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Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 4, 726–729

Persistent URL: http://dml.cz/dmlcz/102268

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

LOCAL SPECTRAL RADIUS FORMULA FOR OPERATORS IN BANACH SPACES

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(Received December 10, 1986)

Let T be a bounded operator acting on a complex Banach space X and let $x \in X$. The local spectrum $\sigma_T(x)$ and the local spectral radius $r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$ were introduced and studied in connection with the theory of decomposable and spectral operators, see e.g. [3], [6].

According to [7] there is a large set $Y \subset X$ (dense and of the second category) of elements $x \in X$ with maximal local spectra $\sigma_T(x) = \sigma(T)$. Also $r_T(x) = r(T)$ for $x \in Y$, see also [4]. In particular,

$$\sup_{x\in X} r_T(x) = \sup_{\substack{x\in X\\ \|x\|=1}} \limsup_{n\to\infty} \|T^n x\|^{1/n} = r(T)$$

(see also [3] p. 38).

We prove that there exists $x \in X$, $\|x\| = 1$ such that $\|T^n x\|^{1/n}$ is arbitrary close to $r(T)$ $(n = 1, 2, \ldots)$. As a corollary we obtain that

$$\sup_{\substack{x\in X\\ |x|=1}} \inf_{n=1,2,\ldots} \|T^n x\|^{1/n} = r(T)$$

so that the supremum and the infimum in the well-known spectral radius formula $r(T) = \inf \|T^n\|^{1/n} = \inf_n \sup_{\substack{x\in X\\ \|x\|=1}} \|T^n x\|^{1/n}$ can be interchanged.

1. Theorem. *Let T be a bounded operator on a Banach space X and let r denote its spectral radius. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of positive numbers satisfying* $\sup\{a_i, i = 1, 2, \ldots\} < 1$ *and* $\lim_{i\to\infty} a_i = 0$. *Then*

1. *there exists $x \in X$, $\|x\| = 1$ such that $\|T^j x\| \geqq r^j a_j$ $(j = 1, 2, \ldots)$;*
2. *there exists a subset $Y \subset X$ dense in X such that for every $y \in Y$ there is a positive integer $j(y)$ with $\|T^j y\| \geqq r^j a_j$ $(j \geqq j(y))$.*

In particular, $\liminf_{j\to\infty} (\|T^j y\|/r^j a_j \geqq 1$ *for every* $y \in Y$.

Proof. We may assume without loss of generality $\|T\| = 1$. Denote by $\sigma_e(T)$ and r_e the essential spectrum and the essential spectral radius of T, respectively. We distinguish two cases:

A) Suppose $r > r_e$. The set $\{\lambda \in \sigma(T), |\lambda| > r_e\}$ is at most countable [2], consists of isolated eigenvalues and the corresponding Riesz subspaces are finite-dimensional. Choose $\lambda \in \sigma(T)$, $|\lambda| = r$. Let x be an eigenvector corresponding to λ, $\|x\| = 1$. Then $\|T^j x\| = r^j$ $(j = 1, 2, \ldots)$.

Let X_λ and P_λ be the spectral subspace corresponding to λ and the Riesz projection onto X_λ, respectively. Put $Y = \{y \in X, P_\lambda y \neq 0\}$. Then Y is a dense subset of the second category in X. Let $y \in Y$. Denote $z = P_\lambda y$, $u = (I - P_\lambda)\, y$. Then

$$T^j y = T^j z + T^j u\,, \quad P_\lambda T^j y = T^j z\,, \quad \text{i.e.}$$

$$\|T^j y\| \geqq \|P_\lambda\|^{-1} \|T^j z\| \quad (j = 1, 2, \ldots)\,. \tag{1}$$

Further, $(T - \lambda)|_{X_\lambda}$ is a finite-dimensional nilpotent operator. Let $k \geqq 1$ be the integer such that $(T - \lambda)^k z = 0$ and $(T - \lambda)^{k-1} z \neq 0$. Let $Q: X_\lambda \to X_\lambda$ be a projection such that $Qz = z$, $Q \operatorname{Ker} (T - \lambda)^{k-1} = 0$. Then $Q(T - \lambda)\, T^{j-1} z = 0$ $(j = 1, 2, \ldots)$, i.e. $QT^j z = \lambda Q T^{j-1} z$, and by induction $QT^j z = \lambda^j Q z = \lambda^j z$. Thus

$$\|T^j z\| \geqq \|Q\|^{-1}\, r^j \|z\| \quad (j = 1, 2, \ldots)\,. \tag{2}$$

Together with (1) this gives

$$\|T^j y\| \geqq \frac{r^j \|P_\lambda y\|}{\|Q\|\, \|P_\lambda\|} \quad (j = 1, 2, \ldots)\,,$$

hence $\|T^j y\| \geqq r^j a_j$ for all j sufficiently large.

