Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 3, 473–478

Persistent URL: http://dml.cz/dmlcz/102243

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

ON CERTAIN INFINITESIMAL ISOMETRIES OF SURFACES

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(Received June 9, 1986)

In the following, I am going to characterize the surfaces of the Euclidean 3-space which admit non-trivial infinitesimal isometries preserving the mean curvature. In this sense, the paper is an infinitesimal version of the recent paper [1] of S. S. Chern; nevertheless, the results are quite different and many problems remain open.

We consider, in the Euclidean space E^3, a piece of a surface M. With each point $m \in M$, let us associate an orthonormal frame $\{m; v_1, v_2, v_3\}$ such that v_3 is a unit normal vector at m. Then we may write

(1) $$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,, \quad \mathrm{d}v_1 = \omega_1^2 v_2 + \omega_1^3 v_3\,, \quad \mathrm{d}v_2 = -\omega_1^2 v_1 + \omega_2^3 v_3\,,$$
$$\mathrm{d}v_3 = -\omega_1^3 v_1 - \omega_2^3 v_2$$

with the usual integrability conditions.

Further, let us consider a 1-parametric family of surfaces $M(t)$, $t \in (-\varepsilon, \varepsilon)$, such that $M(0) = M$; for each t, let an isometry $\iota_t \colon M \to M(t)$ be given. On $M(t)$, take the field of orthonormal frames $\{m(t); v_1(t), v_2(t), v_3(t)\}$ with $m(t) = \iota_t(m)$, $v_1(t) = \mathrm{d}\iota_t(v_1)$, $v_2 = \mathrm{d}\iota_t(v_2)$. Then

(2) $$\mathrm{d}m(t) = \omega^1 v_1(t) + \omega^2 v_2(t)\,, \qquad \mathrm{d}v_1(t) = \omega_1^2(t)\, v_2(t) + \omega_1^3(t)\, v_3(t)\,,$$
$$\mathrm{d}v_2(t) = -\omega_1^2(t)\, v_1(t) + \omega_2^3(t)\, v_3(t)\,, \quad \mathrm{d}v_3(t) = -\omega_1^3(t)\, v_1(t) - \omega_2^3(t)\, v_2(t)$$

with $\omega_i^j(0) = \omega_i^j$ and the integrability conditions

(3) $$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega_1^2(t)\,,$$
$$\mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2(t)\,, \quad \omega^1 \wedge \omega_1^3(t) + \omega^2 \wedge \omega_2^3(t) = 0\,,$$
$$\mathrm{d}\omega_1^2(t) = -\omega_1^3(t) \wedge \omega_2^3(t)\,, \quad \mathrm{d}\omega_1^3(t) = \omega_1^2(t) \wedge \omega_2^3(t)\,,$$
$$\mathrm{d}\omega_2^3(t) = -\omega_1^2(t) \wedge \omega_1^3(t)\,.$$

Let

(4) $$\varphi_i^j := (\mathrm{d}\omega_i^j(t)/\mathrm{d}t)_{t=0}\,.$$

From $(3_{1,2})$, we get $\omega^2 \wedge \varphi_1^2 = \omega^1 \wedge \varphi_1^2 = 0$, i.e.,

(5) $$\varphi_1^2 = 0\,.$$

This together with (3_{3-6}) yield

$$(6) \qquad \omega^1 \wedge \varphi_1^3 + \omega^2 \wedge \varphi_2^3 = 0 \,, \quad \omega_1^3 \wedge \varphi_2^3 + \varphi_1^3 \wedge \omega_2^3 = 0 \,,$$

$$(7) \qquad d\varphi_1^3 = \omega_2^1 \wedge \varphi_2^3 \,, \quad d\varphi_2^3 = -\omega_1^2 \wedge \varphi_1^3 \,.$$

Given a surface M, the couple $\{\varphi_1^3, \varphi_2^3\}$ of 1-forms on M satisfying (6) + (7) is called the *infinitesimal isometry* Φ of M.

