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ON THE EQUALITY OF INJECTIVE AND PROJECTIVE TENSOR PRODUCTS

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INTRODUCTION

In this note we will be concerned with the following general problem which essentially goes back to A. Grothendieck [1] (compare also [14] and [3]):

(*) Let E be a locally convex space. Which conditions have to be imposed on E in order that $E \otimes_\varepsilon E = E \otimes_\pi E$ may imply that E is nuclear?

Various such conditions can be found in literature; we refer e.g. to [7] and [9]. For example, it is shown in [9] that $E \otimes_\varepsilon E = E \otimes_\pi E$ implies nuclearity of E provided E has a 0-basis $\mathcal{U}$ such that the Banach spaces associated with the neighbourhoods in $\mathcal{U}$ are $\mathcal{L}_p$-spaces for some fixed $1 < p < \infty$. On the other hand, G. Pisier [20] presented a method of generating Banach spaces X of infinite dimension such that $X \otimes_\varepsilon X = X \otimes_\pi X$ holds (and such that both X and X' have cotype 2). This indicates that the final answer to problem (*) will at least be not easy.

The above problem (*) is of course a special case of the following more general question (which was actually raised in [1]): If E and F are locally convex spaces such that $E \otimes_\varepsilon F = E \otimes_\pi F$, under which conditions on E and/or F is one of these spaces nuclear? Let us mention that in [8] two (different) non-nuclear Fréchet-Schwartz spaces E and F were constructed, each of which is isomorphic to a subspace of l_2^N, such that $E \otimes_\varepsilon F = E \otimes_\pi F$.

In dealing with problem (*), one is led in a natural fashion to the study of operators $T: X \to Y$ between Banach spaces X and Y such that $T \otimes T$ maps $X \otimes_\varepsilon X$ continuously into $Y \otimes_\pi Y$. We begin our discussions by proving certain results on such operators in general. These are then applied in the context of problem (*). In particular, we obtain several generalizations of results from [7] and [9]. Our methods also lead us to prove a criterion for a locally convex space E to be nuclear whenever $E \otimes_\varepsilon X = E \otimes_\pi X$ holds for a fixed arbitrary Banach space X of infinite dimension. Further we examine certain locally convex topologies on Banach spaces with respect to (*). Among others we shall see that, on every infinite-dimensional Banach space X,

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there exist locally convex topologies $\mathcal{T}$ which are consistent with the dual pairing $\langle X, X' \rangle$ but for which the ε-tensor product of $[X, \mathcal{T}]$ with itself does not coincide with the corresponding π-tensor product. This applies in particular to the Banach spaces constructed in [20].

1. NOTATION

We shall employ standard notation and terminology on Banach spaces, ideals of operators between such spaces, and locally convex spaces. Unexplained terminology and facts may be found e.g. in [12], [15], and [3], respectively. To simplify reading, however, let us briefly recall some basic facts and fix some notation which will be used in the sequel. Continuous linear mappings between locally convex spaces (lcs) will be simply called *operators*. Unless otherwise stated, subspaces are linear submanifolds and, in the Banach space case, assumed to be closed. The ideals (of Banach space operators) consisting of all integral operators, nuclear operators, strongly nuclear operators, $\mathcal{L}_p$-factorable operators, (r, p, q)-summing operators, operators of type r and operators of cotype s, will be denoted by $\mathcal{I}$, $\mathcal{N}$, Γ_p, $\mathcal{P}_{r,p,q}$, $\mathcal{T}_r$ and $\mathcal{C}_s$, respectively. We shall write $\mathcal{P}_{r,p}$ instead of $\mathcal{P}_{r,p,\infty}$, $\mathcal{P}_r$ instead of $\mathcal{P}_{r,r}$, and $\mathcal{P}$ instead of $\mathcal{P}_{2,2,2}$. Recall that $\mathcal{P}$ is the largest extension of Hilbert-Schmidt operators to an operator ideal on Banach spaces.

