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THE REAL K-RING OF SOME CW-COMPLEXES OF SMALL DIMENSION

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In 1981, L. M. Woodward in [8] classified the stable classes of orientable vector bundles over CW-complexes of small dimension. Using his results and some algebraic arguments, we describe the real K-ring of some CW-complexes of a dimension $\leqq 7$ in terms of cohomology and characteristic classes. We also show that our description can be really used for explicit calculation of K-rings.

1. PRELIMINARY NOTES

In this section we introduce the special symbols needed in the sequel. Let $\varrho_m: Z \to Z_m$ denote the reduction mod m, $\varrho_{42}: Z_4 \to Z_2$ the reduction mod 2 and $i: Z_2 \to Z_4$ the injection. The same symbols will abbreviate the induced homomorphisms in cohomology. The symbols δ and Δ are used to denote the Bockstein coboundary homomorphisms of the sequences $0 \to Z \to Z \to Z_2 \to 0$ and $0 \to \to Z_4 \to Z_8 \to Z_2 \to 0$, respectively.

The Pontrjagin square $P: H^2(X; Z_2) \to H^4(X; Z_4)$ (see [5; Chapt. 2, exercises] or [7; 10]) is the cohomology operation satisfying

$$(1.1)\qquad \begin{aligned} \varrho_{42}(P(a)) &= i(a^2)\,, \quad a \in H^2(X; Z_2)\,,\\ P(a + b) &= P(a) + P(b) + i(ab)\,, \quad a, b \in H^2(X; Z_2)\,,\\ P(xy) &= \Delta(x)\,\Delta(y)\,, \quad x, y \in H^1(X; Z_2)\,. \end{aligned}$$

Here the first two equations are precisely [7; 10.2, 10.3] while the third follows from [7; 10.5] and from the fact that $i(x^2) = i(Sq^1 x) = 0$ for $x \in H^1(X; Z_2)$.

We will also use the following relations between the Stiefel-Whitney and the Pontrjagin classes of a fibration ξ ([4], [7]):

$$(1.2)\qquad \begin{aligned} \varrho_2(p_1(\xi)) &= (w_2(\xi))^2\,,\\ \varrho_4(p_1(\xi)) &= P(w_2(\xi)) + i(w_4(\xi) + w_1(\xi)\,.\,Sq^1(w_2(\xi)))\,,\\ w_3(\xi) &= w_1(\xi)\,w_2(\xi) + Sq^1(w_2(\xi))\,. \end{aligned}$$

Note that the first equation can be obtained by applying ϱ_{42} to the second. The characteristic classes obviously define the maps $w_i\colon BO \to K(Z_2; i)$ and $p_j\colon BO \to \to K(Z; 4j)$ of the classifying spaces.

2. RESULTS

We shall deal with the representable K-theory, i.e., with the set $KO^{\sim}(X) = [X; BO]$. This set is endowed with the natural structure of a ring (see [6; 13]). Of course, if X is a finite-dimensional CW-complex, the elements of $KO^{\sim}(X)$ can be viewed as stable equivalence classes of vector fibrations over X. In this case, addition is defined by the Whitney sum while multiplication is given by the tensor product as in [3; 2].

From now on, we shall write for brevity $H^*(X)$ instead of $H^*(X; Z)$ and $\bar{H}^*(X)$ instead of $H^*(X; Z_2)$. Let us define

$$F(X) = \{(a, b, c) \in \bar{H}^1(X) \oplus \bar{H}^2(X) \oplus H^4(X);\ \varrho_2(c) = b^2\}\,,$$

and define operations $\boxplus$ and $*$ on the set $F(X)$ by

$$(a, b, c) \boxplus (x, y, z) = (a + x,\ b + ax + y,\ c + \delta a \delta x + z)\,,$$

$$(a, b, c) * (x, y, z) = (0, ax, \delta a \delta x)\,.$$

Finally, denote by $U = (w_1, w_2, p_1)\colon KO^{\sim}(X) \to F(X)$ the map induced by w_1, w_2 and p_1. In the next section we prove (compare [8; Theorem 1]):

2.1. Theorem. *The system $(F(X), \boxplus, *)$ forms a commutative ring with the zero element $(0, 0, 0)$. The map $U\colon KO^{\sim}(X) \to F(X)$ is a homomorphism of rings for each CW-complex X.*

If X is a CW-complex of a dimension $\leqq 7$, the map U is an epimorphism. If, in addition, $H^4(X)$ has no 2-torsion, the map U is an isomorphism.