The *second form* of $M(t)$ is given by

$$(8) \qquad \mathrm{II}(t) = \omega^1 \, \omega_1^3(t) + \omega^2 \, \omega_2^3(t) \,;$$

its *Gauss curvature* $K(t)$ and its *mean curvature* $H(t)$ by

$$(9) \qquad K(t) \, \omega^1 \wedge \omega^2 = \omega_1^3(t) \wedge \omega_2^3(t) \,,$$

$$2 \, H(t) \, \omega^1 \wedge \omega^2 = \omega_1^3(t) \wedge \omega^2 + \omega^1 \wedge \omega_2^3(t) \,,$$

resp. Let us define the *variations*

$$(10) \qquad \delta \mathrm{II} := (\mathrm{dII}(t)/\mathrm{d}t)_{t=0} \,, \quad \delta K := (\mathrm{d}K(t)/\mathrm{d}t)_{t=0} \,, \quad \delta H := (\mathrm{d}H(t)/\mathrm{d}t)_{t=0} \,;$$

we get

$$(11) \qquad \delta \mathrm{II} = \omega^1 \varphi_1^3 + \omega^2 \wedge \varphi_2^3 \,,$$

$$(12) \qquad \delta K = 0 \,, \quad 2 \delta H \,.\, \omega^1 \wedge \omega^2 = \varphi_1^3 \wedge \omega^2 + \omega^1 \wedge \varphi_2^3 \,.$$

The equation (12_1) is the consequence of (6_2); it is the infinitesimal version of the theorema egregium.

Consider the surface M. The equation (3_3) for $t = 0$ yields the existence of functions a, b, c such that

$$(13) \qquad \omega_1^3 = a\omega^1 + b\omega^2 \,, \quad \omega_2^3 = b\omega^1 + c\omega^2 \,;$$

we have, again on M,

$$(14) \qquad ds^2 = (\omega^1)^2 + (\omega^2)^2 \,, \quad \mathrm{II} = a(\omega^1)^2 + 2b\omega^1\omega^2 + c(\omega^2)^2 \,;$$

$$(15) \qquad 2H = a + c \,, \quad K = ac - b^2 \,.$$

It is easy to see that the *lines of curvature* of M are given by

$$(16) \qquad b(\omega^1)^2 + (c - a) \, \omega^1 \omega^2 - b(\omega^2)^2 = 0 \,.$$

The *Euler function* E on M be defined by $E := H^2 - K$, i.e.,

$$(17) \qquad 4E = (a - c)^2 + 4b^2 \,.$$

A point $m \in M$ is *umbilical* if and only if $E(m) = 0$.

Let $f \colon M \to \mathbb{R}$ be a function. Its first *covariant derivatives* f_i with respect to the coframes $\{\omega^1, \omega^2\}$ are defined by

$$(18) \qquad df = f_1 \omega^1 + f_2 \omega^2 \,.$$

From this,

$$(19) \qquad (df_1 - f_2 \omega_1^2) \wedge \omega^1 + (df_2 + f_1 \omega_1^2) \wedge \omega^2 = 0 \,,$$

and we get the existence of the second covariant derivatives $f_{ij} = f_{ji}$ such that

(20) $$\mathrm{d}f_1 - f_2\omega_1^2 = f_{11}\omega^1 + f_{12}\omega^2\,, \quad \mathrm{d}f_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2\,.$$

Theorem. *Let $M \subset E^3$ be a surface without umbilical points; M admits a non-trivial infinitesimal isometry Φ with*

(21) $$\delta H = 0$$

if and only if

(22) $$2b(H_1E_1 - H_{11}E) + \\ + (c - a)(H_2E_1 + H_1E_2 - 2H_{12}E) - 2b(H_2E_2 - H_{22}E) = 0\,.$$

If a general surface $M \subset E^3$ admits a non-trivial infinitesimal isometry Φ satisfying (21), *we have* (22). *The surfaces admitting non-trivial infinitesimal isometries Φ with* (21) *depend on* 4 *functions of* 1 *variable in the sense of E. Cartan.*

Proof. From (6) and (13), we get the existence of functions R_1, R_2, R_3 on M such that

(23) $$\varphi_1^3 = R_1\omega^1 + R_2\omega^2\,, \quad \varphi_2^3 = R_2\omega^1 + R_3\omega^2\,,$$

