Valter Šeda

Some remarks to coincidence theory

Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 3, 554–572

Persistent URL: http://dml.cz/dmlcz/102250

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

SOME REMARKS TO COINCIDENCE THEORY

VALTER ŠEDA, Bratislava

(Received September 25, 1986)

Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

The aim of this paper is to show the links between the coincidence theory developed by J. Mawhin (see [8]) and the results obtained by L. Cesari and R. Kannan in [2], J. R. Ward, Jr. in [11], S. Ahmad in [1]. A result by J. Mawhin [9] is extended to Banach lattices, and a surjectivity result closely related to that by S. Fučík [5] is established.

1. SOME RESULTS ON LINEAR OPERATORS

Let $(X, |\cdot|_X)$, $(Z, |\cdot|_Z)$ be real Banach spaces with the property that $X \subset Z$ and there is an $\alpha > 0$ such that

$$|x|_Z \leqq \alpha |x|_X \quad \text{for all} \quad x \in X . \tag{1}$$

Hence $(X, |\cdot|_X) \hookrightarrow (Z, |\cdot|_Z)$.

Let L: $D(L) \subset X \to Z$ be a linear mapping which satisfies the following condition:

(L_1) L is a Fredholm mapping of index zero, that is, (i) ker L has a finite dimension; (ii) Im L is closed and has a finite codimension such that Ind L = dim ker L − − codim Im $L = 0$.

Then there exist continuous projectors $P: X \to X$, $Q: Z \to Z$ such that

$$\operatorname{Im} P = \ker L , \quad \ker Q = \operatorname{Im} L \tag{2}$$

and

$$X = \ker L \oplus \ker P , \quad Z = \operatorname{Im} Q \oplus \operatorname{Im} L \tag{3}$$

as topological direct sums. Consequently, the restriction L_P of L to $D(L) \cap \ker P$ is one-to-one and onto Im L so that its algebraic inverse K_P: Im $L \to D(L) \cap \ker P$ is defined.

The properties of K_P will be determined by one of the following assumptions:

(L_2) K_P: Im $L \subset Z \to X$ is continuous.

(L_2') There exists a continuous linear operator $A: X \to Z$ such that $L - A$: $D(L) \subset$ $\subset X \to Z$ is one-to-one and onto and such that the inverse operator $(L - A)^{-1}$:

$Z \to D(L) \subset X$ is continuous. This means that for some $M > 0$ and all $y \in Z$,

$$|(L - A)^{-1} y|_X \leqq M|y|_Z .$$

Lemma 1. *If the conditions* (L_1), (L_2') *hold, then*

(i) *both operators* L, L_P *are closed;*

(ii) *the operator* K_P *is continuous, that is, there exists a* $\Lambda > 0$ *such that for each* $x \in D(L)$, $x = \bar{x} + \tilde{x}$ *with* $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$,

$$|\tilde{x}|_X \leqq \Lambda|Lx|_Z .$$

Proof. (i) Since the proof of closedness of L is similar to that of closedness of L_P, we only prove that the operator L_P is closed. Let $\tilde{x}_n$ be a sequence in $D(L) \cap \ker P$ such that $\tilde{x}_n \to \tilde{x} \in \ker P$ in X and $L\tilde{x}_n \to \tilde{y}$ in Z for $n \to \infty$. By (L_2'), $A\tilde{x}_n \to A\tilde{x}$ in Z and hence $(L - A)\tilde{x}_n \to \tilde{y} - A\tilde{x}$. Since $(L - A)^{-1}$ is continuous as a mapping from Z to X, $\tilde{x}_n \to (L - A)^{-1}(\tilde{y} - A\tilde{x})$ and thus, $\tilde{x} = (L - A)^{-1}(\tilde{y} - A\tilde{x}) \in D(L)$ as well as $(L - A)\tilde{x} = \tilde{y} - A\tilde{x}$ which implies $L\tilde{x} = \tilde{y}$.

(ii) By the closed graph theorem, $D(K_P) = \operatorname{Im} L$ is a closed subset of the Banach space Z. Consequently, the continuity of K_P follows from the closedness of the mapping L_P which is the inverse of K_P.

Denote by $\|K_P\|$ the norm of the operator K_P.

(L_2'') There exists a continuous linear operator $A: X \to Z$ such that $L - A: D(L) \subset X \to Z$ is one-to-one and onto and such that the inverse operator $(L - A)^{-1}: Z \to D(L) \subset X$ is completely continuous.

(L_3) The operator K_P: $\operatorname{Im} L \subset Z \to X$ is completely continuous.

Remark 1. By Lemma 1, $(L_1), (L_2')$ imply the assumptions $(L_1), (L_2)$. The conditions $(L_1), (L_2'')$ guarantee that $(L_1), (L_2')$ and (L_3) hold. In fact, if $y_n \in \operatorname{Im} L$ is a bounded sequence and $\tilde{x}_n = K_P y_n$, then, by Lemma 1, $\tilde{x}_n$ is also bounded, and $(L - A)\tilde{x}_n = y_n - A\tilde{x}_n$ is a bounded sequence. Hence, by (L_2''), the sequence $\tilde{x}_n = (L - A)^{-1}(y_n - A\tilde{x}_n)$ contains a convergent subsequence and the implication $(L_1), (L_2'') \Rightarrow (L_1), (L_2'), (L_3)$ is proved.

Further, we will assume

(L_4) $\operatorname{Im} L \cap \ker L = \{0\}$

or a stronger condition

(L_4') $Z = \ker L \oplus \operatorname{Im} L$.

A sufficient condition for $(L_1), (L_2''), (L_3)$ and (L_4') will result from the following theorem. I will mean the identity on Z, while $\varrho(A)$ and $\sigma(A)$ will denote the resolvent set and the spectrum, respectively, of a linear operator A: $D(A) \subset Z \to Z$.

Theorem 1. *Let* $(Z, |\cdot|_Z)$ *be an infinitely dimensional real Banach space, let* L: $D(L) \subset Z \to Z$ *be a linear mapping such that*

(H_0) *for some* $\lambda \in (-\infty, \infty)$ *the operator* $L - \lambda I$ *is one-to-one and onto* Z, $(L - \lambda I)^{-1}$ *is completely continuous on* Z

and

(L_4) $\operatorname{Im} L \cap \ker L = \{0\}$.

Denote by $\{\lambda_n\}$ *the sequence of all eigenvalues of* $(L - \lambda I)^{-1}$ (*it may be finite or even void*) *and by* $\{x_n\}$ *the corresponding sequence of eigenvectors* (*of* $(L - \lambda I)^{-1}$), *where each term* λ_n *occurs in the sequence* $\{\lambda_n\}$ *so many times as its multiplicity indicates.*

Then the following statements are true:

(i) *The operator L is closed, its resolvent set* $\varrho(L)$ *is non void and for each* $\lambda_0 \in \varrho(L)$ *the resolvent* $(L - \lambda_0 I)^{-1}$ *is a completely continuous operator defined everywhere on Z.*

(ii) *The spectrum* $\sigma(L)$ *consists of the eigenvalues* $\mu_n = \lambda + 1/\lambda_n$ *of L only, and* x_n *are the corresponding eigenvectors.*

(iii) *L is a Fredholm mapping.*

(iv) $Z = \ker L \oplus \operatorname{Im} L$.

Remark 2. By the Riesz-Schauder theory, [12], p. 284, $\sigma((L - \lambda I)^{-1})$ consists of an at most countable set of points of the complex plane with at most one accumulation point which may be 0. Clearly $0 \in \sigma((L - \lambda I)^{-1})$, but it is not an eigenvalue of $(L - \lambda I)^{-1}$. Each nonzero number of $\sigma((L - \lambda I)^{-1})$ is an eigenvalue of $(L - \lambda I)^{-1}$ of finite multiplicity (there exist only finitely many linearly independent eigenvectors of $(L - \lambda I)^{-1}$ corresponding to a given eigenvalue).

Proof of the theorem. (i) By (H_0), $\lambda \in \varrho(L)$. Further, (H_0) implies that $(L - \lambda I)^{-1}$ is a closed operator and thus, $L - \lambda I$ is closed, too. The sum of a closed and of an everywhere (on Z) continuous operator is closed and therefore $L = (L - \lambda I) + \lambda I$ is closed.

