H_1.

Proof. Clearly we have $G \cap H_1 = \{0\}$. Let $z \in H$. Put $g = f(z)$, $h_1 = z - g$. Then $h_1 \in H_1$. Assume that $g' \in G$, $h'_1 \in H_1$ and $z = g' + h'_1$. Thus $g = f(z) = f(g') + f(h'_1) = g'$, and then $h_1 = h'_1$. Therefore the group H is the direct sum of G and H_1.

DIRECT PRODUCT DECOMPOSITIONS

Again, let H be an abelian lattice ordered group. Assume that the relation (1) above is valid.

2.1. Lemma. *Let* A_1 *be a retract of* A *(with the corresponding retract mapping* g_1*) and let* B_1 *be a retract of* B *(with the corresponding retract mapping* g_2*). Put*

$$f(h) = g_1(h(A)) + g_2(h(B)) \quad \textit{for each} \quad h \in H\,,$$

and $f(H) = G$. *Then* G *is a retract of* H *(with the corresponding retract mapping* f*). Moreover,* $G = (\mathrm{i})\, A_1 \times B_1$.

The proof is routine; it will be omitted.

Let $f: H \to G$ be a retract mapping. Let $x \in H$, $x(A) = a$, $x(B) = b$. Then $f(x) = f(a) + f(b)$. Denote

$$f(a)(A) = a_1\,, \quad f(a)(B) = b_1\,, \quad f(b)(A) = a_2\,, \quad f(b)(B) = b_2\,.$$

Hence

$$f(x) = a_1 + b_1 + a_2 + b_2\,.$$

2.2. Lemma. *Under the above notation we have*

$$f(a_1) = a_1 + b_1\,, \quad f(b_2) = a_2 + b_2\,,$$
$$f(b_1) = f(a_2) = 0\,.$$

Proof. First we assume that $x \geqq 0$. Because of $a_1 \geqq 0$ and $b_1 \geqq 0$ we obtain $f(a_1) \geqq 0$ and $f(b_1) \geqq 0$. Since $f(a) = a_1 + b_1$, we have $f(a) = f(f(a)) = f(a_1) + f(b_1)$. Thus $f(b_1) \leqq f(a)$. But $a \wedge b_1 = 0$, whence $f(a) \wedge f(b_1) = 0$ and thus $f(b_1) = 0$. Therefore $f(a_1) = a_1 + b_1$.

Similarly we can verify the validity of the relations $f(a_2) = 0$ and $f(b_2) = a_2 + b_2$.

Now let x be any element of H. Since x can be expressed in the form $x = x_1 - x_2$ where $x_1, x_2 \in H$ and $0 \leqq x_1, 0 \leqq x_2$, we infer that the assertion of the lemma holds for x as well.

Denote $K_1 = f(A)$, $K_2 = f(B)$.

2.3. Lemma. *K_1 and K_2 are l-subgroups of G; moreover, $G = (\mathrm{i})\, K_1 \times K_2$.*

Proof. Since A is an l-subgroup of H and $f\colon H \to G$ is a homomorphism, K_1 is an l-subgroup of G. Similarly, K_2 is an l-subgroup of G.

Let $0 \leqq x \in G$. Then under the above notation we have

$$x = f(x) = f(a) + f(b)\,, \quad 0 \leqq f(a) \in K_1\,, \quad 0 \leqq f(b) \in K_2\,.$$

Since $a \wedge b = 0$, we obtain $f(a) \wedge f(b) = 0$, whence

$$x = f(a) \vee f(b)\,.$$

If x' is another element of G^+, then (under analogous notation) we have

$$x' = f(a') \vee f(b')\,:$$

because $a \wedge b' = b \wedge a' = 0$, we infer that

$$x \wedge x' = (f(a) \wedge f(a')) \vee (f(b) \wedge f(b')) = f(a \wedge a') \vee f(b \wedge b')\,,$$

and clearly

$$x \vee x' = f(a \vee a') \vee f(b \vee b')\,.$$

Hence the mapping $x \to (f(a), f(b))\, (x \in G^+)$ is an isomorphism of the lattice G^+ onto the direct product $K_1^+ \times K_2^+$ of the lattices K_1^+ and K_2^+. Thus in view of Theorem 2 in [6], the relation $G = (\mathrm{i})\, K_1 \times K_2$ is valid.

