Czechoslovak Mathematical Journal

Alois Švec

A projective characterization of the Veronese surface

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 3, 459–469

Persistent URL: http://dml.cz/dmlcz/102317

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

A PROJECTIVE CHARACTERIZATION OF THE VERONESE SURFACE

ALOIS ŠVEC, Brno

(Received August 31, 1987)

There are many metric characterizations of the Veronese surface $\mathscr{V}$; see [4], [8] and the literature therein, see also a profound paper [1]. Nevertheless, in the following I try to treat a global purely projective characterization of $\mathscr{V}$. It seems that there exists no satisfactory local theory of surfaces in $P^4(\mathbb{R})$; the treatise of them in [3] excludes wide classes of surfaces (among them $\mathscr{V}$). Thus a systematic study of global projective properties of surfaces in $P^n(\mathbb{R})$ is needed; the present paper is an initial first step in this direction.

1. First of all, let us explain what we mean by a Veronese surface $\mathscr{V} \subset P^4(\mathbb{R})$, $P^4(\mathbb{R})$ being the projective 4-dimensional space over reals. Let (x, y, z) be orthonormal coordinates in E^3 and $(u_1, \ldots, u_5)$ orthonormal coordinates in E^5, E^p being the p-dimensional Euclidean space. Denote by $S^s(r) \subset E^{s+1}$ the hypersphere of radius r. The mapping $\sigma\colon S^2(\sqrt{3}) \to S^4(1) \subset E^5$ let be given by

$$\sigma(x, y, z) = (\alpha yz, \alpha xz, \alpha xy, \tfrac{1}{2}\alpha(x^2 - y^2), \tfrac{1}{6}(x^2 + y^2 - 2z^2)) \tag{1.1}$$

with

$$\alpha = \tfrac{1}{3}\sqrt{3}\,. \tag{1.2}$$

To each point $\sigma(p)$, $p \in S^2(\sqrt{3})$ we may associate an orthonormal frame $\{m; v_1, \ldots, v_5\}$ in E^5 such that

$$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,; \tag{1.3}$$

$$\mathrm{d}v_1 = \omega_1^2 v_2 + \alpha\omega^2 v_3 + \alpha\omega^1 v_4 - \omega^1 v_5\,, \quad \mathrm{d}v_2 = -\omega_1^2 v_1 + \alpha\omega^1 v_3 - \alpha\omega^2 v_4 - \omega^2 v_5\,,$$

$$\mathrm{d}v_3 = -\alpha\omega^2 v_1 - \alpha\omega^1 v_2 - 2\omega_1^2 v_4\,, \quad \mathrm{d}v_4 = -\alpha\omega^1 v_1 + \alpha\omega^2 v_2 + 2\omega_1^2 v_3\,,$$

$$\mathrm{d}v_5 = \omega^1 v_1 + \omega^2 v_2\,;$$

for this, see [7] where we have to change the sign of v_5.

Let $\tau\colon S^4(1) \to P^4(\mathbb{R})$ be the usual mapping; then $\tau \circ \sigma\colon S^2(\sqrt{3}) \to P^4(\mathbb{R})$ is exactly what we are going to call the *Veronese surface* $\mathscr{V}$.

2. Let us explain several elementary facts from the theory of Laplace transforms in the projective space; for the hyperbolic case, see [5], Chap. IV.

Given a surface $D \to P^n(\mathbb{R})$, D being a 2-dimensional manifold, let us suppose that the points of the surface satisfy exactly one hyperbolic partial differential equation of order 2. Then we may choose local coordinates u, v on (a domain U of) D in such a way that our surface is given by $x = x(u, v)$, and we have

$$x_{uv} = ax_u + bx_v + cx \,, \tag{2.1}$$

the subscripts denoting derivatives. The *Laplace transform* of our surface $x(u, v)$ is a mapping $\tilde{x}: U \to P^n(\mathbb{R})$, $\tilde{x} = \tilde{x}(u, v)$, such that $\tilde{x}(u, v) \in \{x(u, v), x_u(u, v), x_v(u, v)\}$, $\tilde{x}(u, v) \neq x(u, v)$ for each $(u, v) \in U$, and there is a tangent field $t = t(u, v)$ on U satisfying $t(u, v)\, \tilde{x}(u, v) \in \{x(u, v), \tilde{x}(u, v)\}$ on U; by $\{z_1, \ldots, z_p\}$, we denote the projective subspace through $z_1, \ldots, z_p$. It is known that our surface $x(u, v)$ has exactly two Laplace transforms

$$x_1 = x_v - ax \,, \quad x_{-1} = x_u - bx \,. \tag{2.2}$$

Indeed,

$$\frac{\partial}{\partial u} x_1 = bx_1 + hx \,, \quad \frac{\partial}{\partial v} x_{-1} = ax_{-1} + kx \tag{2.3}$$

with

$$h = c + ab - a_u \,, \quad k = c + ab - b_v \,. \tag{2.4}$$

The functions h, k are the so-called *Laplace-Darboux invariants*. In fact, they are not invariants with respect to the transformation of proportionality factor $x \to \varrho x'$ and the transformation of parameters $u' = u'(u)$, $v' = v'(v)$, but the so-called *point forms*

$$\varphi_1 = h \, du \, dv \,, \quad \varphi_{-1} = k \, du \, dv \tag{2.5}$$

are; for their geometrical meaning, see [2].