Let $\lambda_0 \in \varrho(L)$. Then $L - \lambda_0 I$ is a closed operator, its inverse $(L - \lambda_0 I)^{-1}$ is continuous, hence, by Theorem 4.2-E in [10], p. 174, the range $\operatorname{Im}(L - \lambda_0 I)$ of $L - \lambda_0 I$ is a closed subset of Z. Since it is dense in Z, it is equal to Z. Therefore we can apply the resolvent identity (Theorem 5.1-C in [10], p. 245)

$$(L - \lambda_0 I)^{-1} - (L - \lambda I)^{-1} = (\lambda_0 - \lambda)(L - \lambda I)^{-1} \circ (L - \lambda_0 I)$$

and again by (H_0), $(L - \lambda_0 I)^{-1}$ is completely continuous.

(ii) Consider the operators $(L - \lambda I)^{-1}$ and $(L - \lambda I)$. By (H_0), 0 is not an eigenvalue of $(L - \lambda I)^{-1}$ and $0 \in \varrho(L - \lambda I)$. Further, λ_n is an eigenvalue of $(L - \lambda I)^{-1}$ and x_n is the corresponding eigenvector of this operator iff $\varkappa_n = 1/\lambda_n$ is an eigenvalue of the operator $L - \lambda I$ and x_n is its eigenvector. Consequently, $\mu_n = \lambda + 1/\lambda_n$ are eigenvalues of L, x_n are the corresponding eigenvectors, and there are no other eigenvalues of L.

Let $\mu = (\lambda + v) \in C$, where $v \neq 0$, $v \neq 1/\lambda_n$. We can write

$$L - (\lambda + v) I = \left[\frac{1}{v} I - (L - \lambda I)^{-1}\right] \circ v(L - \lambda I)$$

and since $1/v \neq \lambda_n$, the operator

$$\frac{1}{v} I - (L - \lambda I)^{-1}$$

is one-to-one. Therefore

$$[L-(\lambda+\nu)I]^{-1}=\frac{1}{\nu}(L-\lambda I)^{-1}\circ\left[\frac{1}{\nu}I-(L-\lambda I)^{-1}\right]^{-1}$$

and, by Theorem 5.5-F in [10], p. 266,

$$\left[\frac{1}{\nu}I-(L-\lambda I)^{-1}\right]^{-1}$$

is continuous on Z. Hence $\mu\in\varrho(L)$. The statement is proved.

(iii) If $0\in\varrho(L)$, then by (i), L is one-to-one, $\operatorname{Im}L=Z$ and hence (iii) and (iv) are true. Thus we will consider the nontrivial case $0\in\sigma(L)$.

Let $\sigma_e(L)$ be the extended spectrum of the operator L. If L were bounded, then so would be $L-\lambda I$, and $I=(L-\lambda I)\circ(L-\lambda I)^{-1}$ would be compact which contradicts the fact that Z is an infinitely dimensional Banach space. Hence $\sigma_e(L)=\sigma(L)\cup\{\infty\}$. Since 0 is an isolated point of $\sigma_e(L)$, the sets

$$\sigma_1=\{0\}\,,\quad \sigma_2=\sigma_e(L)-\sigma_1$$

are two spectral sets of the operator L. Denote by E_{σ_1}, $E_{\sigma_2}=I-E_{\sigma_1}$ the corresponding projections and let $Z_1=\operatorname{Im}E_{\sigma_1}$, $Z_2=\operatorname{Im}E_{\sigma_2}$ be their ranges. Then Z_1,Z_2 are closed vector subspaces of Z and

(4) $$Z=Z_1\oplus Z_2\,.$$

Let $L_1=L|_{D(L)\cap Z_1}$, $L_2=L|_{D(L)\cap Z_2}$. Then L is reduced by the subspaces Z_1,Z_2 and by Theorem 5.4-A, [10], p. 256,

(5) $$D(L)=D(L_1)\oplus D(L_2)\,,\quad \operatorname{Im}L=\operatorname{Im}L_1\oplus\operatorname{Im}L_2$$

and

$$\operatorname{Im}L_1\subset Z_1\,,\quad \operatorname{Im}L_2\subset Z_2\,.$$

Since L has only the point spectrum by Theorem 5.7-B, [10], p. 283, the operators L_1,L_2 have the same property and $\sigma_e(L_i)=\sigma_i$, $i=1,2$. As $\infty\notin\sigma_1$, $D(L_1)=Z_1$ and L_1 is continuous on Z_1, 0 is the unique eigenvalue of L_1. The operator $L_1-\lambda I$: $Z_1\to Z_1$ is also bounded, and $(L_1-\lambda I)^{-1}\circ(L_1-\lambda I)=I|_{Z_1}$ is compact. Therefore

(6) Z_1 is a finitely dimensional vector subspace of Z.

As to the operator L_2, $L_2\colon D(L_2)\subset Z_2\to Z_2$, we have $0\notin\sigma_e(L_2)$, hence $0\in\varrho(L_2)$. Since the reduction of a closed operator to a closed vector subspace is again a closed operator, L_2 is closed and the relation $0\in\varrho(L_2)$ implies that the inverse operator L_2^{-1} exists. Moreover, it is continuous and again by Theorem 4.2-E, [10], p. 174, $\operatorname{Im}L_2=$ $=Z_2$. So we have

$$L_2\colon D(L_2)\subset Z_2\to Z_2\,,\quad L_2 \text{ is one-to-one and onto } Z_2\,,$$

and

(7) $$Z=Z_1\oplus\operatorname{Im}L_2\,.$$

Two cases should be considered. If $L_1 x = 0$ for all $x \in Z_1$, then by (7),

$$Z = \ker L_1 \oplus \operatorname{Im} L_2 . \tag{8}$$

Let $Lx = 0$. Then $x = x_1 + x_2$, $x_1 \in \ker L_1$, $x_2 \in \operatorname{Im} L_2$, and $Lx_2 = 0$, which implies $L_2 x_2 = 0$ and $x_2 = 0$. Thus $\ker L \subset \ker L_1 = Z_1$ and since $\ker L_1 \subset \ker L$ is obviously true, $\ker L = \ker L_1$. By (5) we have $\operatorname{Im} L = \operatorname{Im} L_2$ and, by virtue of (8), the last two equalities give

$$Z = \ker L \oplus \operatorname{Im} L . \tag{9}$$

Thus, L is a Fredholm mapping of index zero.

Suppose now that $\ker L_1 \neq Z_1$. Then for an $x \in \ker L$ we have $L^n x/\varepsilon^n = 0$ for any $\varepsilon > 0$ and $n \geqq 1$. Hence $\lim_{n \to \infty} (L^n x/\varepsilon^n) = 0$ for each $\varepsilon > 0$. By Lemma 5.8-C, [10], p. 292, this implies that $x \in Z_1$ and thus $\ker L \subset Z_1$. In view of (6), this implies that $\ker L$ is finitely dimensional and Z_1 can be decomposed into the sum $Z_1 = \ker L \oplus$ $\oplus Z_{12}$ where $L_{12} = L_1|_{Z_{12}}$ is one-to-one and by virtue of (L_4), $L_{12}\colon Z_{12} \to Z_{12}$. But Z_{12} is a finitely dimensional vector space and L_{12} is one-to-one. Hence L_{12} maps Z_{12} onto Z_{12}, $Z_{12} = \operatorname{Im} L_{12} = \operatorname{Im} L_1$ and we have $Z_1 = \ker L \oplus \operatorname{Im} L_1$. By (4), (7), (5), $Z = Z_1 \oplus Z_2 = \ker L \oplus \operatorname{Im} L_1 \oplus \operatorname{Im} L_2 = \ker L \oplus \operatorname{Im} L$. As (9) is true, each class of the factor space $Z|_{\operatorname{Im} L}$ is a set $z_1 + \operatorname{Im} L$ where $z \in_1 \ker L$. Hence $\dim \ker L = \dim Z|_{\operatorname{Im} L} = \operatorname{codim} \operatorname{Im} L$. Since $\ker L$ has a finite dimension and $\operatorname{Im} L$ is closed, L is a Fredholm mapping of index zero.

Remark 3. If we replace (H_0) by a stronger assumption (compare with [11], p. 233)

(H_1) for some $\lambda \in (-\infty, \infty)$ the operator $L - \lambda I$ is one-to-one and onto Z, $(L - \lambda I)^{-1}$ is completely continuous as a mapping from Z into X;

then this condition implies (L_2'') and, in view of Theorem 1, (H_1) and (L_4) imply the conditions (L_1), (L_2''), (L_3) and (L_4').