Throughout, we shall tacitly assume that all our lcs are Hausdorff. Also, we shall only consider 0-neighbourhoods in such spaces which are absolutely convex and closed. Given a 0-neighbourhood U in an lcs E, we write $E_{(U)}$ for the Banach space canonically associated with U, and Φ_U for the corresponding natural operator $E \to \to E_{(U)}$. If V is another 0-neighbourhood in E and if $V \subset U$, then there is a unique operator $\Phi_{UV}\colon E_{(V)} \to E_{(U)}$ such that $\Phi_U = \Phi_{UV} \circ \Phi_V$.

Given an ideal $\mathcal{A}$ (of operators between Banach spaces), we call E an lc *$\mathcal{A}$-space* if for some (and then every) 0-basis $\mathcal{U}$ of E the following holds: For every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ contained in U such that $\Phi_{UV} \in \mathcal{A}(E_{(V)}, E_{(U)})$. If $\mathcal{A}$ is $\mathcal{N}$ or $\mathcal{I}$, then the lc $\mathcal{A}$-spaces are precisely the nuclear lcs; in case $\mathcal{A} = \mathcal{S}_0$ one obtains the strongly nuclear lcs, and taking for $\mathcal{A}$ the ideal of compact operators one arrives at the Schwartz spaces, etc.

2. ON (ε, π)-CONTINUOUS OPERATORS

Let X and Y be Banach spaces. An operator $T\colon X \to Y$ is said to be *(ε, π)-continuous* if $T \otimes T$ is continuous as a map from the normed space $X \otimes_\varepsilon X$ into the normed space $Y \otimes_\pi Y$. The relevance of this concept for our purpose is due to the fact that an lcs E with (no matter which) 0-basis $\mathcal{U}$ satisfies $E \otimes_\varepsilon E = E \otimes_\pi E$ if and only if every $U \in \mathcal{U}$ contains a $V \in \mathcal{U}$ such that $\Phi_{UV}\colon E_{(V)} \to E_{(U)}$ is (ε, π)-continuous. This follows e.g. from the well-known representations of π- and ε-tensor products as projective limits; see e.g. [3], 15.4.3 and 16.3.3.

We shall need the following

1. Lemma. *Let X_j and Y_j $(j = 1, 2)$ be infinite-dimensional Banach spaces. Let $S \in \mathscr{L}(X_1, X_2)$ and $T \in \mathscr{L}(Y_1, Y_2)$ be such that $S \otimes T: X_1 \otimes_\varepsilon Y_1 \to X_2 \otimes_\pi Y_2$ is continuous.*

(a) *If $X_1 = Y_1 = X$, $X_2 = Y_2 = Y$ and $S = T$, then $S \in \mathscr{P}(X, Y)$.*

(b) *If T is an isomorphic embedding, then $S \in \mathscr{P}_{1,2,1}(X_1, X_2)$.*

(c) *If T is a surjection, then $S \in \mathscr{P}_{1,1,2}(X_1, X_2)$.*

Proof. (a) cf. [4] and [7]; for the sake of completeness, we give the simple argument. Identify $(Y \otimes_\pi Y)'$ with $\mathscr{L}(Y, Y')$, and $(X \otimes_\varepsilon X)'$ with $\mathscr{I}(X, X')$, in the usual way. The adjoint of $S \otimes S$ then appears to be the map $\mathscr{L}(Y, Y') \to \mathscr{I}(X, X')$: $R \mapsto S'RS$. Thus, given $A \in \mathscr{L}(Y, l_2)$ and $B \in \mathscr{L}(l_2, X)$, $(ASB)'(ASB): l_2 \to l_2$ is nuclear, hence $ASB: l_2 \to l_2$ is a Hilbert-Schmidt operator. Since A and B have been arbitrary, $S \in \mathscr{P}(X, Y)$ follows.