B) Suppose $r = r_e$. Fix $\lambda \in \sigma_e(T)$, $|\lambda| = r$. Then by [1], $\inf\limits_{\substack{x \in Y \\ \|x\| = 1}} \|(T - \lambda)\, x\| = 0$ for every closed subspace $Y \subset X$ of finite codimension. We need the following two lemmas:

2. Lemma. (see Proposition 3 of [5]). *Let $T \in B(X)$, $\lambda \in \sigma_e(T)$, $|\lambda| = r_e$. Let $E \subset X$ be a finite-dimensional subspace of X and let $\varepsilon_1, \varepsilon_2 > 0$. Then there exists $z \in X$, $\|z\| = 1$ such that*

1) $\|(T - \lambda)\, z\| \leqq \varepsilon_1$,
2) $\|x + \alpha z\| \geqq \max \{\|x\|\, (1 - \varepsilon_2),\ \frac{1}{2}|\alpha|\, (1 - \varepsilon_2)\}$ *for every $x \in E$ and for every complex number α.*

3. Lemma. *Let $T \in B(X)$, $\|T\| = 1$, $r_e = r$, $x \in X$ and let $\{a_i\}_{i=1}^\infty$ be a sequence of positive numbers satisfying $\sup\{a_i,\ i = 1, 2, \ldots\} < 1$ and $\lim\limits_{i \to \infty} a_i = 0$. Let $0 \leqq m_0 \leqq m_1 \leqq m_2$ be integers and let $\delta > 0$ satisfy $a_j < \frac{1}{3}\delta$ $(j \geqq m_1 + 1)$. Suppose $\|T^j x\| > r^j a_j$ $(j = m_0 + 1, \ldots, m_1)$. Then there exists $y \in X$ such that $\|x - y\| \leqq \delta$ and $\|T^j y\| > r^j a_j$ $(j = m_0 + 1, \ldots, m_2)$.*

Proof. Fix $\lambda \in \sigma_e(T)$, $|\lambda| = r$. Let $E = \bigvee\{T^j x,\ j = m_0 + 1, \ldots, m_1\}$. Choose $\varepsilon_1, \varepsilon_2 > 0$ such that $\|T^j x\|\, (1 - \varepsilon_2) - m_1 \delta \varepsilon_1 > r^j a_j$ $(j = m_0 + 1, \ldots, m_1)$ and $\varepsilon_1 \leqq \min\{r^j/6j,\ j = m_1 + 1, \ldots, m_2\}$. Let z be the vector from the previous lemma and put $y = x + \delta z$. Clearly, $\|x - y\| = \delta$. Further, for $j = m_0 + 1, \ldots, m_1$, $\|T^j y\| = \|T^j x + \delta T^j z\| \geqq \|T^j x + \delta \lambda^j z\| - \delta \|T^j z - \lambda^j z\| \geqq \|T^j x\|\, (1 - \varepsilon_2) -$

$- \delta\|(T^{j-1} + \lambda T^{j-2} + \ldots + \lambda^{j-1})(T - \lambda) z \geqq \|T^j x\| (1 - \varepsilon_2) - \delta j \varepsilon_1 > r^j a_j$.

Similarly, for $j = m_1 + 1, \ldots, m_2$,

$$\|T^j y\| = \|T^j x + \delta T^j z\| \geqq \|T^j x + \delta \lambda^j z\| - \delta\|T^j z - \lambda^j z\| \geqq$$
$$\geqq \tfrac{1}{2}\delta\, r^j - \delta j \varepsilon_1 \geqq \tfrac{1}{2}\delta\, r^j - \tfrac{1}{6}\delta\, r^j = \tfrac{1}{3}\delta\, r^j > r^j a_j\,.$$

Proof of Theorem 1 (continued):

1. Put $d = 1 - \sup\{a_i,\ i = 1, 2, \ldots\} > 0$. Let $a'_i = a_i(1 + d)$ $(i = 1, 2, \ldots)$. Clearly, $\lim_{i \to \infty} a'_i = 0$ and $\sup\{a'_i,\ i = 1, 2, \ldots\} \leqq (1 - d)(1 + d) = 1 - d^2 < 1$. Denote by n_i $(i = 1, 1, \ldots)$ the smallest index satisfying