2.4. Lemma. *Let x, a, a_1 and b_1 be as above. Next, let x', a', a_1' and b_1' have an analogous meaning. If $a_1 = a_1'$, then $b_1 = b_1'$.*

Proof. Assume that $a_1 = a_1'$. According to 2.2 we have

$$a_1 + b_1 = f(a_1) = f(a_1') = a_1' + b_1'\,,$$

hence $b_1 = b_1'$.

2.5. Lemma. *For $k_1 \in K_1$ put $f_1(k_1) = k_1(A)$. Then f_1 is an isomorphism of K_1 into A.*

Proof. The mapping f_1 is a homomorphism of K_1 into A. Let $k_1 \in K_1$ such that $f_1(k_1) = 0$. Thus (under the notation as above) $a_1 = 0$ and then in view of 2.4 we have also $b_1 = 0$, hence $k_1 = 0$. Thus the kernel of f_1 is a one-element set and therefore f_1 is an isomorphism of K_1 into A.

Similarly we define the mapping f_2 of K_2 into B; the results are analogous.

2.6. Lemma. *For each $a \in A$ we put $\varphi_1(a) = f_1(f(a))$. Then φ_1 is a homomorphism of A onto $f(K_1)$ and $\varphi_1(a_1) = a_1$ for each $a_1 \in f(K_1)$.*

Proof. According to the definition of K_1, the mapping f reduced to A is a homomorphism of A onto K_1. Hence in view of 2.5, φ_1 is a homomorphism of A onto $f_1(K_1)$.

Let $a_1 \in f_1(K_1)$. There exists $k_1 \in K_1$ such that (under the above notation) we have $k_1 = a_1 + b_1$. Next, according to 2.2 the relation $f(a_1) = a_1 + b_1$ is valid, whence $\varphi_1(a_1) = a_1$.

2.7. Corollary. *$f_1(K_1)$ is a retract of A.*

Similarly we can define the mapping φ_2 of K_2 into B; we obtain analogous results.

From 2.1, 2.3, 2.5 and 2.7 we infer:

2.8. Theorem. *Let H and H' be lattice ordered groups, $H = (\mathrm{i})\, A \times B$. Then the following conditions are equivalent:*

(i) *H' is isomorphic to a retract of H.*

(ii) *There are retracts A_1 of A and B_1 of B such that H' is isomorphic to $A_1 \times B_1$.*

This result can be generalized by an obvious induction for the case when H is an internal direct product of a finite number of lattice ordered groups.

Again, let $G = (\mathrm{i})\, A \times B$ and let R be a retract of G. The following example shows that R need not be a direct product of a retract of A and a retract of B.

2.9. Example. Let H be the set of all pairs (x, y) of integers; the operations $+$, $\wedge$, $\vee$ on H are defined componentwise. Put

$$A = \{(x, y) \in H: y = 0\}, \quad B = \{(x, y) \in H: x = 0\}.$$

We have $H = (\mathrm{i})\, A \times B$. For each $(x, y) \in H$ let $f((x, y)) = (x, x)$. Then f is a retract mapping and $f(H)$ is a retract of H.

Since $f(H)(A) = A$ and A fails to be a subset of $f(H)$, in view of 1.1 the lattice ordered group $f(H)$ cannot be represented as an internal direct product $A_1 \times B_1$, where A_1 and B_1 are l-subgroups of A or B, respectively.

3. LINEARLY ORDERED GROUPS

In this section we assume that H is an abelian linearly ordered group.

3.1. Lemma. *Assume that $H = (\mathrm{i})\, A \circ B$. For each $h \in H$ put $f(h) = h(A)$. Then f is a retract mapping of H onto A.*

Proof. This is an immediate consequence of the definition of the lexicographic product.

Assume that $f: H \to G$ is a retract mapping and let H_1 be the kernel of f.

3.2. Lemma. *Let $h_1 \in H_1$ and $0 < g \in G$. Then $h_1 < g$.*

Proof. By way of contradiction, assume that there are $0 < g \in G$ and $h_1 \in H_1$ such that $g \leqq h_1$. Since H_1 is an l-ideal in H we obtain $g \in H_1$, which is impossible.