(b) Without loss of generality, we may suppose T is isometric. Fix $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X_1$ and $u_1, \ldots, u_n \in X_2'$. Given $\varepsilon > 0$, use Dvoretzky's theorem to produce $y_1, \ldots, y_n \in Y_1$ and $v_1, \ldots, v_n \in Y_2'$ such that $\langle Ty_i, v_j\rangle = \delta_{ij}$, $\|v_j\| \leqq 1 + \varepsilon$ $(1 \leqq i, j \leqq n)$, and $w_2(y_i) \leqq 1$. Here and later, $w_s(z_i)$ denotes the weak l_s-norm of a sequence of vectors (z_i) for $1 \leqq s \leqq \infty$.

For an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of scalars, consider $\beta_\lambda := \sum_{j=1}^n \lambda_j u_j \otimes v_j$ as a bilinear form on $X_2 \times Y_2$. Clearly, $\|\beta_\lambda\| \leqq (1 + \varepsilon) \,.\, \|\lambda\|_\infty \,.\, w_1(u_i)$, hence

$$\begin{aligned}\sum_{i=1}^n |\langle Sx_i, u_i\rangle| &= \sup\{|\sum_{i,j} \langle Sx_i, u_j\rangle \,.\, \langle Ty_i, v_j\rangle\, \lambda_j| \mid \|\lambda\|_\infty \leqq 1\} = \\ &= \sup\{|\sum_i \beta_\lambda(Sx_i, Ty_i)| \; \|\lambda\|_\infty \leqq 1\} \leqq \\ &\leqq (1 + \varepsilon)\, w_1(u_i)\, \|\sum_i Sx_i \otimes Ty_i\|_{X_2 \otimes_\pi Y_2} \leqq \\ &\leqq (1 + \varepsilon)\, \|S \otimes T\|\, w_1(u_i)\, \|\sum_i x_i \otimes y_i\|_{X_1 \otimes_\varepsilon Y_1} \leqq \\ &\leqq (1 + \varepsilon)\, \|S \otimes T\|\, w_2(x_i)\, w_1(u_i)\,.\end{aligned}$$

This proves (b).

(c) We proceed similarly. Without loss of generality, let us assume that T is a quotient map. Use Dvoretzky's theorem and local reflexivity to find $y_i \in Y_1$ and $v_j \in Y_2'$ such that $\langle Ty_i, v_j\rangle = \delta_{ij}$, $\|y_i\| \leqq 1 + \varepsilon$ $(1 \leqq i, j \leqq n)$, and $w_2(v_i) \leqq 1$. With β_λ as above, we now have $\|\beta_\lambda\| \leqq \|\lambda\|_\infty\, w_2(u_i)$ from where $\sum_{i=1}^n |\langle Sx_i, u_i\rangle| \leqq (1 + \varepsilon) \,.\, \|S \otimes T\|\, w_1(x_i)\, w_2(u_i)$ follows as before. ■

We note that $S' \in \mathscr{P}_{2,1} \cap \mathscr{P}$ follows in the situation of 1(b), and $S \in \mathscr{P}_{2,1} \cap \mathscr{P}$ in the situation of 1(c). Note also that 1(a) is far from being sufficient, just look at the identity map of an $\mathscr{L}_1$- or $\mathscr{L}_\infty$-space.

Let us pass to (ε, π)-continuity. We shall use the main result in [20] (cf. also [21]) to show that in certain interesting cases all operators are (ε, π)-continuous. For some particular cases (e.g., $X = \mathscr{L}_1$, $Y = \mathscr{L}_\infty$) a simpler argument is available.

