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A STRONG CONVERGENCE IN L^p AND UPPER q-CONTINUOUS OPERATORS

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In [1] S. Banach and S. Saks proved a theorem which can be formulated as follows (see also Banach-Saks' Theorem in [5]):

Theorem 1. *Let* $w\text{-}\lim_{n\to+\infty} x_n = x_\infty$ *(i.e.,* $\{x_n\}$ *weakly converges to* x_∞*) in a space* L^p *(*$p \in (1, +\infty)$*). Then there exists a subsequence* $\{x_{n_k}\}$ *of the sequence* $\{x_n\}$ *such that*

$$\frac{1}{k}(x_{n_1} + x_{n_2} + \ldots + x_{n_k})$$

converges to x_∞ *in the norm of* L^p.

In the same paper S. Banach and S. Saks gave an example which shows that Theorem 1 cannot be extended to L^1.

The aim of this note is to present a sufficient condition under which the hypothesis of Banach-Saks' Theorem in fulfilled in the case $p = 1$. This is used to generalize Theorem 7 of [2] which is useful for example in the theory of differential inclusions.

Now we shall introduce the notation which will be needed in the note.

Let E^n be the space of Euclidean n-vectors. Let us denote by $\text{cf}(E^n)$ the set of all nonempty closed convex subsets of E^n. If $A \subset E^n$ then $|A| = \sup\{|a| : a \in A\}$. $C^n(I)$ denotes the space of all continuous functions mapping the interval $I \subset E$ into E^n. By $B(I)$ we shall denote the Banach space of all continuous and bounded real functions on I with the maximum norm, and $2^{B(I)}$ will stand for the family of all nonempty subsets of $B(I)$.

Definition 1. The sequence $f_k \in B(I)$ *quasi-converges* (q-converges) to $f \in B(I)$ iff $\lim_{k\to+\infty} f_k(t) = f(t)$ for every $t \in I$. This will be denoted by $f_k \to^q f$.

Definition 2. The operator $T: B(I) \to 2^{B(I)}$ is *upper q-continuous* iff the assumptions

$$f_k \to^q f\,, \quad f_k, f \in B(I)\,, \quad y_k \in T(f_k)$$

imply that there exists a subsequence of the sequence $\{y_k\}$ convergent to some $y \in T(f)$ (in the norm).

Corollary 1. *If T is an upper q-continuous operator, then T is upper semicompact.*

Definition 3. Let X and Y be normed linear spaces. A mapping $F: X \to 2^Y$ is *weakly upper q-continuous* at a point $x \in X$ iff the assumptions

$$x_k, x \in X\,, \quad x_k \to^q x\,, \quad y_k \in F(x_k)$$

imply that there is a subsequence of the sequence $\{y_k\}$ which weakly converges to some $y \in F(x)$.

Now we shall formulate and prove a theorem which is an extension of Banach-Saks' Theorem in some subset of L^1.

Theorem 2. *Let* $\underset{n\to+\infty}{w\text{-lim}}\, x_n = x_\infty$ *in* $L^1(\langle a, +\infty))$, *and let there exist a function* $g \in L^1(\langle a, +\infty))$ *such that*

$$|x_n(t)| \leqq g(t) \quad \text{a.e. on} \quad \langle a, +\infty)\,, \quad n = 1, 2, \ldots\,.$$

Then there exists a subsequence $\{x_{n_k}\}$ *of the sequence* $\{x_n\}$ *such that*

$$\frac{1}{k}\,(x_{n_1} + x_{n_2} + \ldots + x_{n_k})$$

converges to x_∞ *in the norm of* $L^1(\langle a, +\infty))$.

Proof. Let us define the sequence $\{y_n\}$, $y_n \in L^2(\langle a, +\infty))$ by

$$y_n(t) = \frac{x_n(t)}{\sqrt{(1 + g(t))}}\,, \quad t \in \langle a, +\infty) \quad \text{and} \quad n = 1, 2, \ldots\,.$$