2. EXISTENCE STATEMENTS

Let $F\colon X \to Z$ be a mapping which is continuous and bounded, i.e., F maps bounded sets into bounded sets. The existence statements for such a mapping will be based on a lemma which follows from Theorem IV.3 and Proposition II.18 in [8], pp. 43 and 22.

Lemma 2. *Let the following conditions be satisfied. X and Z are real normed vector spaces, $L\colon D(L) \subset X \to Z$ is a Fredholm mapping of index zero, Ω is an open bounded subset of X such that $0 \in \Omega$, Ω is symmetric with respect to 0. Further, $F\colon \bar{\Omega} \to Z$ is an L-compact mapping, $G\colon \bar{\Omega} \to Z$ is an L-compact mapping which is odd and such that $0 \notin (L - G)(D(L) \cap \partial\Omega)$ where $\bar{\Omega}$ is the closure of Ω and $\partial\Omega$ is the boundary of Ω (with respect to X).*

Further, let

$$\lambda(L - F)x + (1 - \lambda)(L - G)x \neq 0 \quad \textit{for all} \quad (x, \lambda) \in (D(L) \cap \partial\Omega) \times (0, 1).$$

Then the equation

$$(10) \qquad Lx = Fx$$

has at least one root in $D(L) \cap \bar{\Omega}$.

We recall that $F: \bar{\Omega} \to Z$ is L-compact iff $Q \circ F$ is continuous, $(Q \circ F)(\bar{\Omega})$ is bounded and $K_P \circ (I - Q) \circ F$ is compact, where Q and K_P have the meaning as above.

Theorem 2. *Let L satisfy the conditions (L_1), (L_3) and (L_4) and let F fulfil the following conditions:*

(F_1) *There exist constants $a, b > 0$ such that*

$$(11) \qquad a\|K_P\| < 1$$

and

$$|Fx|_Z \leqq a|\tilde{x}|_X + b$$

for all $x = \bar{x} + \tilde{x} \in D(L)$ with $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$;

(F_2) *Let $\varepsilon = \pm 1$ and denote $R_1 = \|K_P\| \, b/(1 - \|K_P\| \, a)$. There exists an $R_2 > 0$ with the following property: for all $x = \bar{x} + \tilde{x} \in D(L)$ and $k \in R$ such that $|\bar{x}|_X \geqq R_2$, $|\tilde{x}|_X < R_1$*

and

$$(12) \qquad \varepsilon F(\bar{x} + \tilde{x}) + k\bar{x} \in \operatorname{Im} L$$

one has

$$(13) \qquad k \leqq 0 .$$

Then the problem (10) *has at least one solution.*

Proof. First of all we assert that F is L-completely continuous on X. This means that F is L-compact on each bounded subset of X. The same is true for the mapping $G: X \to Z$ which is defined by

$$Gx = \varepsilon \frac{b}{2} \frac{1}{1 + |\bar{x}|_Z} \bar{x}, \quad x = \bar{x} + \tilde{x}, \quad \bar{x} \in \ker L, \quad \tilde{x} \in \ker P .$$

We see that G is odd and L-completely continuous.

Now we shall show that the set of all possible solutions of the family of equations

$$(14) \qquad Lx = (1 - \lambda)\, Gx + \lambda\, Fx, \quad \lambda \in [0, 1)$$

is a priori bounded independently of λ. Let $\lambda \in [0, 1)$ and let x be a possible solution of (14). Denote $x = \bar{x} + \tilde{x}$, $\bar{x} \in \ker L$, $\tilde{x} \in D(L) \cap \ker P$. Then by (F_1),

$$|Lx|_Z \leqq (1 - \lambda) \frac{b}{2} \frac{|\bar{x}|_Z}{1 + |\bar{x}|_Z} + \lambda(a|\tilde{x}|_X + b) < a|\tilde{x}|_X + b .$$

Hence (L_2) gives

$$(15) \qquad |\tilde{x}|_X \leqq \|K_P\| \, |Lx|_Z < \|K_P\| \, a|\tilde{x}|_X + \|K_P\| \, b ,$$

which yields

$$(16) \qquad |\tilde{x}|_X < R_1 .$$

R_1 does not depend on λ. (14) implies

$$(17) \qquad (1 - \lambda)\, \varepsilon \frac{b}{2} \frac{1}{1 + |\bar{x}|_Z}\, \bar{x} + \lambda F(\bar{x} + \tilde{x}) \in \operatorname{Im} L .$$

If $\lambda = 0$, then $\bar{x} \in \operatorname{Im} L \cap \ker L$ and, by (L_4), $\bar{x} = 0$. For $0 < \lambda < 1$ we have

$$F(\bar{x} + \tilde{x}) + \frac{1 - \lambda}{\lambda}\, \varepsilon \frac{b}{2} \frac{1}{1 + |\bar{x}|_Z}\, \bar{x} \in \operatorname{Im} L .$$

By (F_2),

$$|\bar{x}|_X < R_2 .$$

Now we choose $\Omega = \{x \in X: x = \bar{x} + \tilde{x},\ \bar{x} \in \ker L,\ \tilde{x} \in \ker P,\ |\bar{x}|_X < R_2, |\tilde{x}|_X < R_1\}$. Then we see that all assumptions of Lemma 2 are satisfied and hence there is a solution of (10) in Ω.

Remark 4. In [2], L. Cesari and R. Kannan have proved an existence theorem. In that theorem $X = Z = S$ and S is a real separable Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $|\cdot|_S$. Further, $P = Q$ and the assumptions (L_1), (L_2), (L_3) and (L'_4) are satisfied. The condition (F_1) is supposed with $a = 0$ ($F(S)$ is bounded) and instead of (F_2) the following assumption is used.

(F'_2) Let $\varepsilon = \pm 1$ and denote $R_1 = \|K_P\|\, b$. Then there exists an $R_2 > 0$ such that for all $x \in S$, $x = \bar{x} + \tilde{x}$, $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$, $|\bar{x}|_S \geqq R_2$, $|\tilde{x}|_S \leqq R_1$ we have

$$(18) \qquad (\varepsilon\, F(\bar{x} + \tilde{x}), \bar{x}) \geqq 0 .$$

We show that (F'_2) implies (F_2) if P is an orthogonal projection. Indeed, the condition (12) is equivalent to $(\varepsilon\, F(\bar{x} + \tilde{x}) + k\bar{x}, \bar{x}) = 0$, since $PS = \ker L$ and $(I - P)\, S = \operatorname{Im} L$ are orthogonal to each other (Theorem 1, [12], p. 82). Then (18) implies (13). Hence Theorem 2 extends the result of the Cesari and Kannan theorem.

Remark 5. Theorem 2 still holds if the conditions (F_1) and (F_2) are replaced by the conditions (F_3), (F_4) given below, from which (F_3) is weaker than (F_1), but (F_4) is stronger than (F_2).

(F_3) There exist constants $a, b > 0$ such that (11) holds and

$$|Fx|_Z \leqq a|x|_X + b$$

for all $x = \bar{x} + \tilde{x} \in D(L)$ with $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$.

(F_4) Let $\varepsilon = \pm 1$ and denote $R_1 = \|K_P\|\, b/(1 - \|K_P\|\, a)$, $d = \|K_P\| .\, a/(1 - \|K_P\|\, a)$. There exists an $R_2 > 0$ with the following property: for all $x = \bar{x} + \tilde{x} \in D(L)$ and $k \in R$ such that

$$|\bar{x}|_X \geqq R_2 , \quad |\tilde{x}|_X < d|\bar{x}|_X + R_1$$

and (12) holds, the inequality (13) is true.

Indeed, by (F_3), for each solution $x = \bar{x} + \tilde{x}$ of (14) we have instead of (15) the inequality

$$|\tilde{x}|_X < \|K_P\| \, a|x|_X + \|K_P\| \, b \, .$$

Since $|x|_X \leqq |\bar{x}|_X + |\tilde{x}|_X$, this solution satisfies $|\tilde{x}|_X < d|\bar{x}|_X + R_1$. Now either $|\bar{x}|_X < R_2$ and we proceed similarly as in the proof of Theorem 2 or $|\bar{x}|_X \geqq R_2$. By (F_4), the last case cannot arise as can be easily seen by inspecting (17).

Instead of (F_3) and (F_4) a couple of modified assumptions can be used. Thus the following theorem is true.