The Laplace transform $x_1(u, v)$ is a surface if and only if $h \neq 0$ on U; it satisfies the equation

$$x_{1uv} = a_1 x_{1u} + b_1 x_{1v} + c_1 x_1 \tag{2.6}$$

with

$$a_1 = a + (\log h)_v \,, \quad b_1 = b \,, \quad c_1 = c + h - k - b(\log h)_v \,; \qquad (\log h)_v := h^{-1} h_v \,. \tag{2.7}$$

Thus it has once again two Laplace transforms, and they are

$$x_2 = (x_1)_1 = x_{1v} - a_1 x_1 \,, \quad (x_1)_{-1} = x_{1u} - b_1 x_1 = hx \,. \tag{2.8}$$

For the Laplace transform $x_{-1}(u, v)$, $k \neq 0$ on U means that x_{-1} is a surface. The points x_{-1} satisfy

$$x_{-1uv} = a_{-1} x_{-1u} + b_{-1} x_{-1v} + c_{-1} x_{-1} \tag{2.9}$$

with

$$(2.10)\qquad a_{-1} = a\,,\quad b_{-1} = b + (\log k)_u\,,\quad c_{-1} = c + k - h - a(\log k)_u\,;$$
$$(\log k)_u := k^{-1}k_u\,;$$

and the Laplace transforms of the surface $x_{-1}(u, v)$ are

$$(2.11)\qquad x_{-2} = (x_{-1})_{-1} = x_{-1u} - b_{-1}x_{-1}\,,\quad (x_{-1})_1 = x_{-1v} - a_{-1}x_{-1} = kx\,.$$

The Laplace-Darboux invariants of (2.6) and (2.9) are

$$(2.12)\qquad h_1 = 2h - k - (\log h)_{uv}\,,\quad k_1 = h\,;\quad h_{-1} = k\,,$$
$$k_{-1} = 2k - h - (\log k)_{uv}\,,$$

respectively. Thus we get two new invariant point forms

$$(2.13)\qquad \varphi_2 = h_1\,du\,dv\,,\quad \varphi_{-2} = k_{-1}\,du\,dv\,.$$

We may say that, because of (2.3), φ_1 is associated to the line congruence $\{x, x_1\}$, φ_{-1} to $\{x, x_{-1}\}$, φ_2 to $\{x_1, x_2\}$ and φ_{-2} to $\{x_{-1}, x_{-2}\}$. Of course, $x_2(u, v)$ being a surface, we may construct its Laplace transform $x_3(u, v)$, etc.

Now, let the surface $y\colon D \to P^n(\mathbb{R})$ satisfy exactly one elliptic partial differential equation of order 2; we say that it has an *elliptic conjugate net*. It is known that we may choose local coordinates (u, v) in such a way that

$$(2.14)\qquad y_{uu} + y_{vv} = Ay_u + By_v + Cy\,.$$

I did not find this case to be mentioned and studied in the literature, but its theory follows easily. We have to pass to the complexification $P^n(\mathbb{C})$ and complexify the tangent bundle of D. Using the complex coordinate $z = u + \mathrm{i}v$ and the usual vector fields

$$(2.15)\qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - \mathrm{i}\frac{\partial}{\partial v}\right),\quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right)$$

on U, (2.14) may be rewritten as

$$(2.16)\qquad y_{z\bar z} = \mathscr{A}y_z + \bar{\mathscr{A}}y_{\bar z} + \mathscr{C}y\,;\quad \mathscr{A} = \tfrac{1}{4}(A + \mathrm{i}B)\,,\quad \mathscr{C} = \tfrac{1}{4}C\,.$$

The Laplace transforms are then

$$(2.17)\qquad y_1 = y_{\bar z} - \mathscr{A}y\,,\quad y_{-1} = y_z - \bar{\mathscr{A}}y$$

with

$$(2.18)\qquad \frac{\partial}{\partial z}y_1 = \bar{\mathscr{A}}y_1 + Hy\,,\quad \frac{\partial}{\partial \bar z}y_{-1} = \mathscr{A}y_{-1} + Ky\,;$$

$$(2.19)\qquad H = \mathscr{C} + \mathscr{A}\bar{\mathscr{A}} - \mathscr{A}_z\,,\quad K = \mathscr{C} + \mathscr{A}\bar{\mathscr{A}} - \bar{\mathscr{A}}_{\bar z} = \bar H\,.$$

From (2.15), we see that

$$(2.20)\qquad y_{-1} = \bar y_1\,.$$

Further, our point forms are

(2.21) $$\psi_1 = H \, dz \, d\bar{z} \,, \quad \psi_{-1} = K \, dz \, d\bar{z} = \bar{\psi}_1 \,.$$