Remark. Improving slightly [8; Theorem 1], it is possible to prove that the kernel of U is isomorphic with $\mathrm{Tor}(H^4(X); Z_2)$ for each CW-complex of a dimension $\leqq 7$.

In the following example, $C_k(a)$ and $C(a)$ abbreviate the cyclic group of the order k and the infinite cyclic group generated by a, respectively.

Example. The orthogonal K-ring of the Grassmann manifold $G(2, 2)$ of all two-dimensional linear subspaces in R^4 is isomorphic with the direct sum $C_4(f) \oplus \oplus C(g)$ for some $f, g \in KO^{\sim}(G(2, 2))$ satisfying $f^2 = 2f$ and $g^2 = fg = 0$.

Indeed, $G(2, 2)$ is known to be a four-dimensional oriented compact manifold, hence $H^4(G(2, 2)) \cong C(s)$ for some $s \in H^4(G(2, 2))$. It can be easily deduced from the description of the ring $\bar{H}^*(G(2, 2))$, given in [4; exercise 7.B] or [2], that there are unique $w_i \in \bar{H}^i(G(2, 2))$, $i = 1, 2$, with $\bar{H}^i(G(2, 2)) = C_2(w_i)$. It is not hard to compute that $w_2^2 = \varrho_2(s)$ and $w_1^4 = 0$. Now, it is clear that $F(G(2, 2)) \cong \cong C_4((w_1, 0, 0)) \oplus C((0, 0, s))$ as Abelian groups. By definition, $(w_1, 0, 0) * * (w_1, 0, 0) = (0, w_1^2, 0) = 2(w_1, 0, 0)$ and $(0, 0, s)^2 = (w_1, 0, 0) * (0, 0, s) = 0$.

Since $H^4(G(2,2))$ has no 2-torsion, our statement follows from Theorem 2.1 (with $f = U^{-1}((w_1, 0, 0))$ and $g = U^{-1}((0, 0, s)))$.

The next theorem, as well as the previous one, compares $KO^{\sim}(X)$ with some ring created from the cohomology of the space X, but the very restrictive assumption on the non-existence of a 2-torsion is replaced by a weaker one. We introduce the following notation:

$$G(X) = \{(a, b, c, d) \in \bar{H}^1(X) \oplus \bar{H}^2(X) \oplus \bar{H}^4(X) \oplus H^4(X) ;$$
$$\varrho_4(d) = P(b) + i(c + a \,.\, Sq^1(b))\} .$$

The operations $\boxplus$ and $*$ are defined by

$$(a, b, c, d) \boxplus (x, y, z, w) = (a + x,\ b + ax + y,\ c + (Sq^1 b + ab)\,x + by + a(Sq^1 y + xy) + z,\ d + \delta a \delta x + w)\,,$$
$$(a, b, c, d) * (x, y, z, w) = (0, ax, a^2(x^2 + y) + a(x^3 + xy) + (a^3 + ab)\,x + (a^2 + b)\,x^2, \delta a \delta x)\,.$$

Finally, denote by V the map (w_1, w_2, w_4, p_1): $KO^{\sim}(X) \to G(X)$. We prove the following theorem (compare [8; p. 178]):

2.2. Theorem. *The system* $(G(X), \boxplus, *)$ *forms a commutative ring with* $(0, 0, 0, 0)$ *as the zero element. The map* $V: KO^{\sim}(X) \to G(X)$ *is a homomorphism of rings for each CW-complex X.*

If X is a CW-complex of a dimension $\leqq 7$, *the map V is an epimorphism. If, in addition,* $H^4(X)$ *has no 4-torsion, the map V is an isomorphism.*

Remark. Using an improved form of the results of [8] we can establish the existence of the natural exact sequence

$$0 \to \operatorname{Tor}(H^4(X); Z_2) \to \operatorname{Tor}(H^4(X); Z_4) \to KO^{\sim}(X) \to^V G(X) \to 0\,.$$

In the next example we compute anew the real K-ring of the real projective spaces $P^k = P^k(R)$ for $k \leqq 7$ (see [1] or [3; 4.6]).