Theorem 3. *Let L satisfy the conditions $(L_1), (L_3)$ and (L_4) and let F fulfil the conditions*

(F_5) *there exist constants $a, b > 0$ such that*

$$|Fx|_Z \leqq a|x|_X + b$$

for all $x = \bar{x} + \tilde{x} \in D(L)$ with $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$;

(F_6) *let $\varepsilon = \pm 1$ and $d_1 > 0$. The constant a in the assumption (F_5) satisfies*

$$a < \frac{1}{\|K_P\|} \frac{d_1}{1 + d_1} \, . \tag{19}$$

There exists an $R_2 > 0$ with the following property: for all $x = \bar{x} + \tilde{x} \in D(L)$ and $k \in R$ such that

$$|\bar{x}|_X \geqq R_2 \, , \quad |\tilde{x}|_X \leqq d_1|\bar{x}|_X$$

and (12) holds, the inequality (13) is true.

Then the problem (10) has at least one solution.

Proof. With respect to Remark 5 it suffices to show that (F_3) and (F_4) are satisfied. Clearly (19) implies (11). Thus (F_3) holds. Put $R_1 = \|K_P\| \, b/(1 - \|K_P\| \, a)$, $d = \|K_P\| \, a/(1 - \|K_P\| \, a)$. Then $d < d_1$ and since the inequality $d|\bar{x}|_X + R_1 \leqq d_1|\bar{x}|_X$ holds for all sufficiently great $|\bar{x}|_X$, (F_4) is satisfied as well.

Corollary 1. *Let $X = Z = S$ be a real Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $|\cdot|_S = |\cdot|_X = |\cdot|_Z$. Let L satisfy the conditions $(L_1), (L_3)$ and (L_4), let $P: S \to \ker L$ be the orthogonal projection, and let F satisfy the conditions (F_5), and*

(F_7) *let $\varepsilon = \pm 1$, and $d_1 > 0$. The constant a in the assumption (F_5) satisfies (19) and there exists an $R_2 > 0$ such that for all $x = \bar{x} + \tilde{x} \in D(L)$ with $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$,*

$$|\bar{x}|_S \geqq R_2 \, , \quad |\tilde{x}|_S \leqq d_1|\bar{x}|_S \, ,$$

(18) is true.

Then the problem (10) has at least one solution.

Proof. Similarly as in Remark 4 we get that (F_6) is satisfied.

With help of this corollary the following theorem on surjectivity can be proved which is closely related to that in [5], p. 74, given by S. Fučík.

Theorem 4. *Let X, Z, S, L and P satisfy the same conditions as in Corollary* 1 *and let F satisfy the conditions* (F_5),

(F_8) *let $\varepsilon = \pm 1$. There exists a $d_1 > 0$ satisfying the following condition: For any $K > 0$ we have a $t_K > 0$ such that*

$$(\varepsilon\, F(t(\bar{y} + \tilde{y})), \bar{y}) \geqq K \tag{20}$$

for all $t \geqq t_K$, $|\bar{y}|_S = 1$, $\bar{y} \in \ker L$, $\tilde{y} \in D(L) \cap \ker P$, $|\tilde{y}|_S \leqq d_1$. The constant a in the assumption (F_5) *satisfies* (19).

Then the equation

$$Lx = Fx + h$$

has a solution $x \in D(L)$ for each $h \in S$.

Proof. Let $h \in S$ be arbitrary but fixed. It is sufficient to show that the equation $Lx = Gx$ is solvable, where $Gx = Fx + h$ for each $x \in S$. Clearly G is continuous and bounded. Further, $|Gx|_S \leqq a|x|_S + (b + |h|_S)$ for each $x \in S$. Thus G satisfies (F_5). As to (F_7), we proceed as follows. S is reflexive, therefore by the Eberlein-Shmulyan theorem, there exist real numbers k_1, k_2 such that

$$k_1 = \min_{\substack{\bar{y} \in \ker L \\ |\bar{y}|_S = 1}} (h, \bar{y}), \quad k_2 = \max_{\substack{\bar{y} \in \ker L \\ |\bar{y}|_S = 1}} (h, \bar{y}).$$

Clearly $k_1 \leqq 0$, $k_2 \geqq 0$. Denote $-K_1 = \min(k_1, -1)$, $K_2 = \max(k_2, 1)$. Consider the case $\varepsilon = 1$. Since $K_1 > 0$, by (F_8) there exists a $t_{K_1} > 0$ with the above mentioned properties. Put $R_2 = t_{K_1}$ and consider an arbitrary $x = \bar{x} + \tilde{x} \in D(L)$ with $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$, $|\bar{x}|_S \geqq R_2$, $|\tilde{x}|_S \leqq d_1 |\bar{x}|_S$. We denote $|\bar{x}|_S = t$ and define $\bar{y}$, $\tilde{y}$ by $\bar{x} = t\bar{y}$, $\tilde{x} = t\tilde{y}$. Clearly $t \geqq t_{K_1}$, $|\bar{y}|_S = 1$, $\bar{y} \in \ker L$, $\tilde{y} \in D(L) \cap \ker P$, $|\tilde{y}|_S \leqq d_1$. Then (20) yields

$$(G(\bar{x} + \tilde{x}), \bar{x}) = (G(t(\bar{y} + \tilde{y})), t\bar{y}) = t[(F(t(\bar{y} + \tilde{y})), \bar{y}) + (h, \bar{y})] \geqq$$
$$\geqq t[(F(t(\bar{y} + \tilde{y}), \bar{y}) - K_1] \geqq 0$$

and hence (F_7) is satisfied. In the case $\varepsilon = -1$ we put $R_2 = t_{K_2}$ and proceed similarly. For a suitable $x = \bar{x} + \tilde{x}$ we come to the inequality

$$(\varepsilon\, G(\bar{x} + \tilde{x}), \bar{x}) = t[\varepsilon(F(t(\bar{y} + \tilde{y})), \bar{y}) + \varepsilon(h, y)] \geqq$$
$$\geqq t[\varepsilon(F(t(\bar{y} + \tilde{y})), \bar{y}) + \varepsilon K_2] \geqq 0$$

and again (F_7) holds. By Corollary 1 the theorem follows.

Theorem 5. *Let L satisfy the conditions* (L_1), (L_3) *and* (L_4). *Let F satisfy the condition*

(F_9) *Let $\varepsilon = \pm 1$. There exists an $R > 0$ with the following property: for all $x = \bar{x} + \tilde{x} \in D(L)$, $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$ such that $|x|_X = R$ and* (12) *holds, the inequality* (13) *is true.*

Then the problem (10) *has at least one solution in $\bar{\Omega} \cap D(L)$ where $\Omega = \{x \in X: |x|_X < R\}$.*

Proof. Consider the mapping $Gx = \varepsilon k_1 x$, $k_1 > 0$, $x = \bar{x} + \tilde{x} \in X$, $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$. G is odd and L-completely continuous on X. By (L_4), the only solution of $Lx = Gx$ is $x = 0$. As to the solutions of (14) for $0 < \lambda < 1$, they satisfy $\varepsilon Fx + ((1 - \lambda)/\lambda)\, k_1 \bar{x} \in \operatorname{Im} L$ and, in view of (F_9), they do not belong to the boundary $\partial\Omega$ of Ω. By Lemma 2 the statement of the theorem follows.

Corollary 2. *Let X, Z, S, L and P satisfy the same condition as in Corollary 1 and let F fulfil the condition:*

(F10) *Let $\varepsilon = \pm 1$. There exists an $R > 0$ such that for all $x = \bar{x} + \tilde{x} \in D(L)$, $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$ with $|x|_X = R$, (18) is true.*

Then the problem (10) has at least one solution in $\bar{\Omega} \cap D(L)$ where Ω is defined above.

Proof. Similarly as in Remark 4, we get by (F_{10}) that the implication "(12) implies (13)" is true and hence (F_9) is satisfied.

Remark 6. In view of Remark 3 we see that Corollary 2 under the assumptions (H_1), (L_4) and (F_{10}) brings a result which is similar to that of Theorem 1 in [11], p. 233.