Let us suppose $H \neq 0$ on $U \subset D$; then $y_1(u, v)$ is a surface and $y_{-1}(u, v)$ is a surface as well. Then we get the second Laplace transforms

(2.22) $$y_2 = y_{1\bar{z}} - \mathscr{A}_1 y_1 \,, \quad y_{-2} = y_{-1z} - \bar{\mathscr{A}}_1 y_{-1} \,; \quad \mathscr{A}_1 = \mathscr{A} + (\log H)_{\bar{z}} \,;$$

see (2.7)–(2.11). Thus again, using (2.20),

(2.23) $$y_{-2} = \bar{y}_2 \,,$$

and we get the point forms

(2.24) $$\psi_2 = H_1 \, dz \, d\bar{z} \,, \quad \psi_{-2} = K_{-1} \, dz \, d\bar{z} \,;$$

$$H_1 = 2H - K - (\log H)_{z\bar{z}} \,, \quad K_{-1} = 2K - H - (\log K)_{z\bar{z}} = \bar{H}_1$$

satisfying, because of (2.19),

(2.25) $$\psi_{-2} = \bar{\psi}_2 \,.$$

3. Consider a surface $D \to P^4(\mathbb{R})$, D being a 2-dimensional manifold; we are going to restrict ourselves to its coordinate neighborhood U. Let us suppose that our surface carries exactly one elliptic conjugate net and its first and second Laplace transforms exist.

It follows easily that our surface si not contained in a $P^3(\mathbb{R})$. To each point m of our surface (in U), let us associate a frame $\{m_0, \ldots, m_4\}$ consisting of analytic points m_i such that the geometric point m_0 coincides with m and m_1, m_2 are situated in the tangent plane of our surface at m. Further, let

(3.1) $$\det \|m_0, \ldots, m_4\| = 1 \,.$$

We have the fundamental equations

(3.2) $$dm_0 = \omega_0^0 m_0 + \omega^1 m_1 + \omega^2 m_2 \,, \quad dm_i = \omega_i^0 m_0 + \ldots + \omega_i^4 m_4$$
$$(i = 1, \ldots, 4)$$

with the usual integrability conditions

(3.3) $$d\omega_i^j = \omega_i^k \wedge \omega_k^j \quad (i, j = 0, \ldots, 4) \,;$$

of course,

(3.4) $$\omega^1 := \omega_0^1 \,, \quad \omega^2 := \omega_0^2 \,; \quad \omega_0^3 = \omega_0^4 = 0 \,.$$

From (3.1), we get

(3.5) $$\omega_0^0 + \ldots + \omega_4^4 = 0 \,.$$

Let us choose the frames in such a way that

(3.6) $$M_1 := m_1 + i m_2 \,, \quad M_2 := m_4 + i m_3$$

are the first and the second Laplace transforms of our surface, respectively. The equation (3.2_1) may be written as

$$\mathrm{d}m_0 = \omega_0^0 m_0 + \tau^1 M_1 + \bar{\tau}^1 \overline{M}_1 \tag{3.7}$$

with

$$\tau^1 = \tfrac{1}{2}(\omega^1 - \mathrm{i}\omega^2)\,. \tag{3.8}$$

Then it is easy to see that

$$\begin{aligned} \mathrm{d}M_1 &= (\omega_1^0 + \mathrm{i}\omega_2^0)\, m_0 + \tfrac{1}{2}\{\omega_1^1 + \omega_2^2 + \mathrm{i}(\omega_2^1 - \omega_1^2)\}\, M_1 + \\ &+ \tfrac{1}{2}\{\omega_1^1 - \omega_2^2 + \mathrm{i}(\omega_2^1 + \omega_1^2)\}\, \overline{M}_1 + \tfrac{1}{2}\{\omega_1^4 + \omega_2^3 + \mathrm{i}(\omega_2^4 - \omega_1^3)\}\, M_2 + \\ &+ \tfrac{1}{2}\{\omega_1^4 - \omega_2^3 + \mathrm{i}(\omega_2^4 + \omega_1^3)\}\, \overline{M}_2\,, \\ \mathrm{d}M_2 &= (\omega_4^0 + \mathrm{i}\omega_3^0)\, m_0 + \tfrac{1}{2}\{\omega_4^1 + \omega_3^2 + \mathrm{i}(\omega_3^1 - \omega_4^2)\}\, M_1 + \\ &+ \tfrac{1}{2}\{\omega_4^1 - \omega_3^2 + \mathrm{i}(\omega_3^1 + \omega_4^2)\}\, \overline{M}_1 + \tfrac{1}{2}\{\omega_4^4 + \omega_3^3 + \mathrm{i}(\omega_3^4 - \omega_4^3)\}\, M_2 + \\ &+ \tfrac{1}{2}\{\omega_4^4 - \omega_3^3 + \mathrm{i}(\omega_3^4 + \omega_4^3)\}\, \overline{M}_2\,. \end{aligned} \tag{3.9}$$