Example. The ring $KO^{\sim}(P^k)$ is, for $k \leqq 7$, isomorphic with $C_{j(k)}(\lambda)$, where $j(1) = 2$, $j(2) = j(3) = 4$ and $j(4) = \ldots = j(7) = 8$. The element $\lambda \in KO^{\sim}(P^k)$ corresponds to the canonical linear bundle over P^k and the multiplication is characterized by $\lambda^2 = -2\lambda$.

To prove the above statement, recall the existence of $w_1 \in \bar{H}^1(P^k)$ with $\bar{H}^*(P^k) \cong Z_2[w_1]/(w_1^{k+1} = 0)$. Clearly, there exists $s \in H^4(P^k)$, $k \geqq 4$, such that $H^4(P^k) \cong C_2(s)$ and $\varrho_2(s) = w_1^4$. As $\varrho_{42}: H^4(P^k; Z_4) \to \bar{H}^4(P^k)$ is an isomorphism, the equation in the definition of $G(P^k)$ is equivalent simply with $\varrho_2(d) = b^2$ (see the remark following 1.2).

Now, using the above comments, we can easily verify that $G(P^k) \cong C_{j(k)}((w_1, 0, 0, 0))$ and that $(w_1, 0, 0, 0)^2 = -2(w_1, 0, 0, 0)$. Because $H^4(P^k) \cong Z_2$, the ring $G(P^k)$ is, by Theorem 2.2, isomorphic with $KO^{\sim}(X)$.

3. PROOFS

This section contains the proof of Theorems 2.1 and 2.2. In the following lemma we verify the algebraic properties of the maps U and V.

3.1. Lemma. *The sets $F(X)$ and $G(X)$ with the operations $\boxplus$ and $*$ form commutative rings. If X is a CW-complex, the maps $U\colon KO^{\sim}(X) \to F(X)$ and $V\colon KO^{\sim}(X) \to G(X)$ are homomorphisms of rings.*

Proof. It can be verfied directly by using 1.1 and carrying out a long but elementary computation that the sets $F(X)$ and $G(X)$ really satisfy the axioms of commutative rings.

In order to prove the additivity and the multiplicativity of U and V, it is sufficient to do this for finite-dimensional CW-complexes only. Indeed, the additivity (multiplicativity) of the map U means that the natural transformation $A\colon KO^{\sim}(X) \times \times KO^{\sim}(X) \to F(X)$ defined by $A(x, y) = U(x) \boxplus U(y) - U(x + y)$ ($A(x, y) = = U(x) * U(y) - U(xy)$) is zero. For each CW-complex we have the following commutative diagram (vertical maps are induced by the inclusions):

$$\begin{array}{ccc} KO^{\sim}(X) \times KO^{\sim}(X) & \xrightarrow{A} & F(X) \\ \downarrow & & \downarrow\cong \\ KO^{\sim}(X^5) \times KO^{\sim}(X^5) & \xrightarrow{A} & F(X^5) \end{array}$$

where $X^5 \to X$ is the 5-skeleton, i.e., a finite-dimensional CW-complex. By the diagram, the map A is zero if the induced map on the skeleton is. The argument for V is similar.

So, we can suppose that $\dim(X) < \infty$, hence the elements of $KO^{\sim}(X)$ can be viewed as the stable equivalence classes of vector fibrations over X. Such an equivalence class will be denoted by square brackets. The addition and multiplication are defined by

$$[\xi] + [\eta] = [\xi \oplus \eta],$$
$$[\xi^m] \cdot [\eta^n] = [\xi^m \otimes \eta^n] - [\varepsilon^m \otimes \eta^n] - [\xi^m \otimes \varepsilon^n],$$

where ε^k denotes the k-dimensional trivial vector bundle over X. To verify the algebraic properties of U and V, we need only to express the characteristic classes of $\xi \oplus \eta$ and $[\xi] \cdot [\eta]$ in terms of those of ξ and η. By [4; § 4] we have

$$w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta), \quad w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi)\, w_1(\eta) + w_2(\eta),$$
$$w_4(\xi \oplus \eta) = w_4(\xi) + w_3(\xi)\, w_1(\eta) + w_2(\xi)\, w_2(\eta) + w_1(\xi)\, w_3(\eta) + w_4(\eta),$$

where $w_3 = w_1 w_2 + Sq^1 w_2$ by 1.2.