3.3. Lemma. *Let* $h \in H, f(h) = g, h - g = h_1$. *Then the following conditions are equivalent*:

(i) $h \geqq 0$;

(ii) $g > 0$, *or* $g = 0$ *and* $h_1 \geqq 0$.

Proof. Let (i) be valid. If $g < 0$, then $h_1 \geqq -g > 0$; in view of 3.2, this is a contradiction. Thus $g \geqq 0$. If $g = 0$, then clearly $h_1 \geqq 0$. Hence (ii) holds. The implication (ii) $\Rightarrow$ (i) is a consequence of 3.2.

An l-subgroup G of H will be called a *large lexicographic factor of* H if there is an l-subgroup G_1 of H such that $H = (\mathrm{i})\, G \circ G_1$.

From 1.3, 3.1 and 3.3 we obtain:

3.4. Theorem. *Let* H *be an abelian linearly ordered group and let* G *be an* l-*subgroup of* H. *Then the following conditions are equivalent*:

(i) G *is a retract of* H;

(ii) *either* $G = \{0\}$ *or* G *is a large lexicographic factor of* H.

The above theorem and Theorem 2.8 (generalized to a finite number of factors) yield

3.5. Theorem. *Let* H *be a lattice ordered group,* $H = (\mathrm{i})\, A_1 \times A_2 \times \ldots \times A_n$, *where* $A_1, A_2, \ldots, A_n$ *are linearly ordered groups. Let* H' *be a lattice ordered group. Then the following conditions are equivalent*:

(i) H' *is isomorphic to a retract of* H.

(ii) *There are linearly ordered groups* $B_1, B_2, \ldots, B_n$ *such that* H' *is isomorphic to* $B_1 \times B_2 \times \ldots \times B_n$, *and for each* $i \in \{1, 2, \ldots, n\}$, *either* $B_i = \{0\}$ *or* B_i *is a large lexicographic factor of* A_i.

4. COMPLETE RETRACT MAPPINGS

A retract mapping $f: H \to G$ of a lattice ordered group H onto a lattice ordered group G will be called *complete* if f is a complete homomorphism, i.e., if, whenever $\{h_i\}_{i \in I} \subseteq H$ and $\bigwedge_{i \in I} h_i = h$, then $\bigwedge_{i \in I} f(h_i) = f(h)$ (and dually).

In this section we assume that H is a complete lattice ordered group and that $f: H \to G$ is a complete retract mapping.

It will be shown that if

$$H = (\mathrm{i})\, A \times B \tag{1}$$

is valid and if f is a complete retract mapping on H, then there is a refinement

$$H = (\mathrm{i})\, A_1 \times A_2 \times B_1 \times B_2 \tag{2}$$

of the internal direct product decomposition of (1) such that f can be constructed by means of certain complete retract mappings on A_1 and B_1, and complete homomorphisms

(3) $\qquad \varphi_1: A_2 \to B_1\,, \quad \varphi_2: A_2 \to A_1\,, \quad \psi_1: B_2 \to A_1\,, \quad \psi_2: B_2 \to B_1\,.$

The following result is easy to verify.

4.1. Lemma. *Let H be a lattice ordered group, $H = (\mathrm{i})\, A_1 \times A_2 \times B_1 \times B_2$. Assume that mappings $f_1: A_1 \to A_1, f_2: B_1 \to B_1$ and mappings (3) are given such that the following conditions are satisfied:*

(i) *f_1 and f_2 are retract mappings;*

(ii) *$\varphi_1, \varphi_2, \psi_1$ and ψ_2 are homomorphisms;*

(iii) *for each $a_2 \in A_2$ and each $b_2 \in B_2$ the relations $f_2(\varphi_1(a_2)) = 0 = f_1(\varphi_2(a_2))$ and $f_1(\psi_1(b_2)) = 0 = f_2(\psi_2(b_2))$ are valid.*

For each $h \in H$ put

$$(*) \qquad f(h) = f_1(h(A_1)) + \varphi_2(h(A_2)) + h(A_2) + \varphi_1(h(A_2)) + \\ + f_2(h(B_1)) + \psi_2(h(B_2)) + h(B_2) + \psi_1(h(B_2))\,.$$

Then f is a retract mapping of H. If, moreover, all the homomorphisms $f_1, f_2, \varphi_1, \varphi_2, \psi_1$ and ψ_2 are complete, then f is a complete retract mapping.