The definition of M_1 and M_2 yields

$$\omega_1^1 - \omega_2^2 + \mathrm{i}(\omega_2^1 + \omega_1^2) = 0\,, \quad \omega_1^4 - \omega_2^3 + \mathrm{i}(\omega_2^4 + \omega_1^3) = 0\,, \tag{3.10}$$

$$\begin{aligned} &(\omega_1^0 + \mathrm{i}\omega_2^0) \wedge \bar{\tau}^1 = \{\omega_1^4 + \omega_2^3 + \mathrm{i}(\omega_2^4 - \omega_1^3)\} \wedge \tau^1 = 0\,, \\ &(\omega_4^0 + \mathrm{i}\omega_3^0) \wedge \tau^1 = \{\omega_4^1 - \omega_3^2 + \mathrm{i}(\omega_3^1 + \omega_4^2)\} \wedge \tau^1 = \\ &= \{\omega_4^4 - \omega_3^3 + \mathrm{i}(\omega_3^4 + \omega_4^3)\} \wedge \tau^1 = 0\,. \end{aligned} \tag{3.11}$$

Thus

$$\omega_1^1 - \omega_2^2 = \omega_2^1 + \omega_1^2 = \omega_1^4 - \omega_2^3 = \omega_2^4 + \omega_1^3 = 0\,, \tag{3.12}$$

and there are real-valued functions $A_1, \ldots, F_2$ such that

$$\begin{aligned} &\omega_1^4 + \omega_2^3 = 2(A_1\omega^1 + A_2\omega^2)\,, \quad \omega_2^4 - \omega_1^3 = 2(A_2\omega^1 - A_1\omega^2)\,, \\ &\omega_4^1 - \omega_3^2 = 2(B_1\omega^1 + B_2\omega^2)\,, \quad \omega_3^1 + \omega_4^2 = 2(B_2\omega^1 - B_1\omega^2)\,, \end{aligned} \tag{3.13}$$

$$\omega_4^4 - \omega_3^3 = C_1\omega^1 + C_2\omega^2\,, \quad \omega_3^4 + \omega_4^3 = C_2\omega^1 - C_1\omega^2\,, \tag{3.14}$$

$$\begin{aligned} &\omega_4^0 = E_1\omega^1 + E_2\omega^2\,, \quad \omega_3^0 = E_2\omega^1 - E_1\omega^2\,, \\ &\omega_1^0 = F_1\omega^1 - F_2\omega^2\,, \quad \omega_2^0 = F_2\omega^1 + F_1\omega^2\,. \end{aligned} \tag{3.15}$$

Let the functions $G_1, \ldots, H_2$ be defined by

$$\begin{aligned} &\omega_4^1 + \omega_3^2 = (G_1 + H_1)\,\omega^1 + (G_2 - H_2)\,\omega^2\,, \\ &\omega_3^1 - \omega_4^2 = (G_2 + H_2)\,\omega^1 + (H_1 - G_1)\,\omega^2\,. \end{aligned} \tag{3.16}$$

Then the system (3.12)–(3.14), (3.16) is equivalent to the system

$$\begin{aligned} &\omega_2^2 = \omega_1^1\,, \quad \omega_2^1 = -\omega_1^2\,, \\ &\omega_4^4 = \omega_3^3 + C_1\omega^1 + C_2\omega^2\,, \quad \omega_4^3 = -\omega_3^4 + C_2\omega^1 - C_1\omega^2\,, \\ &\omega_1^4 = A_1\omega^1 + A_2\omega^2\,, \quad \omega_2^3 = A_1\omega^1 + A_2\omega^2\,, \end{aligned} \tag{3.17}$$

$$\omega_1^3 = -A_2\omega^1 + A_1\omega^2, \quad \omega_2^4 = A_2\omega^1 - A_1\omega^2,$$
$$\omega_4^1 = (B_1 + G_1 + H_1)\,\omega^1 + (B_2 + G_2 - H_2)\,\omega^2,$$
$$\omega_3^2 = (G_1 + H_1 - B_1)\,\omega^1 + (G_2 - H_2 - B_2)\,\omega^2,$$
$$\omega_3^1 = (G_2 + H_2 + B_2)\,\omega^1 + (H_1 - G_1 - B_1)\,\omega^2,$$
$$\omega_4^2 = (B_2 - G_2 - H_2)\,\omega^1 + (G_1 - H_1 - B_1)\,\omega^2.$$