2. Proposition. *Let X be of cotype 2 and Y such that $\mathscr{L}(Y, Y') = \Gamma_2(Y, Y')$. If $\mathscr{L}(X, Y) = \mathscr{P}(X, Y)$ (e.g., if X or Y is a Hilbert-Schmidt space [4]), then $\mathscr{L}(X, Y)$ consists of (ε, π)-continuous operators only.*

Proof. It is easy to see than it suffices to consider the case where X is separable. Let us fix $S \in \mathscr{L}(Y, Y')$. Write $S = A \circ B$, $A \in \mathscr{L}(H, Y')$, $B \in \mathscr{L}(Y, H)$, H a suitable Hilbert space. Note that A is the adjoint of $C := A'/Y$.

By [20] there is a Banach space Z containing X such that $Z \otimes_\varepsilon Z = Z \otimes_\pi Z$. Let $J: X \to Z$ be the corresponding embedding. Then $B \circ T \in \mathscr{P}_2(X, H)$ has an extension $U \in \mathscr{L}(Z, H)$, and $C \circ T$ has an extension $V \in \mathscr{L}(Z, H)$. Because of $Z \otimes_\varepsilon Z = Z \otimes_\pi Z$, $V' \circ U$ is integral, hence $T' \circ S \circ T = J' \circ V' \circ U \circ J$ belongs to $\mathscr{I}(X, X')$. Since S has been arbitrary, T must be (ε, π)-continuous. ■

So, if X is of cotype 2 and Y is an $\mathscr{L}_\infty$-space, or if X is an $\mathscr{L}_1$-space, Y' has cotype 2, and Y satisfies some weak form of the approximation property (cf. [17]), then every operator $X \to Y$ is (ε, π)-continuous. In particular, for $1 \leqq p \leqq 2 \leqq q \leqq \infty$, the spaces $\mathscr{L}(\mathscr{L}_p, \mathscr{L}_\infty)$ and $\mathscr{L}(\mathscr{L}_1, \mathscr{L}_q)$ consist of (ε, π)-continuous operators only.

Let us say that an operator $T: X \to Y$ *fixes a subspace* Z if there are isomorphic embeddings $J_1: Z \to X$ and $J_2: Z \to Y$ such that $T \circ J_1 = J_2$. Dually, we say that T *fixes a quotient* Z if there are surjective operators $Q_1: X \to Z$ and $Q_2: Y \to Z$ such that $Q_2 \circ T = Q_1$.

3. Proposition. *Suppose $T: X \to Y$ is (ε, π)-continuous. If T fixes a quotient Z, then $\mathscr{L}(Z, Z') = \Gamma_2(Z, Z')$. If T fixes a subspace Z and if X' has the bounded approximation property, then $\mathscr{L}(Z', Z) = \Gamma_2(Z', Z)$.*

Proof. Assume first that T fixes a quotient Z. Let Q_1 and Q_2 be as above. Then $Q_1' S Q_1 = T' Q_2' S Q_2 T$ is integral for any $S \in \mathscr{L}(Z, Z')$, which can only happen if $S \in \Gamma_2(Z, Z')$.

Next assume that T fixes a subspace Z and that X' has the bounded approximation property. Then $(T \otimes T)'$ maps $(Y \otimes_\pi Y)' = \mathscr{L}(Y, Y')$ continuously into $(X \otimes_\varepsilon X)' = \mathscr{I}(X, X')$, By our assumption on X', this space contains $X' \otimes_\pi X'$ isomorphically, so that T' is (ε, π)-continuous and fixes the quotient Z'. Hence $\mathscr{L}(Z', Z'') = \Gamma_2(Z', Z'')$ by the previous observation, and so $\mathscr{L}(Z', Z) = \Gamma_2(Z', Z)$. ■

Given a Banach space X, let us as usual denote by $q(X)$ the infimum of all $2 \leqq q \leqq \infty$ such that X has cotype q.