3. EXISTENCE THEOREMS IN BANACH LATTICES

In this part we generalize a result given by J. Mawhin in [9]. Let $(Z, |\cdot|_Z, \leqq)$ be a real partially ordered Banach space with the positive cone $K = \{z \in Z \colon z \geqq 0\}$. Further, let $(Z, |\cdot|_Z, \leqq)$ be a Banach lattice, that is, for each pair of elements $x, y \in Z$ there exist

$$\sup(x, y) = \text{the l.u.b. of } x, y$$

and

$$\inf(x, y) = \text{the g.l.b. of } x, y\,,$$

and for the modulus $|x|$ of the element $x \in Z$ which is defined by the relation

$$|x| = \sup(x, -x)$$

the statement

(21) $$|x| \leqq |y| \quad \text{implies} \quad |x|_Z \leqq |y|_Z$$

is true. In particular,

(22) $$\text{if} \quad |x| = |y|\,, \quad \text{then} \quad |x|_Z = |y|_Z\,.$$

Further, for each $x \in Z$ we define

$$x^+ = \sup(x, 0)\,, \quad x^- = \sup(-x, 0)\,.$$

Since $\inf(x, 0) = -\sup(-x, 0)$, by [12], pp. 361–356 we have

(23) $$x = x^+ - x^-\,, \quad x = x^+ + x^-\,.$$

Further,

$$(24) \qquad |x| \geqq 0 \,, \quad \text{and} \quad |x| = 0 \quad \text{iff} \quad x = 0 \,,$$

$$(25) \qquad |x + y| \leqq |x| + |y| \,, \quad |cx| = |c| \, |x| \quad \text{for each} \quad c \in R \,.$$

Since

$$(26) \qquad x \geqq 0 \quad \text{implies} \quad |x| = x$$

and thus $|(|x|)| = |x|$, by (22) we have

$$(27) \qquad | \, |x| \, |_Z = |x|_Z \,.$$

Remark 7. In view of (21), (26), the cone K of the Banach lattice Z is normal by [3], p. 219, and by virtue of (23) it is reproducing. Of course, it is minihedral as well.

Lemma 3. *Let $\psi\colon Z \to R$ be a continuous, linear, strictly positive functional. Let*

$$(28) \qquad |x|_\psi = \psi(|x|) \quad \textit{for each} \quad x \in Z \,.$$

Then the following statements are true:

(i) *$|\cdot|_\psi$ is a norm defined on Z with the property*

$$(29) \qquad |x|_\psi \leqq \|\psi\| \, |x|_Z$$

where $\|\psi\|$ is the norm of the functional ψ and hence, $(Z, |\cdot|_Z)$ is continuously embedded into the linear normed space $(Z, |\cdot|_\psi)$.

(ii) *$(Z, |\cdot|_\psi)$ is a Banach space iff there is a $k > 0$ such that*

$$(30) \qquad \psi(|x|) \geqq k | \, |x| \, |_Z \quad \textit{for all} \quad x \in Z \,.$$

Proof. (i) By the strict positivity of the functional ψ and by (24) it follows that $\psi(|x|) \geqq 0$, $\psi(|x|) = 0$ iff $x = 0$. Further, (25) implies that $\psi(|x + y|) \leqq \leqq \psi(|x| + |y|) = \psi(|x|) + \psi(|y|)$, $\psi(|cx|) = \psi(|c| \, |x|) = |c| \, \psi(|x|)$. Hence the functional $|\cdot|_\psi$ which is defined by (28) has all properties of the norm. Further, by (27), $|x|_\psi = \psi(|x|) \leqq \|\psi\| \, | \, |x| \, |_Z = \|\psi\| \, |x|_Z$ which proves (29).

(ii) In view of (29), the identity mapping id: $(Z, |\cdot|_Z) \to (Z, |\cdot|_\psi)$ is continuous as well as one-to-one. If $(Z, |\cdot|_\psi)$ is a Banach space, then by the Banach open mapping theorem, the inverse mapping id^{-1} is also continuous. This means that there is a $k > 0$ such that

$$(31) \qquad |x|_\psi \geqq k |x|_Z \quad \text{for all} \quad x \in Z \,,$$

which in view of (28), (29) can be written in the form (30).

Conversely, if (30) and hence (31) is true, then (29) and (31) imply that the norms $|\cdot|_Z$, $|\cdot|_\psi$ are equivalent to each other.

Remark 8. By (24) and (26), the inequality (30) means that $\psi(x) \geqq k|x|_Z$ for all $x \in K$ and hence $(Z, |\cdot|_\psi)$ is a Banach space iff ψ is uniformly positive. Theorem 19.3 in [3], p. 222, asserts that a uniformly positive linear continuous functional on Z exists iff K allows plastering. Let us recall that the cone K allows plastering if there exists a cone K_1 and a $c > 0$ such that the closed ball $\overline{B(x; c|x|_Z)} = \{y \in Z: |y - x|_Z \leqq \leqq c|x|_Z\} \subset K_1$ for all $x \in K - \{0\}$.

In addition to (1) and the conditions (L_1), (L_4') which are always assumed in this section, some of the conditions (L_2), (L_2'), (L_2''), (L_3) will be also used. Further, we shall use some of the following conditions:

(L_5) $\ker L = \operatorname{span}\{\mathscr{C}\}$ with $\mathscr{C} \in D(L) \cap (K - \{0\})$;

(L_6) there is a strictly positive, linear continuous functional $\psi: Z \to R$ such that

$$\text{(32)} \qquad \operatorname{Im} L = \ker \psi\,, \quad \psi(\mathscr{C}) = 1\,;$$

(L_6') there is a uniformly positive, linear continuous functional $\psi: Z \to R$ such that (32) is true.

A sufficient condition for (L_6) is given by the following lemma.

Lemma 4. *Let the conditions* (L_1), (L_4') *and* (L_5) *hold. Then* $\psi: Z \to R$ *is a linear, continuous, strictly positive functional satisfying* (32) *iff*

$$\text{(33)} \qquad \psi(c\mathscr{C} + \tilde{x}) = c \quad \textit{for each} \quad c \in R\,, \quad \tilde{x} \in \operatorname{Im} L$$

and

$$\text{(34)} \qquad \operatorname{Im} L \cap K = \{0\}\,.$$

Moreover, if ψ *is uniformly positive, then* K *allows plastering and is strongly minihedral (i.e., every subset of* Z *which is bounded from above has a supremum).*

Proof. By (L_4') and (L_5) each element $x \in Z$ has a unique representation in the form $x = c\mathscr{C} + \tilde{x}$, $c \in R$, $\tilde{x} \in \operatorname{Im} L$. Hence the linear functional ψ satisfying (32) fulfils (33), and thus, if $x \in \operatorname{Im} L \cap K$, then $\psi(x) = 0$, and by the strict positivity of ψ, $x = 0$ follows. Hence (34) is true. In the case that ψ is uniformly positive, Remark 8 states that K allows plastering. Then by Proposition 19.2, [3], p. 220, K is fully regular and hence regular as well. (K is regular if every increasing sequence which is bounded from above is convergent.) Since ψ is strictly positive and continuous, by Theorem 6.1, [7], p. 46, K is strongly minihedral.

Conversely, suppose that (33) and (34) are true. Then the functional ψ determined by (33) is well defined. Its linearity follows directly from (33). Further, (33) implies (32). Let $x_n = \bar{x}_n + \tilde{x}_n \in Z$, $\bar{x}_n \in \ker L$, $\tilde{x}_n \in \operatorname{Im} L$, $n = 1, 2, \ldots$, be a sequence such that $\lim_{n \to \infty} x_n = x = \bar{x} + \tilde{x}$, $\bar{x} \in \ker L$, $\tilde{x} \in \operatorname{Im} L$. As the projector $Q: Z \to Z$ is continuous, $Q(x_n) = \bar{x}_n \to Q(x) = \bar{x}$ and hence $\bar{x}_n = c_n\mathscr{C} \to c\mathscr{C}$ as $n \to \infty$. But this occurs iff $c_n \to c$ for $n \to \infty$. This implies that $c_n = \psi(c_n\mathscr{C} + \tilde{x}_n) \to c = \psi(c\mathscr{C} + \tilde{x})$ and the continuity of ψ is proved.

Let $x = \bar{x} + \tilde{x} \in K - \{0\}$. If $\bar{x} = c\mathscr{C} \neq 0$, then $c \neq 0$ and $\psi(c\mathscr{C} + \tilde{x}) = c \neq 0$. The case $c < 0$ cannot occur, since otherwise $\mathscr{C} \in K$, $(c\mathscr{C} + \tilde{x}) \in K$ would imply $\tilde{x} \in K \cap \operatorname{Im} L$ and hence, by (34), $\tilde{x} = 0$ which would lead to $-\mathscr{C}, \mathscr{C} \in K$, therefore $\mathscr{C} = 0$ (by a property of the cone) which contradicts $c\mathscr{C} \neq 0$. Thus $\psi(c\mathscr{C} + \tilde{x}) > 0$. In view of (34) the case $\bar{x} = 0$ cannot occur and the proof of the lemma is complete.