Similarly, using the formula for the Chern classes of the Whitney sum [4; § 14] and the definition of the Pontrjagin class, we obtain

$$p_1(\xi \oplus \eta) = -c_2(\xi_C \oplus \eta_C) = p_1(\xi) - c_1(\xi_C)\, c_1(\eta_C) + p_1(\eta),$$

where $c_1(\xi_C)\, c_1(\eta_C) = \delta w_1(\xi)\, \delta w_1(\eta)$ [4; exercise 15D]. These formulas make the additivity obvious. Writing formally

$$w(\xi) = \prod_{1 \leqq i \leqq m} (1 + x_i)\,, \quad w(\eta) = \prod_{1 \leqq j \leqq n} (1 + y_j)$$

and using [4; exercise 7C] we can write

$$w([\xi] \cdot [\eta]) = \prod_{\substack{1 \leqq i \leqq m \\ 1 \leqq j \leqq n}} (1 + x_i + y_j) \{ \prod_{1 \leqq f \leqq m} (1 + x_f)^n \prod_{1 \leqq g \leqq n} (1 + y_g)^m \}^{-1} =$$

$$= \prod_{\substack{1 \leqq i \leqq m \\ 1 \leqq j \leqq n}} ((1 + x_i + y_j)(1 + x_i)^{-1} (1 + y_j)^{-1})\,.$$

The last polynomial can be expressed in terms of the elementary symmetric polynomials in the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, from which we shall deduce that

$$w_1([\xi] \cdot [\eta]) = 0\,, \quad w_2([\xi] \cdot [\eta]) = w_1(\xi)\, w_1(\eta)\,,$$

$$w_4([\xi] \cdot [\eta]) = w_1^2(\xi)\, w_1^2(\eta) + w_1^2(\xi)\, w_2(\eta) + w_2(\xi)\, w_1^2(\eta) +$$

$$+ w_1(\xi)\,(w_1^3(\eta) + w_1(\eta)\, w_2(\eta)) + (w_1^3(\xi) + w_1(\xi)\, w_2(\xi))\, w_1(\eta)\,.$$

Again we omit this long but elementary computation. Using a similar formula for the Chern classes we can verify that $p_1([\xi] \cdot [\eta]) = \delta w_1(\xi)\, \delta w_1(\eta)$ and the lemma follows.

Let us consider the rings $\tilde{F}(X) = \{(a, b, c) \in F(X);\ a = 0\}$ and $\tilde{G}(X) = \{(a, b, c, d) \in G(X);\ a = 0\}$. Clearly, the maps U and V restrict to $\tilde{U}: [X; BSO] \to \tilde{F}(X)$ and $\tilde{V}: [X; BSO] \to \tilde{G}(X)$. The results [8; Theorem 1 and the note at the top of p. 178] can be reformulated as follows:

3.2. Proposition. *The maps $\tilde{U}$ and $\tilde{V}$ are homomorphisms of Abelian groups. If X is a CW-complex with* $\dim(X) \leqq 7$, *our maps are epimorphisms. If, in addition, the group $H^4(X)$ has no* 2*-torsion, the map $\tilde{U}$ is an isomorphism. If $H^4(X)$ has no* 4*-torsion, the map $\tilde{V}$ is an isomorphism.*

Now, we are able to complete our proofs. We can form the commutative diagram of Abelian groups:

$$\begin{array}{ccccc}
\tilde{F}(X) & \subset & F(X) & \xrightarrow{q} & F(X)/\tilde{F}(X) \\
{\scriptstyle \tilde{U}}\uparrow & & {\scriptstyle U}\uparrow & & {\scriptstyle B}\uparrow \\
[X; BSO] & \subset & KO^{\sim}(X) & \xrightarrow{p} & KO^{\sim}(X)/[X; BSO]
\end{array}$$

where $\subset$ are the natural inclusions, p, q are projections and the map B is defined by $B(p([\xi])) = q(U([\xi]))$.

Notice that there are identifications $KO^{\sim}(X)/[X; BSO] \cong [X; BO(1)]$ and $F(X)/\tilde{F}(X) \cong \bar{H}^1(X)$ such that $B(p([\xi])) = w_1(\xi)$. Because $BO(1) \cong K(1; Z_2)$, the map B is an isomorphism. It is not hard to deduce from the above diagram that U is an epimorphism if $\tilde{U}$ is, and that U is an isomorphism if $\tilde{U}$ is. This concludes the proof of Theorem 2.1.

In order to prove Theorem 2.2 we can form the diagram analogous to the above for the rings $G(X)$ and $\tilde{G}(X)$. Then Theorem 2.2 follows by the same argument as Theorem 2.1.

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