4. Proposition. *Let X and Y be Banach spaces.*

(a) *If $q(X') > 2$ and every infinite-dimensional subspace of X' contains a copy of X', then every (ε, π)-continuous operator $T: X \to Y$ is strictly cosingular.*

(b) *If $q(X) > 2$, X' has the bounded approximation property, and every infinite-dimensional subspace of X contains a copy of X, then every (ε, π)-continuous operator $T: X \to Y$ is strictly singular.*

Proof. (a) If Z is an infinite-dimensional quotient of X, then our hypothesis yields $q(Z') = q(X') > 2$. This implies $\mathscr{L}(Z, Z') \neq \Gamma_2(Z, Z')$, cf. [17] or [21]. By Proposition 3, T cannot fix any infinite-dimensional quotient, i.e., T is strictly cosingular.

(b) This time we have $q(Z) = q(X) > 2$ for every infinite-dimensional subspace Z of X. Again this yields $\mathscr{L}(Z', Z) \neq \Gamma_2(Z', Z)$. Since X' has the bounded approximation property, Proposition 3 tells us that T cannot fix Z. Consequently, T must be strictly singular. ∎

In particular, all (ε, π)-continuous operators from l_p into any Banach space Y are strictly cosingular if $1 < p \leqq 2$; to cover the case $p = 2$, use the fact (ε, π)-continuous operators belong to $\mathscr{P}$. However, by Proposition 2, this does not extend to $p = 1$. Similarly, for $2 < q < \infty$, all (ε, π)-continuous operators $l_q \to Y$ are strictly singular. It follows again from Proposition 3 that this does not extend to $q \leqq 2$, or to $q = \infty$. By duality, all (ε, π)-continuous operators from an arbitrary Banach space into l_r are strictly singular if $2 \leqq r < \infty$, and strictly cosingular if $1 < r < 2$; cf. [13].

In addition, we may state

5. Proposition. *Let X and Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$ be (ε, π)-continuous.*

(a) *If Y is an $\mathscr{L}_1$-space, then T is strictly singular.*

(b) *If X is an $\mathscr{L}_\infty$-space, then T is strictly cosingular.*

Proof. (b) follows from (a) by a duality argument. To prove (a) let Z be a subspace of X which is fixed by T. Let $J_1: Z \to X$ and $J_2: Z \to Y$ be isomorphic embeddings such that $T \circ J_1 = J_2$. By our choice of Y, $Z \otimes_\pi Y$ is a subspace of $Y \otimes_\pi Y$; cf. [1] or [3]. Also, $T \otimes T$ induces the continuous operator $\mathrm{Id}_Z \otimes J_2$: $Z \otimes_\varepsilon Z \to Z \otimes_\pi Y$. So, if we assume Z to be of infinite dimension, application of Lemma 1(b) yields $\mathrm{Id}_Z \in \mathscr{P}$, i.e., Z must be a Hilbert-Schmidt space [4].

Z cannot contain a copy of l_1, since otherwise $l_1 \otimes_\varepsilon l_1 = l_1 \otimes_\pi l_1$ would follow from our assumptions, which is impossible. Being a subspace of an $\mathscr{L}_1$-space, Z must therefore be reflexive [10], hence super-reflexive [22] (compare [6] for a direct argument). However, there are no super-reflexive Hilbert-Schmidt spaces of infinite dimension [4].

It follows that Z must be finite-dimensional, i.e., T is strictly singular. ∎

Every strictly cosingular operator $L_\infty(\mu) \to X$ is weakly compact (but not conversely: consider any surjection $L_\infty[0, 1] \to l_2$). Similarly for strictly singular operators $X \to L_1(\mu)$. However, as was shown in [13], the strictly cosingular operators $L_\infty(\mu) \to L_\infty(\nu)$, that is, the strictly singular operators $L_1(\mu) \to L_1(\nu)$, are precisely the weakly compact ones. We shall use this below.

Our first result in this section depends on another theorem of G. Pisier [16].