Thus our starting point are the equations (3.17) + (3.15). Let us define

(3.18)
$$\mathscr{D}F_1 = \mathrm{d}F_1 + F_1(\omega_0^0 - \omega_1^1), \quad \mathscr{D}F_2 = \mathrm{d}F_2 + F_2(\omega_0^0 - \omega_1^1),$$
$$\mathscr{D}A_1 = \mathrm{d}A_1 + A_1(\omega_0^0 - 2\omega_1^1 + \omega_3^3) - A_2(2\omega_1^2 + \omega_3^4),$$
$$\mathscr{D}A_2 = \mathrm{d}A_2 + A_2(\omega_0^0 - 2\omega_1^1 + \omega_3^3) + A_1(2\omega_1^2 + \omega_3^4),$$
$$\mathscr{D}E_1 = \mathrm{d}E_1 + E_1(2\omega_0^0 - \omega_1^1 - \omega_3^3) + E_2(\omega_3^4 - \omega_1^2),$$
$$\mathscr{D}E_2 = \mathrm{d}E_2 + E_2(2\omega_0^0 - \omega_1^1 - \omega_3^3) - E_1(\omega_3^4 - \omega_1^2),$$
$$\mathscr{D}B_1 = \mathrm{d}B_1 + B_1(\omega_0^0 - \omega_3^3) - B_2(2\omega_1^2 - \omega_3^4),$$
$$\mathscr{D}B_2 = \mathrm{d}B_2 + B_2(\omega_0^0 - \omega_3^3) + B_1(2\omega_1^2 - \omega_3^4),$$
$$\mathscr{D}G_1 = \mathrm{d}G_1 + G_1(\omega_0^0 - \omega_3^3) + G_2\omega_3^4, \quad \mathscr{D}G_2 = \mathrm{d}G_2 + G_2(\omega_0^0 - \omega_3^3) - G_1\omega_3^4,$$
$$\mathscr{D}H_1 = \mathrm{d}H_1 + H_1(\omega_0^0 - \omega_3^3) + H_2(2\omega_1^2 + \omega_3^4),$$
$$\mathscr{D}H_2 = \mathrm{d}H_2 + H_2(\omega_0^0 - \omega_3^3) - H_1(2\omega_1^2 + \omega_3^4),$$
$$\mathscr{D}C_1 = \mathrm{d}C_1 + C_1(\omega_0^0 - \omega_1^1) - C_2(\omega_1^2 - 2\omega_3^4),$$
$$\mathscr{D}C_2 = \mathrm{d}C_2 + C_2(\omega_0^0 - \omega_1^1) + C_1(\omega_1^2 - 2\omega_3^4).$$

Then the differential consequences of (3.17) + (3.15) are

(3.19)
$$\mathscr{D}F_1 \wedge \omega^1 - \mathscr{D}F_2 \wedge \omega^2 = 0, \quad \mathscr{D}F_2 \wedge \omega^1 + \mathscr{D}F_1 \wedge \omega^2 = 0,$$

(3.20)
$$\mathscr{D}A_1 \wedge \omega^1 + \mathscr{D}A_2 \wedge \omega^2 = (A_1C_2 - A_2C_1)\,\omega^1 \wedge \omega^2,$$
$$\mathscr{D}A_2 \wedge \omega^1 - \mathscr{D}A_1 \wedge \omega^2 = (A_1C_1 + A_2C_2)\,\omega^1 \wedge \omega^2,$$

(3.21)
$$(\mathscr{D}G_1 + \mathscr{D}H_1) \wedge \omega^1 + (\mathscr{D}G_2 - \mathscr{D}H_2) \wedge \omega^2 =$$
$$= \{C_1(B_2 + G_2) - C_2(B_1 + G_1)\}\,\omega^1 \wedge \omega^2,$$
$$(\mathscr{D}G_2 + \mathscr{D}H_2) \wedge \omega^1 - (\mathscr{D}G_1 - \mathscr{D}H_1) \wedge \omega^2 =$$
$$= \{C_1(B_1 - G_1) + C_2(B_2 - G_2)\}\,\omega^1 \wedge \omega^2,$$

(3.22)
$$\mathscr{D}E_1 \wedge \omega^1 + \mathscr{D}E_2 \wedge \omega^2 = 2(E_2C_1 - E_1C_2 - F_1G_2 - F_2G_1)\,\omega^1 \wedge \omega^2,$$
$$\mathscr{D}E_2 \wedge \omega^1 - \mathscr{D}E_1 \wedge \omega^2 = 2(F_1G_1 - F_2G_2)\,\omega^1 \wedge \omega^2,$$
$$\mathscr{D}B_1 \wedge \omega^1 + \mathscr{D}B_2 \wedge \omega^2 = \{C_1(B_2 + G_2) - C_2(B_1 + G_1) - E_2\}\,\omega^1 \wedge \omega^2,$$
$$\mathscr{D}B_2 \wedge \omega^1 - \mathscr{D}B_1 \wedge \omega^2 = \{C_2(G_1 - B_1) + C_2(G_2 - B_2) + E_1\}\,\omega^1 \wedge \omega^2,$$
$$\mathscr{D}C_1 \wedge \omega^1 + \mathscr{D}C_2 \wedge \omega^2 = 4(A_2B_1 - A_1B_2)\,\omega^1 \wedge \omega^2,$$
$$\mathscr{D}C_2 \wedge \omega^1 - \mathscr{D}C_1 \wedge \omega^2 = \{4(A_1B_1 + A_2B_2) - C_1^2 - C_2^2\}\,\omega^1 \wedge \omega^2.$$