(24) $$cR_1 - 2bR_2 + aR_3 = 0\,.$$

The condition (21) is equivalent, see (12_2), to

(25) $$R_1 + R_3 = 0\,.$$

First of all, let us suppose that M contains no umbilical points, i.e., $E \neq 0$ on M. From (24) and (25), we get $(c - a)R_1 = 2bR_2$, and $E \neq 0$ implies the existence of a function R on M such that $R_1 = 2bR$, $R_2 = (c - a)R$. Thus (23) turn out to be

(26) $$\varphi_1^3 = R\{2b\omega^1 + (c - a)\omega^2\}\,, \quad \varphi_2^3 = R\{(c - a)\omega^1 - 2b\omega^2\}\,.$$

Let us mention that, see (11),

(27) $$\delta\mathrm{II} = 2R\{b(\omega^1)^2 + (c - a)\omega^1\omega^2 - b(\omega^2)^2\}\,.$$

The couple $M + \Phi$ is thus given by (13) + (26). The differential consequences are

(28) $$(\mathrm{d}a - 2b\omega_1^2) \wedge \omega^1 + (\mathrm{d}b + (a - c)\omega_1^2) \wedge \omega^2 = 0\,,$$
$$(\mathrm{d}b + (a - c)\omega_1^2) \wedge \omega^1 + (\mathrm{d}c + 2b\omega_1^2) \wedge \omega^2 = 0\,,$$

(29) $$\mathrm{d}R \wedge \{2b\omega^1 + (c - a)\omega^2\} + R\{2(\mathrm{d}b + (a - c)\omega_1^2) \wedge \\ \wedge \omega^1 + (\mathrm{d}c - \mathrm{d}a + 4b\omega_1^2) \wedge \omega^2\} = 0\,,$$
$$\mathrm{d}R \wedge \{(c - a)\omega^1 - 2b\omega^2\} + R\{(\mathrm{d}c - \mathrm{d}a + 4b\omega_1^2) \wedge \\ \wedge \omega^1 - 2(\mathrm{d}b + (a - c)\omega_1^2) \wedge \omega^2\} = 0\,.$$

Using Cartan's lemma, we get the existence of functions $\alpha, \beta, \gamma, \delta, r_1, r_2$ such that

(30) $$\mathrm{d}a - 2b\omega_1^2 = \alpha\omega^1 + \beta\omega^2\,, \quad \mathrm{d}b + (a - c)\omega_1^2 = \beta\omega^1 + \gamma\omega^2\,,$$
$$\mathrm{d}c + 2b\omega_1^2 = \gamma\omega^1 + \delta\omega^2\,,$$

(31) $$\mathrm{d}R = r_1\omega^1 + r_2\omega^2\,,$$

and satisfying

$$(32) \qquad (c - a)\, r_1 - 2br_2 = R(\alpha + \gamma)\,, \quad 2br_1 + (c - a)\, r_2 = -R(\beta + \delta)\,.$$

It is elementary to see that the equations (28) + (29) are linearly independent. Thus the system (13) + (26) is in involution and its solutions depend on 4 functions of 1 variable.

Let us rewrite (32). From $(9)_{t=0}$, we see that

$$(33) \qquad 2H = a + c\,,$$

with the notation (18), the equations $(30_{1,3})$ imply $2H_1 = \alpha + \gamma$, $2H_2 = \beta + \delta$, i.e., (32) may be written as

$$(34) \qquad (c - a)\, r_1 - 2br_2 = 2H_1 R\,, \quad 2br_1 + (c - a)\, r_2 = -2H_2 R\,.$$

Because of $E \neq 0$, we may evaluate r_1, r_2 from them, and (31) turns out to be

$$(35) \qquad \mathrm{d}R = \tfrac{1}{2} R\omega$$

with

$$(36) \qquad \omega := E^{-1}[\{(c - a)\, H_1 - 2bH_2\}\, \omega^1 - \{2bH_1 + (c - a)\, H_2\}\, \omega^2]\,.$$

The integrability condition of (35) being $R\, \mathrm{d}\omega = 0$, there exists a non-trivial function R on M if and only if $\mathrm{d}\omega = 0$. By a direct calculation, this is exactly (22).