6. Proposition. *Let* X_0, X, Y, Y_0 *be Banach spaces, let* $1 < p \leqq 2 \leqq q < \infty$ *be fixed, and let* $R \in \mathscr{C}_q(Y, Y_0)$, $S \in \mathscr{T}_p(X, Y)$ *and* $T \in \mathscr{P}(X_0, X)$ *be given. Put* $\sigma := (1/p) - (1/q)$. *Then there is a constant* C *such that*

$$\Big(\sum_{i=1}^{n} \|RSTx_i\|^2\Big)^{1/2} \leqq Cn^\sigma\, w_2(x_i)$$

for all finite subsets $\{x_1, \ldots, x_n\}$ *of* X_0. *Here we put*

$$w_2(x_i) := \sup \Big\{\Big(\sum_{i=1}^{n} |\langle a, x_i\rangle^2\Big)^{1/2} \Big| \, a \in X_0', \ \|a\| \leqq 1\Big\}\,.$$

Proof. Let us write $\hat{X}$ for the linear span of $\{Tx_1, \ldots, Tx_n\}$, $\hat{Y}$ for $S(\hat{X})$, R_0 for the restriction of R to $\hat{Y}$, and $S_0 \colon \hat{X} \to \hat{Y}$ for the operator induced by S. By Theorem 4.1 of [16], $R_0 S_0$ has an extension $U \colon X \to Y_0$ such that $\gamma_2(U) \leqq K\, \tau_p(S)\, c_q(R)\, n^\sigma$, K being independent of $x_1, \ldots, x_n$. Here γ_2, τ_p and c_q denote the canonical ideal norms on Γ_2, $\mathscr{T}_p$ and $\mathscr{C}_q$, respectively. Let us abbreviate $C_0 := K\, \tau_p(S)\, c_q(R)$.

Given $\varepsilon > 0$, we can find a Hilbert space H and operators $A_1 \colon X \to H$, $A_2 \colon H \to Y_0$ such that $A_2 A_1 = U$ and $\|A_2\|\, \|A_1\| \leqq (1 + \varepsilon)\, \gamma_2(U) \leqq (1 + \varepsilon)\, C_0 n^\sigma$. Let π be the natural ideal norm on $\mathscr{P}$. Then $\pi(T) = \sup \pi_2(AT)$, π_2 being the ideal norm on $\mathscr{P}_2$ and A ranging over all operators from X into a Hilbert space such that $\|A\| \leqq 1$. In particular, we have $\pi_2(A_1 T) \leqq \pi(T)\, \|A_1\|$. Thus

$$\Big(\sum_{1}^{n} \|RSTx_i\|^2\Big)^{1/2} = \Big(\sum_{1}^{n} \|UTx_i\|^2\Big)^{1/2} \leqq \|A_2\| \Big(\sum_{1}^{n} \|A_1 Tx_i\|^2\Big)^{1/2} \leqq$$
$$\leqq \|A_2\|\, \|A_1\|\, \pi(T)\, w_2(x_i) \leqq (1 + \varepsilon)\, C_0\, \pi(T)\, w_2(x_i)\, n^\sigma\,.$$

Put $C := C_0\, \pi(T)$ and let ε pass to zero. ∎

Application of Proposition 3 in [7] yields:

7. Corollary. *Let* R, S, T *be as before. If* $(1/p) - (1/q) < \frac{1}{2}$, *then* RST *belongs to* $\mathscr{P}_{r,2}(X_0, Y_0)$ *for every* r *such that*

$$\frac{1}{r} < \frac{1}{2} - \frac{1}{p} + \frac{1}{q}\,.$$

Further, from Lemma 1(a) we obtain:

8. Corollary. *Let* $\mathscr{A}$ *be an operator ideal which is contained in* $\mathscr{C}_q \cap \mathscr{T}_p$ *where* $1 < p \leqq 2 \leqq q < \infty$ *are such that* $(1/p) - (1/q) < \frac{1}{2}$. *Let* E *be an* lc $\mathscr{A}$*-space. If* $E \otimes_\varepsilon E = E \otimes_\pi E$, *then* E *is nuclear.*

