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# ON A PROPERTY OF PSEUDOMETRICS AND UNIFORMITIES NEAR TO CONVEXITY

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Several attempts to generalize the concept of convexity to the theory of metric spaces appear in the literature. They can be divided roughly into two parts. The first one is the study of "convex-like" sets in a general metric space, the second one is the study of "convex-like" metrics reflecting some properties of the usual convexity in linear spaces (see e.g. [6], [9]). The nature of our work is of the second type. In the first part, we define a very general concept of a so-called preconvex pseudometric. Many methods of the convex calculus may be used mainly to examine the uniform structure of the corresponding spaces. The second part is devoted to uniform spaces the uniformity of which has a basis consisting of preconvex pseudometrics. It is shown that such spaces have very nice characterizations in terms of covers and entourages and that the class of all such spaces contains many natural spaces.

We refer to [4] for basic definitions and results pertaining to uniform spaces. However, we assume no separation axiom, thus a uniform space is a preuniform space in the sense of [4]. Similarly several elementary facts on uniform dimensions $\delta d$ and $\Delta d$ may and will be used in this general case too.

If $d$ is a pseudometric on a set $X$, $r$ a positive number, $x \in X$, we denote $B_d(x, r) = \{y \in X;\ d(x, y) < r\}$ and $\mathscr{B}_d(r)$ stands for the canonical uniform cover $\{B_d(x, r);\ x \in X\}$. If $\mathscr{G}$ is a cover of $X$, $x \in X$, then $\operatorname{St}(x, \mathscr{G})$ denotes the star of $x$ with respect to $\mathscr{G}$, $\operatorname{St} \mathscr{G} = \{\operatorname{St}(x, \mathscr{G});\ x \in X\}$. The letters $\mathsf{N}$, $\mathsf{R}$ stand for the set of all non-negative integers, all reals, respectively. The symbol $\circ$ denotes the usual composition of functions or relations.

## 1. PRECONVEX PSEUDOMETRICS

**1.1. Definition.** Let $d$ be a pseudometric on a set $X$, $c$ a positive number. We call $d$ to be *preconvex for distances less than* $c$ if for any $x, y$ in $X$ with $d(x, y) < c$ and any positive numbers $r, s$ with $d(x, y) < r + s$ there exists $z$ in $X$ with $d(x, z) < r$, $d(z, y) < s$. The pseudometric $d$ will be called *preconvex* if the upper condition holds for at least one $c$, it will be called *globally preconvex* if it holds for all positive $c$.

The following proposition contains several easy reformulations of the definition.

**1.2. Proposition.** *Let $d$ be a pseudometric on a set $X$, $c > 0$. The following properties are equivalent:*

(1) *$d$ is preconvex for distances less than $c$.*

(2) *If $x, y \in X$, $r_i > 0$ for $i = 1, \ldots, m$, $m \geqq 2$ and $d(x, y) < \min\{r_1 + \ldots \ldots + r_m, c\}$, then there are $z_0 = x, z_1, \ldots, z_{m-1}, z_m = y$ in $X$ such that $d(z_{i-1}, z_i) < r_i$ for all $i = 1, \ldots, m$.*

(3) *If $x, y \in X$, $r > 0$, $d(x, y) < 2r \leqq c$, then there is $z \in X$ with $d(x, z) < r$, $d(z, y) < r$.*

(4) *For any $x \in X$, $0 < 2r \leqq c$, we have $\mathrm{St}(x, \mathscr{B}_d(r)) = B_d(x, 2r)$.*