Remark 9. As the assumption (L_5) indicates, in this section we are concerned with the case that the equation $Lx = 0$ has a one dimensional space of eigenfunctions spanned over a nonnegative function $\mathscr{C}$. The problem when a strictly positive functional is uniformly positive will be illustrated in the following example.

Example. Let $I = [0, 1]$, $Z = L^1(I)$ with the usual norm $|\cdot|_{L^1}$ and the usual ordering, $\psi \in L^\infty(I)$ such that $\psi(t) > 0$ a.e. in I. Then $\psi(x) = \int_I x(t)\,\psi(t)\,dt$ is a continuous, linear, strictly positive functional in $L^1(I)$. Two cases may occur.

(i) There is a $k > 0$ such that

$$\psi(t) \geqq k \quad \text{a.e. in } I .$$

Then the norm

$$|x|_\psi = \int_I |x(t)|\,\psi(t)\,dt \quad \text{for each} \quad x \in L^1(I)$$

satisfies the inequalities

$$k|x|_{L^1} \leqq |x|_\psi \leqq |\psi|_{L^\infty}\,|x|_{L^1} ,$$

and hence the norms $|\cdot|_{L^1}$, $|\cdot|_\psi$ are equivalent on $L^1(I)$. This implies that the space $L^1_\psi(I) = (L^1(I), |\cdot|_\psi)$ is a Banach space.

(ii) For each n natural and the set $S_n = \{t \in I : \psi(t) \leqq 1/n\}$, the measure $\mu(S_n) > 0$. Then the sequence

$$y_n(t) = \begin{cases} 0 , & t \in I - S_n , \\ \dfrac{1}{\mu(S_n)} , & t \in S_n , \end{cases}$$

enjoys the properties

$$|y_n|_{L^1} = 1 \quad \text{but} \quad |y_n|_\psi \leqq \frac{1}{n} \quad \text{and hence} \quad y_n \to 0 \quad \text{in} \quad L^1_\psi(I) .$$

Therefore the norms $|\cdot|_{L^1}$, $|\cdot|_\psi$ are not equivalent. We show that L^1_ψ need not be complete. If $1/\psi(t) \notin L^1(I)$, e.g. $\psi(t) = t$, $t \in I$, each term of the sequence

$$z_n(t) = \begin{cases} \dfrac{1}{\psi(t)} , & t \in I - S_n , \\ 0 , & t \in S_n , \end{cases}$$

belongs to $L^1(I)$,

$$\left|\frac{1}{\psi(t)} - z_n(t)\right|_\psi = \int_{S_n} dt = \mu(S_n) . \tag{35}$$

Since $[0, 1] \subset S_1 \subset S_2 \subset \ldots$, we have

$$\mu\left(\bigcap_{n=1}^{\infty} S_n\right) = \lim_{n\to\infty} \mu(S_n)$$

and, by $\mu(\bigcap_{n=1}^{\infty} S_n) = 0$, we have $\lim_{n\to\infty} \mu(S_n) = 0$. In view of (35), this implies that the sequence $\{z_n\}$ is fundamental, but not convergent in L^1_ψ. The space L^1_ψ is not a Banach space. Of course it may be completed.

We shall consider two cases. In the first case we assume that the functional ψ given by (33) is uniformly positive and hence the space $(Z, |\cdot|_\psi)$ is a Banach space.

Theorem 6. *Assume that L satisfies the conditions* $(L_1), (L_2'), (L_3), (L_4'), (L_5)$ *and* (L_6') *and that F satisfies the conditions:*

(F_{11}) *there exists a $u \in Z$ and $\varepsilon = \pm 1$ such that*

$$|Fx| \leqq \varepsilon Fx + u \quad \text{for all} \quad x \in X\,; \tag{36}$$

(F_{12}) *there exists $\delta = \pm 1$ and $\varrho > 0$ such that*

$$\delta[F(c\mathscr{C} + \tilde{v})] \leqq 0$$

whenever $c \leqq -\varrho$ and $|\tilde{v}|_X \leqq \Lambda |u|_\psi$,

and

$$\delta[F(c\mathscr{C} + \tilde{v})] \geqq 0$$

for every $c \geqq \varrho$ and $|\tilde{v}|_X \leqq \Lambda |u|_\psi$ where $\tilde{v} \in D(L) \cap \operatorname{Im} L$ and Λ is given in Lemma 1 in the space $(Z, |\cdot|_\psi)$ which will be denoted by $\tilde{Z}$.

Then the problem (10) has at least one solution.

Proof. Denote $\eta = |u|_\psi / \|\psi\|$ and put

$$Gx = \frac{\delta\eta}{2(1 + |\bar{x}|_Z)}\,\bar{x}\,, \quad x \in X\,, \quad x = \bar{x} + \tilde{x}\,, \quad \bar{x} \in \ker L\,, \quad \tilde{x} \in \operatorname{Im} L\,. \tag{37}$$

Then G is odd, L-completely continuous and

$$|Gx|_Z \leqq \frac{\eta}{2}\,. \tag{38}$$

Again we have to find an apriori bound for all possible solutions of (14). Let $\lambda \in [0, 1)$ and let x be a solution of (14). Then in view of (32) we have

$$0 = (1 - \lambda)\,\psi(Gx) + \lambda\,\psi(Fx) \tag{39}$$

and by virtue of (25), (28), the strict positivity of the functional ψ and (F_{11}) we come to the inequality

$$|Lx|_\psi \leqq (1 - \lambda)\,|Gx|_\psi + \lambda\varepsilon\,\psi(Fx) + \lambda\,\psi(u)\,.$$

If we use (39), (29) and (38), we obtain

$$|Lx|_\psi \leqq (1 - \lambda)\,\frac{|u|_\psi}{2} - \varepsilon(1 - \lambda)\,\psi(Gx) + \lambda|u|_\psi \leqq \tag{40}$$

$$\leqq (1 - \lambda)\,\frac{|u|_\psi}{2} + (1 - \lambda)\,\|\psi\|\,|Gx|_Z + \lambda|u|_\psi \leqq (1 - \lambda)\,|u|_\psi + \lambda|u|_\psi = |u|_\psi\,.$$

In view of Lemma 3, the inequalities (29), (31) hold. Thus the conditions (L_1), (L_2') and Lemma 1 are true not only in the space Z, but also in $\tilde{Z}$. Hence, there is a $\Lambda > 0$

such that $|\tilde{x}|_X \leqq \Lambda |Lx|_\psi$. This together with (40) gives

(41) $$|\tilde{x}|_X \leqq \Lambda |u|_\psi \, .$$

If we put $\bar{x} = c\mathscr{C}$, then (41), (37) and the condition (F_{12}) for $c \leqq -\varrho$ imply

$$\delta[(1-\lambda)\,\psi(Gx) + \lambda\,\psi(Fx)] = (1-\lambda)\frac{\eta}{2}\,\psi\left(\frac{1}{1+|\tilde{x}|_Z}\,c\mathscr{C}\right) +$$

$$+ \lambda\delta\,\psi[F(c\mathscr{C} + \tilde{x})] \leqq (1-\lambda)\frac{\eta}{2}\,[1 + |\tilde{x}|_Z]^{-1}\,c \leqq -(1-\lambda)\frac{\eta}{2}\,[1 + |\tilde{x}|_Z]^{-1}\,\varrho < 0$$

so that (39) cannot hold. Similarly the assumption $c \geqq \varrho$ leads to a contradiction with (39). Hence we necessarily have $|c| < \varrho$ and this together with (41) gives

(42) $$|x|_X \leqq |c\mathscr{C}|_X + |\tilde{x}|_X < \varrho|\mathscr{C}|_X + \Lambda|u|_\psi \, ,$$

which completes the proof of the theorem.