Now, let us drop the supposition $E \neq 0$ on M. Because of (25), (23) may be written as

$$(37) \qquad \varphi_1^3 = R_1\omega^1 + R_2\omega^2\,, \quad \varphi_2^3 = R_2\omega^1 - R_1\omega^2\,;$$

the condition (24) being

$$(38) \qquad (c - a)\, R_1 - 2bR_2 = 0\,.$$

Of course

$$(39) \qquad \delta \mathrm{II} = R_1(\omega^1)^2 + 2R_2\omega^1\omega^2 - R_1(\omega^2)^2\,.$$

Our problem is thus given by (13) + (37) + (38). The differential consequences of (13) are (28), and from (37) we get

$$(40) \qquad \begin{aligned} &(\mathrm{d}R_1 - 2R_2\omega_1^2) \wedge \omega^1 + (\mathrm{d}R_2 + 2R_1\omega_1^2) \wedge \omega^2 = 0\,,\\ &(\mathrm{d}R_2 + 2R_1\omega_1^2) \wedge \omega^1 - (\mathrm{d}R_1 - 2R_2\omega_1^2) \wedge \omega^2 = 0\,. \end{aligned}$$

Using Cartan's lemma, we get (30) from (28) and, from (40), the existence of functions S_1, S_2 such that

$$(41) \qquad \mathrm{d}R_1 - 2R_2\omega_1^2 = S_1\omega^1 + S_2\omega^2\,, \quad \mathrm{d}R_2 + 2R_1\omega_1^2 = S_2\omega^1 - S_1\omega^2\,.$$

The differential consequences of (38) are then

$$(42) \qquad \begin{aligned} &(c - a)\, S_1 - 2bS_2 + (\gamma - \alpha)\, R_1 - 2\beta R_2 = 0\,,\\ &(c - a)\, S_2 + 2bS_1 + (\delta - \beta)\, R_1 - 2\gamma R_2 = 0\,. \end{aligned}$$

The exterior differentiation of (30) + (41) yields

$$(43)\quad \begin{aligned}&(\mathrm{d}\alpha - 3\beta\omega_1^2) \wedge \omega^1 + (\mathrm{d}\beta + (\alpha - 2\gamma)\,\omega_1^2) \wedge \omega^2 = 2Kb\omega^1 \wedge \omega^2\,,\\ &(\mathrm{d}\beta + (\alpha - 2\gamma)\,\omega_1^2) \wedge \omega^1 + (\mathrm{d}\gamma + (2\beta - \delta)\,\omega_1^2) \wedge \omega^2 = K(c - a)\,\omega^1 \wedge \omega^2\,,\\ &(\mathrm{d}\gamma + (2\beta - \delta)\,\omega_1^2) \wedge \omega^1 + (\mathrm{d}\delta + 3\gamma\omega_1^2) \wedge \omega^2 = -2Kb\omega^1 \wedge \omega^2\,,\\ &(\mathrm{d}S_1 - 3S_2\omega_1^2) \wedge \omega^1 + (\mathrm{d}S_2 + 3S_1\omega_1^2) \wedge \omega^2 = 2KR_2\omega^1 \wedge \omega^2\,,\\ &(\mathrm{d}S_2 + 3S_1\omega_1^2) \wedge \omega^1 - (\mathrm{d}S_1 - 3S_2\omega_1^2) \wedge \omega^2 = -2KR_1\omega^1 \wedge \omega^2\,,\end{aligned}$$

and we get the existence of functions $A, \ldots, E, T_1, T_2$ such that

$$(44)\quad \begin{aligned}&\mathrm{d}\alpha - 3\beta\omega_1^2 = A\omega^1 + (B - bK)\,\omega^2\,,\\ &\mathrm{d}\beta + (\alpha - 2\gamma)\,\omega_1^2 = (B + bK)\,\omega^1 + (C + aK)\,\omega^2\,,\\ &\mathrm{d}\gamma + (2\beta - \delta)\,\omega_1^2 = (C + cK)\,\omega^1 + (D + bK)\,\omega^2\,,\\ &\mathrm{d}\delta + 3\gamma\omega_1^2 = (D - bK)\,\omega^1 + E\omega^2\,,\\ &\mathrm{d}S_1 - 3S_2\omega_1^2 = (T_1 + KR_1)\,\omega^1 + (T_2 - KR_2)\,\omega^2\,,\\ &\mathrm{d}S_2 + 3S_1\omega_1^2 = (T_2 + KR_2)\,\omega^1 - (T_1 - KR_1)\,\omega^2\,.\end{aligned}$$