Proof. (1) $\Rightarrow$ (2) follows easily by induction, the implications (2) $\Rightarrow$ (3), (3) $\Leftrightarrow$ (4) are immediate. To prove (3) $\Rightarrow$ (1) let $d(x, y) < r + s \leqq c$. Choose $b > r$ such that $d(x, y) < b < r + s$. Put $z_0^1 = x$, $z_2^1 = y$ and take $z_1^1 \in X$ such that $d(z_0^1, z_1^1) < 2^{-1}b$, $d(z_1^1, z_2^1) < 2^{-1}b$. Proceeding by induction we construct $z_i^n \in X$, $n = 1, 2, \ldots$, $i = 0, 1, \ldots, 2^n$ such that $z_0^n = x$, $z_{2^n}^n = y$, $d(z_{i-1}^n, z_i^n) < 2^{-n}b$ for $i = 1, \ldots, 2^n$. Choose $n$ such that $(1 + 2^{-n})\, b < r + s$ and take $i \leqq 2^n$ the first integer such that $2^{-n}b(i + 1) \geqq r$. Then we have

$$d(x, z_i^n) \leqq 2^{-n}bi < r\,,$$

$$d(z_i^n, y) \leqq 2^{-n}b(2^n - i) = b(1 + 2^{-n}) - 2^{-n}b(i + 1) < s\,.$$

Note that if $X$ is a convex subset of a normed linear space, the metric induced on $X$ by the norm is (globally) preconvex. On the other hand, $X$ need not look like convex and the mentioned metric is still globally preconvex, take e.g. the rationals in $\mathsf{R}$. Also the uniformly discrete metric is preconvex.

The following two observations show that there are not many discrete or zero-dimensional preconvex pseudometrics. Recall that a pseudometric $d$ on a set $X$ is called quasidiscrete if for each $x \in X$ there is $r > 0$ such that $d(x, y) < r$ implies $d(x, y) = 0$, if $r$ can be chosen independently on $x$, then $d$ is called uniformly quasidiscrete. This simply means that the corresponding metric is discrete or uniformly discrete respectively.

**1.3. Proposition.** *A pseudometric $d$ on a set $X$ is uniformly quasidiscrete if and only if $d$ is quasidiscrete and preconvex.*

Proof. Suppose $d$ is preconvex for distances less than $c$ and there exist $x, y$ with $0 < d(x, y) < c$. Given any $r > 0$, there is $z$ such that $d(x, z) < r$, $d(z, y) < d(x, y)$. But $d(x, z) > 0$, hence $d$ is not quasidiscrete. The rest of proof is evident.

**1.4. Proposition.** *Let $d$ be a preconvex pseudometric on a set $X$. Then either $d$ is uniformly quasidiscrete or $\delta d(X, d) \geqq 1$.*

Proof. Take $c > 0$ such that $d$ is preconvex for distances less than $c$. Suppose $\delta d(X, d) < 1$ and $d$ is not uniformly quasidiscrete. Take $x, y$ with $0 < d(x, y) < c$. Then $\{X \setminus \{y\}, X \setminus \{x\}\}$ is a uniform cover of $(X, d)$, we can refine it by a uniform partition $\{G, H\}$ such that $x \in G$, $y \in H$. The $d$-distance of $G$, $H$ is at least some

$\varepsilon > 0$, by 1.2 (2) for $r_i \leqq \varepsilon$ we find $u \in G$, $v \in H$ with $d(u, v) < \varepsilon$, which is a contradiction.

**1.5. Proposition.** *Let $d$ be a pseudometric on a set $X$, $Y$ a dense subset of $(X, d)$, $c > 0$. Then $d$ is preconvex for distances less than $c$ if and only if its restriction $d_Y$ to $Y$ is preconvex for distances less than $c$.*

Proof. Suppose $d$ is preconvex for distances less than $c$, take $x, y \in Y$, $r, s > 0$, $d(x, y) < r + s \leqq c$. There is $z \in X$ with $d(x, z) < r$, $d(z, y) < s$. Choose $w \in Y$ with $d(z, w) < \min\{r - d(x, z), s - d(z, y)\}$, then $d(x, w) < r$, $d(w, y) < s$.