Below it will be proved that if (1) holds and if f is a complete retract mapping of H, then there are direct decompositions

$$A = (\mathrm{i})\, A_1 \times A_2\,, \quad B = (\mathrm{i})\, B_1 \times B_2$$

and mappings $f_1, f_2, \varphi_1, \varphi_2, \psi_1$ and ψ_2 with the properties as in 4.1 such that all these mappings are complete homomorphisms and for each $h \in H$ the relation $(*)$ holds.

Thus let us suppose that (1) holds and that f is a complete retract mapping of H. Denote

$$A_1 = \{a \in A: f(a) \in A\}\,.$$

From the definition of A_1 we immediately obtain:

4.2. Lemma. *A_1 is a convex l-subgroup of A.*

For $X \subseteq A$ we set

$$X^\delta = \{a \in A: |a| \wedge |x| = 0 \text{ for each } x \in X\}\,.$$

4.3. Lemma. $A_1^{\delta\delta} = A_1$.

Proof. Clearly $A_1 \subseteq A_1^{\delta\delta}$. Hence we have to verify that $A_1^{\delta\delta} \subseteq A_1$. By way of contradiction, suppose that $A_1^{\delta\delta}$ fails to be a subset of A_1. Then there is $0 < a \in A_1^{\delta\delta}$ such that a does not belong to A_1. Hence $f(a) \notin A$. Thus there are $0 \leqq a' \in A$ and $0 < b \in B$ with $f(a) = a' + b$. Then according to 2.2,

$$f(a') = a' + b\,, \quad f(b) = 0\,.$$

Let $\{a_i: i \in I\}$ be the set of all elements $a_i \in A_1$ with $0 \leqq a_i \leqq a'$. Since H is a complete lattice ordered group, there exists

$$a_1 = \bigvee_{i \in I} a_i$$

in A. Because f is a complete retract mapping, we have

$$f(a_1) = \bigvee_{i\in I} f(a_i)\,.$$

Since $f(a_i) \in A$ for each $i \in I$, we infer that $f(a_i) \wedge b' = 0$ whenever $0 \leqq b' \in B$; hence $f(a_1) \wedge b' = 0$ for each $0 < b' \in B$. Thus $f(a_1) \in A$ and hence $a_1 \in A_1$. Next, $0 \leqq a' - a_1$. If there existed $0 < a'' \leqq a' - a_1$ with $a'' \in A_1$, then we would have $a_1 + a'' \in A_1$ and

$$a_1 < a_1 + a'' \leqq a_1 + (a' - a_1) = a'$$

which is a contradiction. Therefore $a' - a_1 \in A_1^\delta$. At the same time, $0 \leqq a' - a_1 \leqq \leqq a' \in A_1^{\delta\delta}$; we obtain $a' - a_1 = 0$, whence $a' = a_1 \in A_1$.

From 4.3 and from the completeness of H we get

4.4. Lemma. $A = (\mathrm{i})\, A_1 \times A_1^\delta$.

For $a \in A_1^\delta$ with $f(a) = a_2 + b$, $a_2 \in A$, $b \in B$ put $\varphi_1(a) = b$ and $\varphi_2(a) = a_2(A_1)$.

4.5. Lemma. *φ_1 is an isomorphism of A_1^δ into B.*

Proof. Let $a \in A_1^\delta$. Let a_2 and b be as above.

Since f is an endomorphism of H and since A_1^δ is an l-subgroup of H, the mapping φ_1 is a homomorphism of A_1^δ into B. Assume that $\varphi_1(a) = 0$. Then $a \in A_1$, hence $a = 0$. Thus the kernel of φ_1 is $\{0\}$, and therefore φ_1 is an isomorphism.