From (3.9) and (3.17) we obtain

$$(3.23)\qquad \mathrm{d}M_1 = F\bar{\tau}^1 m_0 + \tau_1^1 M_1 + A\tau^1 M_2\,,$$
$$\mathrm{d}M_2 = E\tau^1 m_0 + (G\tau^1 + H\bar{\tau}^1)\, M_1 + B\tau^1 \overline{M}_1 + \tau_3^3 M_2 + C\tau^1 M_2$$

with

$$(3.24)\qquad A = 2(A_1 + \mathrm{i}A_2), \ldots, H = 2(H_1 + \mathrm{i}H_2)\,,$$

$$(3.25)\qquad \tau_1^1 = \omega_1^1 - \mathrm{i}\omega_1^2\,,\quad \tau_3^3 = \omega_3^3 + \mathrm{i}\omega_3^4 + (C_1 - \mathrm{i}C_2)\,\bar{\tau}^1\,.$$

The geometrical points m_0, M_1, M_2 are fixed; nevertheless, we may change their factors of proportionality, i.e., choose other analytic points n_0, N_1, N_2 by

$$(3.26)\qquad m_0 = Rn_0\,,\quad M_1 = SN_1\,,\quad M_2 = TN_2$$

with R an $\mathbb{R}$-valued and S, T a $\mathbb{C}$-valued function, respectively. Further, because of (3.1),

$$(3.27)\qquad RS\bar{S}T\bar{T} = 1\,.$$

Then the equations (3.7) + (3.23) become

$$(3.28)\qquad \mathrm{d}n_0 = \varphi_0^0 n_0 + \varrho^1 N_1 + \bar{\varrho}^1 \overline{N}_1\,,$$
$$\mathrm{d}N_1 = F^*\bar{\varrho}^1 n_0 + \varrho_1^1 N_1 + A^*\varrho^1 N_2\,,$$
$$\mathrm{d}N_2 = E^*\varrho^1 n_0 + (G^*\varrho^1 + H^*\bar{\varrho}^1)\, N_1 + B^*\varrho^1 \overline{N}_1 + \varrho_3^3 N_2 + C^*\varrho^1 \overline{N}_2\,,$$

and we have

$$(3.29)\qquad \varrho^1 = R^{-1}S\tau^1\,,\quad F^* = R^2(S\bar{S})^{-1}\,F\,,\quad A^* = RS^{-2}TA\,,$$
$$E^* = R^2S^{-1}T^{-1}E\,,\quad G^* = RT^{-1}G\,,\quad H^* = RS\bar{S}^{-1}T^{-1}H\,,$$
$$B^* = RS^{-1}\bar{S}T^{-1}B\,,\quad C^* = RS^{-1}T^{-1}\bar{T}C\,.$$

Consequently,

$$(3.30)\qquad F^*\varrho^1\bar{\varrho}^1 = F\tau^1\bar{\tau}^1\,,\quad A^*H^*\varrho^1\bar{\varrho}^1 = AH\tau^1\bar{\tau}^1\,,$$

and we see immediately that we get the invariant point forms

$$(3.31)\qquad \psi_1 = F\tau^1\bar{\tau}^1\,,\quad \psi_2 = AH\tau^1\bar{\tau}^1\,;$$

the point forms ψ_{-1} and ψ_{-2} are given by (2.21_3) and (2.25), respectively.

Theorem. *Let $D \subset \mathbb{R}^2$ be a bounded domain, ∂D its boundary. Let $m\colon D \to P^4(\mathbb{R})$ be a surface, and let us suppose* (i) *$m(D)$ has exactly one elliptic conjugate net and its Laplace transforms $M_1, M_2\colon D \to P^4(\mathbb{C})$ exist;* (ii) *for the point forms ψ_1 and and $\psi_{-1} = \bar{\psi}_1$, we have*

$$(3.32)\qquad \psi_1 = \psi_{-1}\,;$$

(iii) *the (now real) point form ψ_1 is negative definite, ψ_2 does not vanish, and the Gauss curvature $\varkappa$ of $|\psi_1|$ satisfies*

$$(3.33)\qquad \varkappa > \tfrac{12}{5}(1 - \tfrac{2}{3}\sqrt{3}) \doteq -0{\cdot}371 \quad \text{on } D\,,\quad \mathrm{d}\varkappa = 0 \quad \text{on } \partial D\,;$$

(iv) *if the Laplace transform* M_3 *exists, it is situated on the straight line* $\{M_2, N\}$ *with* $N \in \{m, M_{-1} = \bar{M}_1, M_{-2} = \bar{M}_2\}$; (v) *the tangent space of* $M_2(p)$ *is 1-dimensional for each* $p \in \partial D$. *Then* $m(D)$ *is a part of the Veronese surface.*

Proof. Let us formulate our conditions analytically: (ii) means that F is an $\mathbb{R}$-valued function, i.e.,

$$F_2 = 0 \quad \text{on} \quad D\,; \tag{3.34}$$

(iv) gives

$$G = 0 \quad \text{on} \quad D\,; \tag{3.35}$$

(v) is equivalent to

$$E = B = C = 0 \quad \text{on} \quad \partial D\,; \tag{3.36}$$

for the last two conditions, see (3.23_2).