Conversely suppose $d_Y$ is preconvex for distances less than $c$, take $x, y \in X$, $r, s$ positive such that $d(x, y) < r + s \leqq c$. Choose $\varepsilon > 0$ such that $d(x, y) + 4\varepsilon < r + s$ and take $u, v \in Y$ with $d(x, u) < \varepsilon$, $d(y, v) < \varepsilon$. Then $d(u, v) \leqq d(u, x) + d(x, y) + d(y, v) < r + s - 2\varepsilon$. Find $z \in Y$ such that $d(u, z) < r - \varepsilon$, $d(z, v) < s - \varepsilon$. Then $d(x, z) < r$, $d(z, y) < s$, hence $d$ is preconvex for distances less than $c$.

**1.6. Proposition.** *Let $d$ be a pseudometric on a set $X$ preconvex for distances less than $c$, $x, y \in X$, $0 < d(x, y) < c$. Let $S$ be an arbitrary countable subset of the interval $[\![0, 1]\!]$ containing $0, 1$. Then for every $\varepsilon > 0$ there is an injective mapping $f: S \to X$ such that $f(0) = x$, $f(1) = y$ and $d(f(s), f(t)) < (d(x, y) + \varepsilon)(t - s)$ for all $t > s$ in $S$.*

Proof. I. At first observe that if $r, s$ are positive, $d(x, y) < r + s \leqq c$, then the set of all $z$ such that $d(x, z) < r$ and $d(z, y) < s$ is infinite. Actually, if it would be finite, take $a = d(x, z_0)$ the minimum of all the numbers $d(x, z)$. Then $d(x, y) \leqq d(x, z_0) + d(z_0, y) < a + s$, hence we can find $v \in X$ with $d(x, v) < a$, $d(v, y) < s$ and we get a contradiction with the minimality of $a$.

II. Put $k = \min\{d(x, y) + \varepsilon, c\}$. Suppose that $S$ is infinite and $S = \{s_n; n \in \mathbb{N}\}$, where $s_0 = 0$, $s_1 = 1$, $s_n \neq s_m$ for $n \neq m$. Define $f(0) = x$, $f(1) = y$ and proceed by induction. Suppose $n > 1$ and $f(s_i)$ have been defined for all $i < n$ and $d(f(s_i), f(s_j)) < k(s_j - s_i)$ for all $i, j < n$, $s_i < s_j$. Take $u = \max\{s_i;\ i < n,\ s_i < s_n\}$, $v = \min\{s_i;\ i < n,\ s_i > s_n\}$. Since $d(f(u), f(v)) < k(v - u) \leqq c$, we can choose $f(s_n) = z$ such that $d(f(u), z) < k(s_n - u)$, $d(z, f(v)) < k(v - s_n)$. Using I) $z$ can be chosen different from all $f(s_i)$ defined before. Now, if $i < n$, $s_i < s_n$, we have $d(f(s_i), f(s_n)) \leqq d(f(s_i), f(u)) + d(f(u), f(s_n)) < k(u - s_i) + k(s_n - u) = k(s_n - s_i)$. Similarly for the case $s_i > s_n$.

As a consequence of the preceding proposition we obtain the following characterization of preconvexity.

**1.7. Theorem.** *Let $d$ be a pseudometric on a set $X$, $c > 0$. The following properties are equivalent:*

(1) *$d$ is preconvex for distances less than $c$.*

(2) *For any $x, y \in X$ with $0 < d(x, y) < c$, for any countable dense subset $S$ of the interval $[\![0, 1]\!]$ containing $0, 1$ endowed with the usual metric and for any*

$k > d(x, y)$ *there exists an injective* $f: S \to X$ *Lipschitz with constant* $k$ *such that* $f(0) = x$, $f(1) = y$.

(3) *For any* $x, y \in X$ *with* $0 < d(x, y) < c$ *and any* $k > d(x, y)$ *there is* $f: \mathsf{Q}_2 \to X$ *Lipschitz with constant* $k$ *such that* $f(0) = x$, $f(1) = y$. (Here $\mathsf{Q}_2$ stands for the set of dyadic rationals in $[\![0, 1]\!]$ with the usual metric.)