Proof. Fix any 0-basis $\mathscr{U}$ in E. From our assumptions, Lemma 1 and Corollary 7, we get that each $U \in \mathscr{U}$ contains a $V \in \mathscr{U}$ such that $\Phi_{UV} \in \mathscr{P}_{r,2}(E_{(V)}, E_{(U)})$. Since the composition of n operators in $\mathscr{P}_{r,2}$ is nuclear if n is sufficiently large (cf. [11]), our lcs E is nuclear. ∎

9. Corollary. *Let E be an lcs with a 0-basis $\mathscr{U}$ such that each $E_{(U)}$, $U \in \mathscr{U}$, is isomorphic to a subspace of a quotient of some $\mathscr{L}_p$-space, $1 < p < \infty$ fixed. If $E \otimes_\varepsilon E = E \otimes_\pi E$, then E is nuclear.*

Of course, we may also allow p to vary over an interval $[s, t]$ for any fixed $1 < < s \leqq t < \infty$. Compare with [9] and [7].

We do not know, however, if for projective limits E of $\mathscr{L}_1$-spaces, or of $\mathscr{L}_\infty$-spaces, it is also true that E is nuclear whenever $E \otimes_\varepsilon E = E \otimes_\pi E$ holds. From Proposition 5 and the fact that $\mathscr{L}_1$-spaces and $\mathscr{L}_\infty$-spaces possess the Dunford-Pettis property we only get that such a space E must be a Schwartz space.

The methods used here can also be employed to obtain informtion on the following problem which is related to (*): Given an lcs E and a Banach space X, when does it follow from $E \otimes_\varepsilon X = E \otimes_\pi X$ that E is nuclear? Various conditions to be imposed on X are known to ensure the answer is affirmative, see e.g. [3] and in particular [7]. We ammend these results by specifying conditions to be put on E in order that the answer to the above question be affirmative no matter which infinite-dimensional Banach space X we have chosen to start with. These conditions are broader than the corresponding ones in Corollary 8: the cases of $\mathscr{L}_1$-spaces and of $\mathscr{L}_\infty$-spaces are included.

10. Proposition. *Let A be an operator ideal which is contained in either of the ideals*
(a) *Γ_1 of all $\mathscr{L}_1$-factorable operators,*
(b) *Γ_∞ of all $\mathscr{L}_\infty$-factorable operators,*
(c) *$\mathscr{T}_p \cap \mathscr{C}_q$ for $1 < p \leqq 2 \leqq q < \infty$ with $(1/p) - (1/q) < \frac{1}{2}$,*
and let E be an lc *$\mathscr{A}$-space. If there is a Banach space X of infinite dimension such that $E \otimes_\varepsilon X = E \otimes_\pi X$, then E is nuclear.*

Proof. (a) follows from (b) by a duality argument. To prove (b), recall that $\mathscr{P}_{2,1} \circ \Gamma_\infty$ is contained in $\mathscr{P}_q$ for every $q > 2$ (cf. [15]), and apply Lemma 1(c). To obtain (c), we may proceed as in Corollary 8, by using Corollary 7 and Lemma 1(b). ∎

4. LOCALLY CONVEX TOPOLOGIES ON BANACH SPACES

Let X be a Banach space and $\mathscr{A}$ an operator ideal. An lc topology $\mathscr{T}_\mathscr{A}$ is defined on X in terms of the seminorms $x \mapsto \|Tx\|$, where T runs through $\mathscr{A}(X, Y)$ and Y varies over all (or sufficiently many) Banach spaces. We shall also write $X_\mathscr{A}$ instead of $[X, \mathscr{T}_\mathscr{A}]$. Note that $X_\mathscr{A}$ does not change if we pass from $\mathscr{A}$ to its injective hull [15]. Therefore we only need to work with injective ideals.