Since

$$|y_n|_2^2 = \int_a^{+\infty} \frac{x_n^2(t)}{1 + g(t)}\,\mathrm{d}t \leqq \int_a^{+\infty} \frac{g(t)}{1 + g(t)}\,g(t)\,\mathrm{d}t \leqq c = \int_a^{+\infty} g(t)\,\mathrm{d}t\,,$$

there is a subsequence $\{y_{1n}\}$ of the sequence $\{y_n\}$ which weakly converges to some $y_0 \in L^2(\langle a, +\infty))$. We shall show that

$$y_0(t) = \frac{x_0(t)}{\sqrt{(1 + g(t))}}\,.$$

Since the set $L_b^2\,(\langle a, +\infty))$ of bounded functions of $L^2(\langle a, +\infty))$ is strongly dense in $L^2(\langle a, +\infty))$, it suffices to show that

$$(1) \qquad \int_a^{+\infty} y_{1n}(t)\,z(t)\,\mathrm{d}t \to \int_a^{+\infty} \frac{x_0(t)}{\sqrt{(1 + g(t))}}\,z(t)\,\mathrm{d}t \quad \text{as} \quad n \to +\infty$$

for each $z \in L_b^2(\langle a, +\infty))$.

We have that $\{x_n\}$ weakly converges to x_0, $z(t)$ is bounded, i.e., $z(t)/\sqrt{(1 + g(t))}$ is bounded, thus

$$\int_a^{+\infty} x_{1n}(t)\,\frac{z(t)}{\sqrt{(1 + g(t))}}\,\mathrm{d}t \to \int_a^{+\infty} x_0(t)\,\frac{z(t)}{\sqrt{(1 + g(t))}}\,\mathrm{d}t\,, \quad \text{as} \quad n \to +\infty\,,$$

i.e., (1) holds.

Further, by Banach-Saks' Theorem there is a subsequence $\{y_{2n}\}$ of the sequence $\{y_{1n}\}$ such that

$$w_k = \frac{1}{k}(y_{21} + y_{22} + \ldots + y_{2k}) \to y_0 \quad \text{as} \quad k \to +\infty$$

in the norm of L^2.

Now, by Riesz' Theorem, there is a subsequence $\{w_{1k}\}$ of the sequence $\{w_k\}$ such that

$$w_{1k}(t) \to y_0(t) \quad \text{a.e. on} \quad \langle a, +\infty) \quad \text{as} \quad k \to +\infty,$$

i.e.,

$$\frac{1}{k}\left(\frac{x_{2\sigma_1}(t)}{\sqrt{(1 + g(t))}} + \frac{x_{2\sigma_2}(t)}{\sqrt{(1 + g(t))}} + \ldots + \frac{x_{2\sigma_k}(t)}{\sqrt{(1 + g(t))}}\right) \to$$

$$\to \frac{x_0(t)}{\sqrt{(1 + g(t))}} \quad \text{a.e. on} \quad \langle a, +\infty) \quad \text{as} \quad k \to +\infty.$$

Thus

$$\frac{1}{k}(x_{2\sigma_1}(t) + x_{2\sigma_2}(t) + \ldots + x_{2\sigma_k}(t)) \to x_0(t) \quad \text{a.e. on} \quad \langle a, +\infty) \quad \text{as} \quad k \to +\infty,$$

i.e.,

$$\left|\frac{1}{k}(x_{2\sigma_1}(t) + x_{2\sigma_2}(t) + \ldots + x_{2\sigma_k}(t)) - x_0(t)\right| \to 0 \quad \text{a.e. on} \quad \langle a, +\infty) \quad \text{as} \quad k \to +\infty.$$

By virtue of

$$\left|\frac{1}{k}(x_{2\sigma_1}(t) + x_{2\sigma_2}(t) + \ldots + x_{2\sigma_k}(t)) - x_0(t)\right| \leqq 2\, g(t) \in L^1(\langle a, +\infty))$$

and the Lebesgue Dominated Theorem, this yields

$$\int_a^{+\infty} \left|\frac{x_{2\sigma_1}(t) + \ldots x_{2\sigma_k}(t)}{k} - x_0(t)\right| \mathrm{d}t \to 0 \quad \text{as} \quad k \to +\infty.$$

The proof of Theorem 2 is complete.

The following lemma will be needed in the proof of Theorem 3.