Remark 10. The condition (F_{11}) says that F is either bounded from below (when $\varepsilon = 1$) or from above (if $\varepsilon = -1$). In fact, in the first case (36) implies that $Fx \leqq |Fx| \leqq Fx + u$ and hence $u \geqq 0$ as well as $-\,Fx \leqq |Fx| \leqq Fx + u$ and

$$Fx \geqq -\frac{u}{2} \quad \text{for all} \quad x \in X \, .$$

In the case $\varepsilon = -1$ we get that $u \geqq 0$ and

$$Fx \leqq \frac{u}{2} \quad \text{for all} \quad x \in X \, .$$

If the functional ψ given by (33) is only strictly positive and thus Z is not complete, then instead of (L_2') we have to assume a stronger condition:

(L_2''') There exists a continuous linear operator $A: X \to Z$ such that $L - A: D(L) \subset X \to Z$ is one-to-one and onto, and the inverse operator $(L-A)^{-1}$: $\tilde{Z} \to D(L) \subset X$ is continuous. This means that for some $M > 0$ and all $y \in Z$,

(43) $$|(L-A)^{-1}\,y|_X \leqq M|y|_\psi \, .$$

Lemma 5. *If the conditions* $(L_1), (L_2''')$ *hold, then* K_P: $\operatorname{Im} L \subset \tilde{Z} \to X$ *is continuous, that is, there exists a* $\Lambda > 0$ *such that for each* $x \in D(L)$, $x = \bar{x} + \tilde{x}$ *with* $\bar{x} \in \ker L$, $\tilde{x} \in \ker P$ *we have*

$$|\tilde{x}|_X \leqq \Lambda |Lx|_\psi \, .$$

Proof. Since $K_P = K_P \circ I|_{\operatorname{Im} L} = K_P \circ (L-A) \circ (L-A)^{-1}|_{\operatorname{Im} L}$, we can write

(44) $$K_P = K_P \circ L \circ (L-A)^{-1}|_{\operatorname{Im} L} - K_P \circ A \circ (L-A)^{-1}|_{\operatorname{Im} L} =$$
$$= (I-P) \circ (L-A)^{-1}|_{\operatorname{Im} L} - K_P \circ A \circ (L-A)^{-1}|_{\operatorname{Im} L} \, .$$

First we remark that on the basis of (29), the inequality (43) implies that

$$|(L-A)^{-1}\,y|_X \leqq M\|\psi\|\,|y|_Z$$

and hence (L_2') is fulfilled. Therefore, by Lemma 1, K_P is continuous as a mapping from Z into X. Since $(L - A)^{-1}$ is a continuous mapping from $\tilde{Z}$ into X, $I - P$ continuously maps X into X and A is continuous as a mapping from X into Z, (44) implies that K_P continuously maps $\operatorname{Im} L \subset \tilde{Z}$ into $D(L) \cap \ker P \subset X$.

Theorem 7. *Assume that* L *satisfies the assumptions* (L_1), (L_2'''), (L_3), (L_4'), (L_5) *and* (34). *Further, let F satisfy the conditions* (F_{11}) *and* (F_{12}). *Then the problem* (10) *has at least one solution.*

Proof. We proceed in the same way as in the proof of Theorem 6, arriving at the inequality (40). On the basis of Lemma 5, this inequality implies (41) which yields (42). This completes the proof of the theorem.

Remark 11. Theorems 6 and 7 are extensions of Mawhin's Theorem 1 in [9], p. 408, to Banach lattices. They use the same idea of proof. Nonetheless, it is useful to distinguish the case of a strictly positive functional ψ from the case of a uniformly positive functional ψ.

4. APPLICATION

Consider the existence of a solution to the boundary value problem

$$x'' + n^2 x = f(t, x)\,, \tag{45}$$

$$x(0) = 0\,, \quad x(\pi) = 0\,, \tag{46}$$

where $f \in C([0, \pi], R)$ and n is a natural number.

Let $X = Z$ be the Banach space $C([0, \pi], R)$ with the sup-norm $\|\cdot\|$. Let $D(L) = \{x \in X : x \in C^2([0, \pi], R),\ x \text{ satisfies } (46)\}$. Let the operators L and F be defined by

$$L: D(L) \subset X \to X\,, \quad x \mapsto x'' + n^2 x\,, \tag{47}$$

$$F: X \to X\,, \quad x \mapsto f(\cdot, x(\cdot))\,. \tag{48}$$

Then the problem (45), (46) takes the form (10). A little calculation yields

$$\ker L = \{x \in X : x = c \sin nt,\ c \in R\} \tag{49}$$

and

$$\operatorname{Im} L = \{y \in X : \textstyle\int_0^\pi y(t) \sin nt \,\mathrm{d}t = 0\}\,. \tag{50}$$

The projectors $P: X \to \ker L$, $I - Q: X \to \operatorname{Im} L$ are determined by the relations

$$(Px)(t) = \frac{2}{\pi} \sin nt \int_0^\pi x(t) \sin nt \,\mathrm{d}t\,,$$

$$(Qx)(t) = (Px)(t) \quad \text{for each} \quad x \in X \quad \text{and} \quad t \in [0, \pi]\,.$$

Then

$$X = \ker L \oplus \operatorname{Im} L$$

and thus, the assumption (L_4') is satisfied.

Consider the operator K_P: $\operatorname{Im} L \to D(L) \cap \operatorname{Im} L$, $y \mapsto x$, x is the unique solution of the problem (46), $x'' + n^2 x = y(t)$, which satisfies the condition $x \in \operatorname{Im} L$. x has the form

$$x(t) = c \sin nt + \frac{\sin nt}{n} \int_0^t \cos ns\, y(s)\, ds - \frac{\cos nt}{n} \int_0^t \sin ns\, y(s)\, ds ,$$

where c is determined by the condition $\int_0^\pi x(t) \sin nt\, dt = 0$.

Since

$$\int_s^\pi \sin^2 nt\, dt = \frac{1}{2}\left[(\pi - s) + \frac{1}{n} \sin ns \cos ns\right],$$

$$\int_s^\pi \sin nt \cos nt\, dt = -\frac{1}{2n} \sin^2 ns , \quad 0 \leqq s \leqq \pi ,$$

we have

$$c = \frac{1}{\pi n}\left[-\int_0^\pi (\pi - s) \cos ns\, y(s)\, ds - \frac{1}{n} \int_0^\pi \sin ns\, y(s)\, ds\right].$$

Hence

(51) $$K_P\, y(t) = \int_0^\pi K(t, s)\, y(s)\, ds , \quad 0 \leqq t \leqq \pi , \quad y \in \operatorname{Im} L$$

where

(52) $$K(t, s) = \begin{cases} \dfrac{1}{\pi n} \sin nt \left[s \cos ns - \dfrac{1}{n} \sin ns\right] - \dfrac{1}{n} \sin nt \cos ns , & 0 \leqq t \leqq s \leqq \pi , \\ \dfrac{1}{\pi n} \sin nt \left[s \cos ns - \dfrac{1}{n} \sin ns\right] - \dfrac{1}{n} \cos nt \sin ns , & 0 \leqq s \leqq t \leqq \pi . \end{cases}$$

Theorem 8. *Let $0 < d_1 < 1$. Suppose that there exist constants $a > 0$, $b > 0$ such that*

(53) $$|f(t, x)| \leqq a|x| + b , \quad t \in [0, \pi] , \quad x \in R ,$$

and a satisfies (19) where K_P is the operator given by (51), (52). Suppose that $\varepsilon = \pm 1$ and that there is an $R_2 > 0$ such that

(54) $$\varepsilon \operatorname{sgn} A \int_0^\pi f([t, A \sin nt + \tilde{x}(t)] \sin nt\, dt \geqq 0$$

for all A and $\tilde{x}(t) \in D(L) \cap \operatorname{Im} L$ such that

(55) $$|A| \geqq R_2 , \quad \|\tilde{x}\| \leqq d_1 |A| .$$

Then exists a solution of (45), (46).

Proof. By the above considerations, it follows that the operator L determined by (47) satisfies (L_1), (L_3) and (L_4'). In view of (53), the operator F which is defined by (48) fulfils (F_5). On the basis of (49), (50), we have

$$\varepsilon F(\bar{x} + \tilde{x}) + k\bar{x} \in \operatorname{Im} L \quad \text{iff} \quad \varepsilon \operatorname{sgn} A \int_0^\pi f[t, A \sin nt + \tilde{x}(t)] \,.$$

$$. \sin nt\, dt + k|A| \frac{\pi}{2} = 0 .$$

Hence, by (54) and (19), the assumption (F_6) is satisfied as well. Then, by Theorem 3, the result follows.

The following corollary gives a sufficient condition for (54) to be satisfied.