From (3.34) and (3.19), we get

$$\mathrm{d}F_1 + F_1(\omega_0^0 - \omega_1^1) = 0\,. \tag{3.37}$$

The exterior differentiation yields, because of $F_1 \neq 0$,

$$A_1 H_2 + A_2 H_1 = 0\,. \tag{3.38}$$

Thus we have $AH = \bar{A}\bar{H}$, and (3.31), (2.25) imply

$$\psi_2 = \psi_{-2}\,; \tag{3.39}$$

thus ψ_2 is a real form and

$$AH \neq 0\,. \tag{3.40}$$

From $(3.29_{2,3})$ and the condition (iii) we see that we may choose the frames in such a way that

$$F_1 = -1\,; \quad A_1 = \alpha = \tfrac{1}{3}\sqrt{3}\,, \quad A_2 = 0\,; \tag{3.41}$$

this and (3.38) imply

$$H_2 = 0\,, \quad H_1 \neq 0\,. \tag{3.42}$$

The condition (3.37) reduces then to

$$\omega_1^1 = \omega_0^0\,, \tag{3.43}$$

and (3.20) are simply

$$\begin{aligned} &(\omega_3^3 - \omega_0^0) \wedge \omega^1 + (2\omega_1^2 + \omega_3^4) \wedge \omega^2 = C_2 \omega^1 \wedge \omega^2\,, \\ &(2\omega_1^2 + \omega_3^4) \wedge \omega^1 - (\omega_3^3 - \omega_0^0) \wedge \omega^2 = C_1 \omega^1 \wedge \omega^2\,. \end{aligned} \tag{3.44}$$

Let functions $f_1, f_2 \colon D \to \mathbb{R}$ satisfy a system of the form

$$\begin{aligned} &\mathrm{d}f_1 \wedge \omega^1 + \mathrm{d}f_2 \wedge \omega^2 = (a_{11} f_1 + a_{12} f_2)\, \omega^1 \wedge \omega^2\,, \\ &\mathrm{d}f_2 \wedge \omega^1 - \mathrm{d}f_1 \wedge \omega^2 = (a_{21} f_1 + a_{22} f_2)\, \omega^1 \wedge \omega^2\,, \end{aligned} \tag{3.45}$$

$a_{ij}: D \to \mathbb{R}$ being given. We may choose the coordinates (u, v) on D in such a way that

(3.46) $$\omega^1 = r\,du\,, \quad \omega^2 = r\,dv\,; \quad r = r(u, v) \neq 0\,.$$

Then the system (3.45) may be rewritten as

(3.47) $$f_{1v} - f_{2u} = -r(a_{11}f_1 + a_{12}f_2)\,, \quad f_{1u} + f_{2v} = -r(a_{21}f_1 + a_{22}f_2)\,.$$

This is clearly an elliptic system on D and $f_1 = f_2 = 0$ on ∂D implies $f_1 = f_2 = 0$ on D; for this, see [9] or any other textbook on pseudoanalytic functions.

Aplying this remark to $(3.22_{1,2})$ with $G_1 = G_2 = 0$, we see that $E_1 = E_2 = 0$ on D; here we use (3.36_1). Similarly, we get $B_1 = B_2 = 0$ and $C_1 = C_2 = 0$. Thus (3.36) implies

(3.48) $$E = B = C = 0 \quad \text{on} \quad D\,.$$

Now the equations (3.21) are

(3.49) $$\{dH_1 + H_1(\omega_0^0 - \omega_3^3)\} \wedge \omega^1 + (2\omega_1^2 + \omega_3^4) \wedge \omega^2 = 0\,,$$
$$-(2\omega_1^2 + \omega_3^4) \wedge \omega^1 + \{dH_1 + H_1(\omega_0^0 - \omega_3^3)\} \wedge \omega^2 = 0\,,$$

and the system (3.19)–(3.22) reduces to (3.49) + (3.44) with $C_1 = C_2 = 0$. But this system immediately implies

(3.50) $$dH_1 + (H_1 + 1)(\omega_0^0 - \omega_3^3) = 0\,,$$

this last equation being completely integrable. Applying Cartan's lemma to (3.44), we get the existence of functions M, N such that

(3.51) $$\omega_3^3 - \omega_0^0 = M\omega^1 + N\omega^2\,, \quad 2\omega_1^2 + \omega_3^4 = N\omega^1 - M\omega^2\,,$$

with the differential consequences

(3.52) $$(dM - N\omega_1^2) \wedge \omega^1 + (dN + M\omega_1^2) \wedge \omega^2 = 0\,,$$
$$(dN + M\omega_1^2) \wedge \omega^1 - (dM - N\omega_1^2) \wedge \omega^2 = -2(1 + 3\alpha H_1)\,\omega^1 \wedge \omega^2\,;$$

for α, see (1.2). Thus we get functions K, L, P such that

(3.53) $$dM - N\omega_1^2 = K\omega^1 + L\omega^2\,, \quad dN + M\omega_1^2 = L\omega^1 + P\omega^2\,,$$