$$a'_n < \frac{d}{3 \,.\, 2^{i+1}} \quad (n > n_i)\,.$$

Fix $\lambda \in \sigma_e(T)$, $|\lambda| = r$. Let $x_0 \in X$, $\|x_0\| = 1$ be an approximative eigenvector corresponding to λ and satisfying $\|T^j x_0\| > r^j a'_j$ $(j = 1, \ldots, n_0)$. Using the previous lemma repeatedly we construct a sequence $\{x_k\}_{k=0}^{\infty}$, $x_k \in X$ such that $\|x_{k+1} - x_k\| \leqq$ $\leqq d/2^{k+1}$ and $\|T^j x_{k+1}\| > r^j a'_j$ $(j = 1, \ldots, n_{k+1})$ (we put $x = x_k$, $y = x_{k+1}$, $\delta =$ $= d/2^{k+1}$, $m_0 = 0$, $m_1 = n_k$, $m_2 = n_{k+1}$ in the $(k + 1)$-st step).

Denote by z the limit of the Cauchy sequence $\{x_k\}_{k=0}^{\infty}$, $z = \lim_{k \to \infty} x_k$. Then

$$\|T^j z\| = \lim_{k \to \infty} \|T^j x_k\| \geqq r^j a'_j \quad (j = 1, 2, \ldots)\,.$$

Further, $\|z\| \leqq \|x_0\| + \|x_1 - x_0\| + \|x_2 - x_1\| + \ldots \leqq 1 + \frac{1}{2}d + \frac{1}{4}d + \ldots =$ $= 1 + d$.

Put $x = z/\|z\|$ (clearly $z \neq 0$). Then $\|x\| = 1$ and $\|T^j x\| \geqq r^j a_j$ $(j = 1, 2, \ldots)$.

2. Let $z \in X$ and $\varepsilon > 0$ be arbitrary. Denote by n_i $(i = 0, 1, 2, \ldots)$ the smallest index such that

$$a_n < \frac{\varepsilon}{3 \,.\, 2^{i+1}} \, (n > n_i)\,.$$

Put $y_0 = z$. Using Lemma 3 repeatedly we construct a sequence $\{y_k\}_{k=0}^{\infty}$ such that $\|y_{k+1} - y_k\| \leqq \varepsilon/2^{k+1}$ and

$$\|T^j y_{k+1}\| > r^j a_j \quad (j = n_0 + 1, \ldots, n_{k+1})$$

(put $x = y_k$, $y = y_{k+1}$, $\delta = \varepsilon/2^{k+1}$, $m_0 = n_0$, $m_1 = n_k$, $m_2 = n_{k+1}$ in the $(k + 1)$-st step).

Let $y = \lim_{k \to \infty} y_k$. Then

$$\|T^j y\| \geqq r^j a_j \quad (j = n_0 + 1, \ldots)$$

and

$$\|y - z\| \leqq \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon + \ldots = \varepsilon\,.$$

Hence the set Y of all $y \in X$ such that $\|T^j y\| \geqq r^j a_j$ $(j \geqq j(y))$ is dense in X.

Remark. The estimate in Theorem is the best possible. Let H be a separable complex Hilbert space with an orthonormal basis $\{e_0, e_1, \ldots\}$ and let T be the backward shift defined by $Te_0 = 0$, $Te_i = e_{i-1}$ $(i = 1, 2, \ldots)$. Then $r(T) = r_e(T) = 1$ and $\lim_{j \to \infty} T^j x = 0$ for every $x \in H$.

4. Corollary. *Let $T \in B(X)$. Then*

$$\sup_{\substack{x \in X \\ \|x\| = 1}} \inf_{n=1,2,\ldots} \|T^n x\|^{1/n} = \inf_{n=1,2,\ldots} \sup_{\substack{x \in X \\ \|x\| = 1}} \|T^n x\|^{1/n} = r(T).$$

References

[1] *C. Apostol:* On the left essential spectrum and non-cyclic operators in Banach spaces, Rev. Roum. de Math. Puris et Appl. *17* (1972), 1141—1147.

[2] *S. R. Carradus, W. E. Pfaffenberger, B. Yood:* Calkin algebras and algebras of operators on Banach spaces, Marcel Pokker, New York, 1974.

[3] *I. Colojara, C. Foias:* Theory of generalized spectral operators, Gordon and Breach, New York, 1968.

[4] *J. Daneš:* On local spectral radius Čas. pěst. mat. *112* (1987), 177—187.

[5] *V. Müller:* On quasialgebraic operators in Banach spaces, J. Oper. Theory *17* (1987), 291—300.

[6] *F. H. Vasilescu:* Residually decomposable operators in Banach spaces, Tôhoku Math. J. *21* (1969), 509—522.

[7] *P. Vrbová:* On local spectral properties of operators in Banach spaces, Czech. Math. J. *23* (1973), 483—492.

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