Observe that if $(X, d)$ is complete, $d$ preconvex for distances less than $c$, $0 < d(x, y) < c$, then by 1.7 and by Mazurkiewicz Theorem (see e.g. [5], § 50), $x$ and $y$ can be joined by an arc, hence there is a (uniformly) homeomorphic copy of the compact interval between $x$ and $y$. In the non-complete case, Theorem 1.7 allows to join every such points by a one-to-one "almost isometric" image of $\mathsf{Q}_2$, but in general it is possible, as the following example shows, to find a non-trivial space with globally preconvex metric that contains no uniformly homeomorphic copy of $\mathsf{Q}_2$.

**1.8. Example.** Let $X$ be the set of all sequences $x = (x_n;\ n \in \mathsf{N})$ of non-negative numbers such that $\Sigma(x_n;\ n \in \mathsf{N}) < \infty$. For $x, y$ in $X$ define $d(x, y)$: if $x \neq y$, let $h$ be the first index with $x_h \neq y_h$ and put $d(x, y) = |x_h - y_h| + \Sigma(x_n;\ n > h) + \Sigma(y_n;\ n > h)$, if $x = y$, we put $d(x, y) = 0$. Let us show that $d$ is a metric on $X$. To prove the triangle inequality let $x \neq y \neq z \neq x$ be points of $X$, let $h$ be as above and let $k$ be the first index with $y_k \neq z_k$. We may suppose $h \leqq k$. If $h < k$, then

$$d(x, z) = |x_h - y_h| + \Sigma(x_n;\ n > h) + \Sigma(z_n;\ h < n < k) + z_k + \Sigma(z_n;\ n > k) \leqq |x_h - y_h| + \Sigma(x_n;\ n > h) + \Sigma(y_n;\ h < n < k) + y_k + |z_k - y_k| + \Sigma(z_n;\ n > k) \leqq d(x, y) + d(y, z)\,.$$

If $h = k$, let $l$ be the first index with $x_l \neq z_l$. If $l > h$, then

$$d(x, z) = |x_l - z_l| + \Sigma(x_n;\ n > l) + \Sigma(z_n;\ n > l) \leqq \Sigma(x_n;\ n \geqq l) + \Sigma(z_n;\ n \geqq l) \leqq d(x, y) + d(y, z)\,.$$

If $l = h$, it suffices to use the inequality $|x_h - z_h| \leqq |x_h - y_h| + |y_h - z_h|$.

Let us prove that $d$ is globally preconvex. Moreover, we prove that $d$ is convex (see below). Let $x, y \in X$, $r > 0$, $s > 0$, $d(x, y) = r + s$. Let $h$ be the first index with $x_h \neq y_h$. Then $d(x, y) = |x_h - y_h| + \Sigma(x_n;\ n > h) + \Sigma(y_n;\ n > h)$. We may suppose $x_h < y_h$. If $r > \Sigma(x_n;\ n > h)$, $s > \Sigma(y_n;\ n > h)$, put $z = (z_n)$ where $z_n = x_n$ for $n < h$, $z_h = r - \Sigma(x_n;\ n > h)$, $z_n = 0$ for $n > h$. Then clearly $d(x, z) = r$, $d(z, y) = s$. Suppose $r \leqq \Sigma(x_n;\ n > h)$. Let $k$ be the first index such that $r > \Sigma(x_n;\ n > k)$ and put $z = (z_n)$ where $z_n = x_n$ for $n < k$, $z_k = r - \Sigma(x_n;\ n > k)$, $z_n = 0$ for $n > k$. Again $d(x, z) = r$, $d(z, y) = s$. The case $s \leqq \Sigma(y_n;\ n > h)$ is completely analogous.