Corollary 3. *Suppose that*

(i) *there exist numbers* $\varrho \geqq 0$, $0 < \delta < \pi/2n$ *and* $a_1 > 0$ *such that for* $\varepsilon = \pm 1$

(a) $\varepsilon f(t, x) \operatorname{sgn} x \geqq 0$ *for each* $t \in [0, \pi]$, $|x| \geqq \varrho$;

(b) *there is a* $t^* \in [0, \pi)$ *with the property* $\varepsilon f(t, x) \operatorname{sgn} x \geqq a_1$ *for each* $t \in [t^*, t^* + \delta]$ *and each* x, $|x| \geqq \varrho$;

(ii) *there are numbers* $M > 0$, $b > 0$ *and numbers* ε_0, K, d_1 *and* $a > 0$ *which are determined by the relations*

(56)
$$\varepsilon_0 = a_1 \frac{\delta}{6} \sin n \frac{\delta}{6}, \quad K = \max_{\substack{0 \leqq t \leqq \pi \\ |x| \leqq \varrho}} |f(t, x)|, \quad d_1 = \min\left(\varepsilon_0 \frac{1}{2\pi(K + M)}, \; \frac{1}{2} \sin n \frac{\delta}{6}\right),$$

(19)
$$a < \frac{1}{\|K_P\|} \frac{d_1}{1 + d_1}$$

where K_P *is given by* (51), (52), *such that*

(c) *the function* $f(t, x)$ *satisfies* (53);

(d) $|f(t, x)| \leqq M$ *for each* $t \in \left(\bigcup_{l=0}^{n} [(l\pi/n) - \delta, (l\pi/n) + \delta]\right) \cap [0, \pi]$ *and each* x, $|x| \geqq \varrho$.

Then there exists a solution of (45), (46).

Proof. Only the case $\varepsilon = 1$ will be proved. The case $\varepsilon = -1$ can be dealt with in a similar way. In view of (56), $0 < d_1 < 1$. Put $R_2 = \varrho/d_1$ and consider the function $A \sin nt + \tilde{x}(t)$ with $\tilde{x}(t) \in D(L) \cap \operatorname{Im} L$ and such that (55) is true.

Let $S_1 = \{t \in [0, \pi] : |\sin nt| \geqq 2d_1\}$. Then in view of (55), for $t \in S_1$ we have

$|A \sin nt + \tilde{x}(t)| \geqq 2|A|\, d_1 - |\tilde{x}(t)| \geqq |A|\, d_1 \geqq \varrho$ and $\operatorname{sgn}[A \sin nt + \tilde{x}(t)] = \operatorname{sgn}(A \sin nt) = \operatorname{sgn} A \operatorname{sgn} \sin nt$, which by (a) implies that $f[t, A \sin nt + \tilde{x}(t)] \,.\, \sin nt \operatorname{sign} A \geqq 0$ and hence $\operatorname{sgn} A \int_{S_1} f[t, A \sin nt + \tilde{x}(t)] \sin nt \, dt \geqq 0$. Consider the interval $[t^*, t^* + \delta]$. As $\delta < \pi/2n$, the function $\sin nt$ has at most one zero in this interval. Therefore one of the intervals $[t^*, t^* + \frac{1}{3}\delta]$, $[t^* + \frac{1}{3}\delta, t^* + \frac{2}{3}\delta]$, $[t^* + \frac{2}{3}\delta, t^* + \delta]$, say $[t^*, t^* + \frac{1}{3}\delta]$, contains no zero of $\sin nt$, and each point of that interval has the distance to the nearest zero point of $\sin nt$ greater than or equal to $\frac{1}{6}\delta$. Hence for $t \in [t^*, t^* + \frac{1}{3}\delta]$ we have $|\sin nt| \geqq \sin n \frac{1}{6}\delta \geqq 2d_1$ and thus, $[t^*, t^* + \frac{1}{3}\delta] \subset S_1$. Moreover, (b) implies that $\operatorname{sgn} A \int_{[t^*, t^* + \frac{1}{3}\delta]} f[t, A \sin nt + \tilde{x}(t)] \,.\, \sin nt \, dt \geqq a_1 \frac{1}{3}\delta \sin n \frac{1}{6}\delta$. This, with respect to (56), means that

(57)
$$\operatorname{sgn} A \int_{S_1} f[t, A \sin nt + \tilde{x}(t)] \sin nt \, dt \geqq 2\varepsilon_0 \,.$$

Now we denote $S_2 = \{t \in [0, \pi] : |\sin nt| < 2d_1\}$, $S_3 = \{t \in S_2 : |A \sin nt + \tilde{x}(t)| \leqq \varrho\}$, $S_4 = \{t \in S_2 : |A \sin nt + \tilde{x}(t)| > \varrho$, $\operatorname{sgn}[A \sin nt + \tilde{x}(t)] = \operatorname{sgn}(A \sin nt)\}$,

$S_5 = \{t \in S_2 \colon |A \sin nt + \tilde{x}(t)| > \varrho, \quad \operatorname{sgn}[A \sin nt + \tilde{x}(t)] \neq \operatorname{sgn}(A \sin nt)\}$. Then in view of (56) we have

$$(58) \qquad |\operatorname{sgn} A \int_{S_3} f[t, A \sin nt + \tilde{x}(t)] \sin nt \, dt| \leqq \mu(S_3)\, K\, 2d_1 \,.$$

The assumption (a) implies

$$(59) \qquad \operatorname{sgn} A \int_{S_4} f[t, A \sin nt + \tilde{x}(t)] \sin nt \, dt \geqq 0 \,.$$

Again by (56), $2d_1 \leqq \sin n\delta$, and therefore $S_2 \subset (\bigcup_{l=0}^{n} [l\pi/n - \delta, l\pi/n + \delta]) \cap [0, \pi]$ which, on the basis of (d), implies

$$(60) \qquad |sgn\, A \int_{S_5} f[t, A \sin nt + \tilde{x}(t)] \sin nt \, dt| \leqq \mu(S_5)\, 2M\, d_1 \,.$$

Using the inequality $\mu(S_3) + \mu(S_5) \leqq \pi$ and respecting (56), we get from (58), (60) that $|\operatorname{sgn} A \int_{S_3 \cup S_5} f[t, A \sin nt + \tilde{x}(t)] \sin nt \, dt \leqq \varepsilon_0$, which together with (57), (59) gives that (54) is satisfied. Then Theorem 8 implies the corollary.

Remark 12. Theorems 34.2 and 34.5 in [6], pp. 270–275, give similar results about the existence of a weak solution to the problem (46), $x'' + n^2 x = g(x) + f(t)$. Proposition 2 in [4], p. 290, is closely related to Theorem 8 and its corollary.

Literature

[1] *S. Ahmad:* A resonance problem in which the nonlinearity may grow linearly. Proc. Amer. Math. Soc. *92* (1984), 381–384.

[2] *L. Cesari* and *R. Kannan:* An abstract existence theorem at resonance. Proc. Amer. Math. Soc. *81* (1977), 221–225.

[3] *K. Deimling:* Nonlinear Functional Analysis. Springer-Verlag, Berlin—Heidelberg, 1985.

[4] *C. Fabry* and *C. Franchetti:* Nonlinear equations with growth restrictions on the nonlinear term. J. Differential Equations *20* (1976), 283–291.

[5] *S. Fučík:* Surjectivity of operators involving linear noninvertible part and nonlinear compact perturbation. Funkcialaj Ekvacioj, *17* (1974), 73–83.

[6] *S. Fučík, A. Kufner:* Nelineární diferenciální rovnice. Praha, SNTL, 1978.

[7] *М. А. Красносельский, Е. А. Лифшиц, А. В. Соболев:* Позитивные линейные системы-Метод положительных операторов, Наука, Москва, 1985.

[8] *J. Mawhin:* Topological Degree Methods in Nonlinear Boundary Value Problems. Regional Confer. Series in Math. No. 40, Amer. Math. Soc., Previdence, 1979.

[9] *J. Mawhin:* Boundary value problems with nonlinearities having infinite jumps. Commentat. Math. Univ. Carol. *25* (1984), 401–414.

[10] *A. Taylor:* Introduction to Functional Analysis. John Wiley and Sons, Inc. New York 1958 (Czech Translation Academia, Praha 1973).

[11] *J. R. Ward, Jr.:* Existence theorems for nonlinear boundary value problems at resonance. J. Differential Equations *35* (1980), 232–247.

[12] *K. Yosida:* Functional Analysis. Springer-Verlag, Berlin, 1980.

Author's address: 842 15 Bratislava, Mlynská dolina, Czechoslovakia (Matematicko-fyzikálna fakulta UK).