Note that $X_\mathscr{A}$ is *not* necessarily an lc $\mathscr{A}$-space. In fact, one easily checks that for every Banach space X and every injective ideal $\mathscr{A}$ the space $X_\mathscr{A}$ is an lc $\mathscr{A}$-space if and only if $\mathscr{A}(X, Y) = \bigcap_{n \in N} \mathscr{A}^n(X, Y)$ holds for every Banach space Y. Here we put of course $\mathscr{A}^n = \mathscr{A} \circ \mathscr{A} \circ \ldots \circ \mathscr{A}$ (n times) for every $n \in \mathbb{N}$.

The above condition is trivially satisfied if the operators in $\mathscr{A}$ factor through Banach spaces whose identity mappings belong to $\mathscr{A}$. It is also satisfied if $\mathscr{A}$ is injective, surjective, and uniformly closed, cf. [2]. And finally, it is satisfied for the ideal $\mathscr{S}_0$ of all strongly nuclear operators, cf. [15]. It is well-known that the topology of $X_{\mathscr{S}_0}$ is in fact the finest (strongly) nuclear topology on X which is consistent with the dual pairing $\langle X, X' \rangle$, cf. [3]. Using this it is easy to see that nuclearity of $X_{\mathscr{A}}$ implies $\mathscr{A}(X, Y) \subset \mathscr{S}_0(X, Y)$ for every Banach space Y.

Note also that $X_{\mathscr{A}}$ is not necessarily nuclear if $X_{\mathscr{A}} \otimes_\varepsilon X_{\mathscr{A}} = X_{\mathscr{A}} \otimes_\pi X_{\mathscr{A}}$ holds: Just consider any of the Banach spaces X constructed in [20] and define $\mathscr{A}$ by any of the properties Id_x possesses anyway (e.g., being of cotype 2, or of dual cotype 2, etc.). Nonetheless, even for these spaces X there are, in between the norm topology and $\mathscr{T}_{\mathscr{S}_0}$, lc topologies $\mathscr{T}$ such that the ε-tensor product of two copies of $[X, \mathscr{T}]$ does not coincide with the corresponding π-tensor product. To prove this, let us denote by $\mathscr{T}_p^\wedge$ and $\mathscr{C}_q^\wedge$ the ideals of operators which factor through a Banach space of type p, that is, of cotype q.

11. Proposition. *Let $1 < p \leqq 2 \leqq q < \infty$ be such that $(1/p) - (1/q) < \frac{1}{2}$, and let $\mathscr{A}$ be the ideal $\mathscr{T}_p^\wedge \cap \mathscr{C}_q^\wedge$. If X is a Banach space such that $X_{\mathscr{A}} \otimes_\varepsilon X_{\mathscr{A}} = X_{\mathscr{A}} \otimes_\pi X_{\mathscr{A}}$, then X is finite-dimensional.*

Proof. $\mathscr{T}_p^\wedge$ and $\mathscr{C}_q^\wedge$, and hence $\mathscr{A}$, are injective. Let $T \in \mathscr{A}(X, Z)$ with dense range be given. By hypothesis, there are operators $S_j \in \mathscr{A}(X, Z_j)$, $j = 1, 2$, with dense range, and operators $R_1 \in \mathscr{P}(Z_1, Z)$, $R_2 \in \mathscr{P}(Z_2, Z_1)$ such that $R_1 S_1 = T$ and $R_2 S_2 = S_1$. We may assume that Z has cotype q and Z_1 has type p. From Corollary 7 we infer $T \in \mathscr{C}_q \circ \mathscr{T}_p \circ \mathscr{P} \subset \mathscr{P}_{r,2}$ for any r such that

$$\frac{1}{r} < \frac{1}{2} - \frac{1}{p} + \frac{1}{q}.$$

As in Corollary 8 we conclude that $X_{\mathscr{A}}$ is nuclear.

Let H be a Hilbert space. Since Id_H belongs to $\mathscr{A}$, we get that $\mathscr{L}(X, H)$ consists of nuclear operators only. As in [5] we obtain $\dim X < \infty$. ■

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