Using these, we get, from (42),

$$(45)\quad \begin{aligned}&(c - a)\,T_1 - 2bT_2 + 2(\gamma - \alpha)\,S_1 - 4\beta S_2 +\\ &+ (C - A + 2cK - aK)\,R_1 - 2(B + 2bK)\,R_2 = 0\,,\\ &2bT_1 + (c - a)\,T_2 + (\beta + \delta)\,S_1 - (\alpha + \gamma)\,S_2 +\\ &+ (D - B)\,R_1 - (2C + aK + cK)\,R_2 = 0\,,\\ &-(c - a)\,T_1 + 2bT_2 + 4\gamma S_1 + 2(\delta - \beta)\,S_2 +\\ &+ (E - C - 2aK + cK)\,R_1 - 2(D + 2bK)\,R_2 = 0\,.\end{aligned}$$

From $(45_{1,3})$,

$$(46)\quad \begin{aligned}&2(3\gamma - \alpha)\,S_1 + 2(\delta - 3\beta)\,S_2 +\\ &+ (E - A + 3cK - 3aK)\,R_1 - 2(B + D + 4bK)\,R_2 = 0\,.\end{aligned}$$

Consider the system (46) + (42) + (38) for S_1, S_2, R_1, R_2. If Φ is non-trivial, this system must have a non-trivial solution – see (39) – and its determinant Δ must vanish. Let us calculate Δ at a point $m_0 \in M$. Because of $\mathrm{II} = a(\omega^1)^2 + 2b\omega^1\omega^2 + c(\omega^2)^2$ – see (8) – we may choose the frames in such a way that $b(m_0) = 0$. Then, at m_0,

$$(47)\quad \Delta(m_0) = 2(c - a)\begin{vmatrix} 6\gamma - 2\alpha & 2\delta - 6\beta & B + D\\ c - a & 0 & \beta\\ 0 & c - a & \gamma \end{vmatrix} =$$

$$= 2(c - a)\,\{(c - a)^2\,(B + D) - 2(c - a)\,(\gamma\delta - \alpha\beta)\}\,.$$

Again at $m_0 \in M$, we have

$$(48)\quad \begin{aligned}&2H_1 = \alpha + \gamma\,,\quad 2H_2 = \beta + \delta\,,\quad 2H_{12} = B + D\,,\\ &4E = (c - a)^2\,,\quad 4E_1 = 2(c - a)\,(\gamma - \alpha)\,,\quad 4E_2 = 2(c - a)\,(\delta - \beta)\,.\end{aligned}$$

Thus

$$(49) \quad 2EH_{12} - H_2E_1 - H_1E_2 = \tfrac{1}{4}\{(c - a)^2 (B + D) - 2(c - a)(\gamma\delta - \alpha\beta)\},$$

and we get, at $m_0 \in M$,

$$(50) \quad \Delta(m_0) = 8(c - a)(2EH_{12} - H_2E_1 - H_1E_2).$$

Thus $\Delta(m_0) = 0$ is equivalent to (22) for $b = 0$. The left-hand side of (22) being an invariant of our surface, we are finished.

Remarks. Evidently, all surfaces of revolution satisfy (22). Indeed, let M be a surface of revolution. The frames of M be chosen in such a way that v_1 be tangent to the circles of M. On each of these circles, $H =$ const. and $E =$ const.; further, $b = 0$ on M. Thus $E_1 = H_1 = 0$ on M, and (22) is satisfied.

For $E \neq 0$, (22) may be written as

$$(51) \quad 2b(E^{-1}H_1)_1 + (c - a)((E^{-1}H_2)_1 + (E^{-1}H_1)_2) - 2b(E^{-1}H_2)_2 = 0.$$

Using the tensor notation, we may write (22) as follows: Let $c_{ij}\,dx^i\,dx^j = 0$ be the lines of curvature on M; (22) is then

$$(52) \quad c^{ij}H_{;i}E_{;j} = Ec^{ij}H_{;ij}.$$

Reference

[1] *S. S. Chern:* Deformation of Surfaces Preserving Principal Curvatures. In: Differential Geometry and Complex Analysis, pp. 156—163; Springer-Verlag, Berlin—Heidelberg, 1985.

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