(3.54) $$K + P = 2(1 + 3\alpha H_1)\,.$$

The exterior differentiation of (3.53) yields

(3.55) $$(dK - 2L\omega_1^2) \wedge \omega^1 + \{dL + (K - P)\,\omega_1^2\} \wedge \omega^2 =$$
$$= 2(1 + 2\alpha H_1)\,N\omega^1 \wedge \omega^2\,,$$
$$\{dL + (K - P)\,\omega_1^2\} \wedge \omega^1 + (dP + 2L\omega_1^2) \wedge \omega^2 =$$
$$= -2(1 + 2\alpha H_1)\,M\omega^1 \wedge \omega^2\,,$$

and we write

(3.56) $$dK - 2L\omega_1^2 = K_1\omega^1 + K_2\omega^2\,, \quad dP + 2L\omega_1^2 = P_1\omega^1 + P_2\omega^2\,,$$
$$dL + (K - P)\,\omega_1^2 = L_1\omega^1 + L_2\omega^2$$

with

$$(3.57) \qquad L_1 - K_2 = 2(1 + 2\alpha H_1) N , \quad P_1 - L_2 = -2(1 + 2\alpha H_1) M .$$

From (3.54) we get, using (3.50)

$$(3.58) \qquad K_1 + P_1 = 6\alpha(H_1 + 1) M , \quad K_2 + P_2 = 6\alpha(H_1 + 1) N .$$

Consider the function $f\colon D \to \mathbb{R}$ defined by

$$(3.59) \qquad 2f = M^2 + N^2 .$$

Then

$$(3.60) \qquad *\mathrm{d}f = -(ML + NP)\,\omega^1 + (MK + NL)\,\omega^2 ,$$

$*$ being the Hodge $*$-operator with respect to the metric

$$(3.61) \qquad \mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2 = -4\psi_1 .$$

From (3.60) we get the Stokes theorem in the form

$$(3.62) \qquad \int_{\partial D} *\mathrm{d}f = \int_D \{K^2 + 2L^2 + P^2 + 2(M^2 + N^2)(1 + 3\alpha + 5\alpha H_1)\}\,\omega^1 \wedge \omega^2 .$$

Let us now calculate the Gauss curvature $\varkappa'$ of the metric (3.61). We have

$$(3.63) \qquad \mathrm{d}\omega^1 = -\omega^2 \wedge \omega_1^2 , \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2 , \quad \mathrm{d}\omega_1^2 = -\varkappa'\omega^1 \wedge \omega^2 ,$$

i.e., $\varkappa' = 1 + 2\alpha H_1$. Thus the Gauss curvature $\varkappa$ of $|\psi_1| = \frac{1}{4}\,\mathrm{d}s^2$ is given by

$$(3.64) \qquad \varkappa = 4(1 + 2\alpha H_1) ,$$

and we have

$$(3.65) \qquad \mathrm{d}\varkappa = 8\alpha(H_1 + 1)(M\omega^1 + N\omega^2)$$

because of (3.50) and (3.51). Because of (3.33_1) and (3.64),

$$(3.66) \qquad 1 + 3\alpha + 5\alpha H_1 > 0 \quad \text{on} \quad D .$$

The equation $H_1 = -1$ contradicts (3.66), and (3.33_2) implies $M = N = 0$ on ∂D. Thus the integral formula (3.62) implies

$$(3.67) \qquad M = N = 0 \quad \text{on} \quad D .$$

The equations (3.51) reduce then to

$$(3.68) \qquad \omega_3^3 - \omega_0^0 = 0 , \quad 2\omega_1^2 + \omega_3^4 = 0$$

with the integrability condition $1 + 3\alpha H_1 = 0$, i.e.,

$$(3.69) \qquad H_1 = -\alpha .$$

Finally, from (3.5), (3.17_1), (3.17_3) + (3.36_3), (3.43) and (3.68) we get

$$(3.70) \qquad \omega_0^0 = \ldots = \omega_4^4 = 0 .$$

Considering now (3.15) + (3.17) with all the specializations made up to now, we see that we get exactly the equations of the form (1.3_{2-6}). QED.

Bibliography

[1] *O. Borůvka:* Sur les surfaces représentées par les fonctions sphériques de première espèce. J. Math. Pures et Appl., 1933, t. XII, 337—383.

[2] *E. Čech:* О точечных изгибаниях конгруэнций прямых, Czech. Math. J., *5* (*80*), 1955, 234—273.

[3] *G. Fubini, E. Čech:* Geometria proiettiva differenziale, t. 2, Zanichelli, 1927.

[4] *M. Kozlowski, U. Simon:* Minimal immersions of 2-manifolds into spheres. Preprint TU Berlin, 115/1983.

[5] *E. P. Lane:* Projective differential geometry of curves and surfaces. Univ. of Chicago Press, 1932.

[6] *A. Švec:* Projective differential geometry of line congruences. Czech. Acad. of Sciences, Prague 1965.

[7] *A. Švec:* Vector fields on hyperspheres. Czech. Math. J., *37* (*112*), 1987, 207—230.

[8] *A. Švec:* On Veronese surfaces. Czech. Math. J., *38* (*113*), 1988, 231—236.

[9] *W. L. Wendland:* Elliptic systems in the plane. Pitman, 1979.

Author's address: Přehradní 10, 635 00 Brno, Czechoslovakia.