4.6. Lemma. *Let $a \in A_1^\delta$. Then $f(a)\,(A_1^\delta) = a$.*

Proof. There are $a' \in A$ and $b \in B$ with $f(a) = a' + b$. According to 2.2 we have $f(a') = a' + b$, whence $f(a' - a) = 0$ and thus $a' - a \in A_1$. Put $a' - a = a_1$. Hence $a'(A_1) = a_1$ and $a'(A_1^\delta) = a$. Next, $a' = f(a)\,(A)$, thus

$$a'(A_1^\delta) = f(a)\,(A)\,(A_1^\delta) = f(a)\,(A_1^\delta)\,.$$

We obtain $f(a)\,(A_1^\delta) = a$.

Let us define B_1 and B_1^δ analogously as we did for A_1 and A_1^δ. Lemmas 4.1 – 4.6 are valid for B_1 and B_1^δ as well; instead of φ_1 and φ_2 we now apply the notation ψ_1 and ψ_2.

For $a \in A_1$ let us put $f_1(a) = f(a)$. We have $f(f(a)) = f(a) \in A$, thus $f(a) \in A_1$. Hence we obtain:

4.7. Lemma. *f_1 is a retract mapping on A_1.*

If f_2 has an analogous meaning with respect to B_1, then 4.7 remains valid with f_1 replaced by f_2 and A_1 by B_1, respectively.

From the definitions of the mappings under consideration we immediately obtain:

4.8. Lemma. *All the mappings $f_1, f_2, \varphi_1, \varphi_2, \psi_1$ and ψ_2 are complete homomorphisms.*

4.9. Lemma. *Let $a \in A_1^\delta$. Then $\varphi_1(a)\,(B_1^\delta) = 0$.*

Proof. Let $a \in A_1^\delta$. Put $f(a)(A) = a'$, $f(a)(B) = b$. Then $\varphi_1(a) = b$. According to 2.2 we have $f(b) = 0$, thus $b \in B_1$. From $\varphi_1(a) \in B_1$ we obtain $\varphi_1(a)(B_1^\delta) = 0$.

Analogously, $\psi_1(b)(A_1^\delta) = 0$ for each $b \in B_1$.

Lemmas 4.5 and 4.9 yield:

4.10. Lemma. *φ_1 is an isomorphism of A_1^δ into B_1; similarly, ψ_1 is an isomorphism of B_1^δ into A_1.*

4.11. Lemma. *For each $a \in A_1^\delta$ and each $b \in B_1^\delta$ we have $f_2(\varphi_1(a)) = 0$ and $f_1(\psi_1(b)) = 0$.*

Proof. In view of 4.10, $\varphi_1(a) = f(a)(B_1)$. Thus $\varphi_1(a) \in B_1$. Hence $f_2(\varphi_1(a)) = f(\varphi_1(a))$. On the other hand, under the notation as in the proof of 4.9 we have $f(\varphi_1(a)) = f(b) = 0$. Therefore $f_2(\varphi_1(a)) = 0$. Analogously we can verify the validity of the relation $f_1(\psi_1(b)) = 0$.

4.12. Lemma. *For each $a \in A_1^\delta$ and each $b \in B_1^\delta$ we have $f_1(\varphi_2(a)) = 0$ and $f_2(\psi_2(b)) = 0$.*

Proof. Let $a \in A_1^\delta$. Put $f(a) = a' + b$, $a' \in A$, $b \in B$. Next, let $a_1 = a'(A_1)$. In view of 4.2, $a'(A_1^\delta) = a$. Denote $a'(A_1) = a_1$. Hence $f(a) = a_1 + a + b$, whence $f(a) = f(f(a)) = f(a_1) + f(a) + f(b)$. According to 2.2, $f(b) = 0$. Therefore $0 = f(a_1) = f_1(a_1) = f_1(\varphi_2(a))$. Analogously we can verify that $f_2(\psi_2(b)) = 0$.

The above results can be summarized as follows.

4.13. Theorem. *Let $f: H \to G$ be a complete retract mapping of a complete lattice ordered group H. Let $H = (\mathrm{i})\, A \times B$. Let us apply the notation introduced above and put $A_1^\delta = A_2$, $B_1^\delta = B_2$. Then the conditions (i), (ii) and (iii) from 4.1 are valid. For each $h \in H$, the relation $(*)$ from 4.1 holds.*

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