Lemma 1 (Lemma 4, A. Haščák [3]). *Let $J = \langle 0, +\infty)$ and let the mapping $F\colon J \times E^n \to \mathrm{cf}(E^n)$ satisfy the following conditions:*

(c_0) *$F(t, x)$ is a non-empty, compact and convex subset of E^n for each $(t, x) \in J \times E^n$,*
(c_1) *for every fixed $t \in J$ the function $F(t, x)$ is upper semicontinuous,*
(c_2) *for each measurable function $x\colon J \to E^n$, there exists a measurable function $f_x\colon J \to E^n$ such that*

$$f_x(t) \in F(t, x(t)) \quad \text{a.e. on} \quad J.$$

Further, suppose that there exists $g\colon J \times J \to J$ such that

i) *$g(t, u)$ is monotone nondecreasing in u for each fixed $t \in J$,*
ii) *$\int_0^{+\infty} g^{p'}(s, c)\, \mathrm{d}s < +\infty$ for any constant $c > 0$ and $p' \geqq 1$,*

iii) *for each* $x \in E^n$,

$$|F(t, x)| \leqq g(t, |x|) \quad \text{a.e.} \quad \text{on} \quad J\,.$$

Given a function $x \in C^n(J)$, *denote by* $M(x)$ *the set of all measurable functions* $y: J \to E^n$ *such that*

$$y(t) \in F(t, x(t)) \quad a.e. \quad on \quad J\,.$$

Then the correspondence $x \to M(x)$ *defines a bounded weakly upper q-continuous mapping of*

$$B^n_\varrho(J) = \{x \in C^n(J) : |x(t)| \leqq \varrho\}\,, \quad \varrho > 0$$

into

$$\text{cf}(L^{p'}_n(J))\,, \quad L^{p'}_n = L^{p'} \times \ldots \times L^{p'}\,.$$

Theorem 3. *Let the hypotheses of Lemma* 1 *be satisfied and let D be a Banach space. Suppose that* $T: L^{p'}_n(J) \to D$ *is a compact linear operator.*

Then the operator TM defined by

$$TM\,x = \{z \in D: z = Ty \text{ and } y \in M\,x\}$$

maps B^n_ϱ *into* cf(D) *and is upper q-continuous.*

Proof. For $p' > 1$, Theorem 3 is proved in [2]. Thus we have to prove this theorem only for $p' = 1$. The proof in this case proceeds analogously as in the case $p' > 1$, but instead of Banach-Saks' Theorem we use Theorem 2.

First we shall prove that the operator TM is upper q-continuous. Let $x_n \to^q x$, $x_n, x \in B^n_\varrho$ and $z_n \in TM\,x_n$. We have to show that there is a subsequence of the sequence $\{z_n\}$ that converges (in the norm of D) to some $z \in TM\,x$. Let $z_i = Ty_i$, $y_i \in M\,x_i$. Since M is weakly upper q-continuous, there is a subsequence $\{y_{1i}\}$ of the sequence $\{y_i\}$ which weakly converges to some $y \in M\,x$. Since $\{y_{1i}\}$ is bounded and T is a compact linear operator there is a subsequence $\{y_{2i}\}$ of the sequence $\{y_{1i}\}$ such that $Ty_{2i} \to z \in D$ as $i \to +\infty$. We shall show that $z = Ty \in TM\,x$. Because $\{y_{1i}\}$ weakly converges to y we have that also $\{y_{2i}\}$ weakly converges to y. By Theorem 2 there is a subsequence $\{y_{3i}\}$ of the sequence $\{y_{2i}\}$ such that

$$\frac{y_{31} + y_{32} + \ldots + y_{3i}}{i} \to y$$

as $i \to +\infty$, in the norm of $L^{p'}_n(J)$.

Since T is compact and linear (hence T is continuous),

$$(2) \qquad T\left(\frac{y_{31} + y_{32} + \ldots + y_{3i}}{i}\right) \to Ty \quad \text{as} \quad i \to +\infty\,.$$

On the other hand, since $Ty_{3i} \to z \in D$ and T is linear we have

$$(3) \qquad z = \lim_{i \to +\infty} Ty_{3i} = \lim_{i \to +\infty} \frac{Ty_{31} + Ty_{32} + \ldots + Ty_{3i}}{i} =$$

$$= \lim_{i \to +\infty} T\left(\frac{y_{31} + y_{32} + \ldots + y_{3i}}{i}\right).$$

By (2) and (3) we get that $z = Ty \in TM\,x$. Thus the operator is upper q-continuous.

From the upper q-continuity of the operator $TM\,x$ we conclude that $TM\,x$ is closed. Further, Mx is a convex set and T is a linear operator. Thus $TM\,x$ is also a convex set.

Remark 1. In Theorem 3, T is a compact linear operator. M. Švec has constructed an example which shows that Theorem 3 is not valid if T is merely a linear operator.

Remark 2. In [4] S. Mazur has proved a theorem which deals with the strong convergence in normed linear spaces (see also [6], Theorem V.1.2). Banach-Saks' Theorem as well as Theorem 2 of this note are stronger variants of Mazur's Theorem.

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