Let $Y$ be the subset of $X$ consisting of all sequences $(x_n;\ n \in \mathsf{N})$ where $x_n > 0$ for infinitely many $n$. It is easy to prove that $Y$ is dense in $(X, d)$. Thus, by 1.5, $d_Y$ is globally preconvex. Further, $\delta d(Y, d_Y) \geqq 1$ by 1.4. Let $\mathrm{pr}_j : (Y, d_Y) \to \mathsf{R}$ be the projection $x \mapsto x_j$ for $j \in \mathsf{N}$. Since, for any $x$, $y$, $|x_j - y_j| \leqq d(x, y)$, $\mathrm{pr}_j$ are Lipschitz with constant 1. Observe that any set $\mathrm{pr}_j^{-1}[c]$ with $j \in \mathrm{N}$, $c \geqq 0$ is both closed and open in $Y$. Hence e.g., the space $(Y, d_Y)$ is not separable. But the subspace of sequences of rational numbers is separable and has all considered properties.

Now, suppose that $S$ is a dense subset of the interval $[\![0, 1]\!]$ and $f: S \to (Y, d_Y)$ is a uniformly continuous injective mapping. We are going to prove that $f$ is not uniformly homeomorphic. In view of the density in any non-degenerate subinterval we may suppose $0 \in S$, $1 \in S$. There exists a smallest index $j$ such that $\mathrm{pr}_j \circ f$ is not constant. As $S$ is dense and $\mathrm{pr}_j \circ f$ is uniformly continuous, the set $(\mathrm{pr}_j \circ f)[S]$ contains at least three numbers. Choose a number $c$ such that the set $C = (\mathrm{pr}_j \circ f)^{-1}[c]$ is non-void and contains neither 0 nor 1. Choose $\gamma \in C$ and put $\alpha = \inf\{\xi;\ s \in S,\ \ \xi < s < \gamma \Rightarrow s \in C\}$, $\beta = \sup\{\xi;\ s \in S,\ \ \gamma < s < \xi \Rightarrow s \in C\}$. Since $C$ is closed and open, we get $0 < \alpha < \beta < 1$. Now, let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $|s - t| < \delta$ implies $d(f(s), f(t)) < \varepsilon$, we may suppose $3\delta < \beta - \alpha$. As $S$ is dense, there exist $u$, $v$ in $C$ and $u'$, $v'$ in $S \setminus C$ such that $u' \leqq \alpha < u < v < \beta \leqq v'$ and $u - u' < \delta$, $v' - v < \delta$. Let $f(u) = (x_n;\ n \in \mathrm{N})$, $f(v) = (y_n;\ n \in \mathrm{N})$. Then $d(f(u'), f(u)) < \varepsilon$ and $u' \notin C$ implies $\Sigma(x_n; n > j) < \varepsilon$. Analogously $\Sigma(y_n; n > j) < \varepsilon$. Thus $d(f(u), f(v)) < 2\varepsilon$. However, $v - u > (\beta - \alpha)/3$, hence the inverse mapping $f^{-1}$ is not uniformly continuous.

Let us mention still two non-trivial properties of the space $(X, d)$, which are not needed now: $(X, d)$ is complete, $\Delta d\,(X, d) \leqq 1$, hence $\delta d\,(Y, d_Y) = \Delta d\,(Y, d_Y) = 1$ (see [4], V.2, V.5).

We will turn our attention to the question, how to construct preconvex pseudometrics on a given pseudometric space. We are going to describe a useful method, roughly speaking defining a new distance as "the length of the shortest path".

Let $d$ be a pseudometric on a set $X$, let $J \subset \mathrm{R}$, $f: J \to X$. Put $\operatorname{var} f = \sup \Sigma(d(f(t_{i-1}), f(t_i));\ i = 1, \ldots, n)$ where the supremum is taken over all sequences $t_0 < t_1 < \ldots < t_n$, where $t_i \in J$ and $n$ is arbitrary. Further, for any $x$, $y$ in $X$, denote by $\mathbf{P}(x, y)$ (= pathes from $x$ to $y$) the collection of all mappings $f$ such that there exist $J \subset \mathrm{R}$, $a \in J$, $b \in J$ with $\bar{J} = [\![a, b]\!]$ such that $J$ is the domain of $f$, $f: J \to (X, d)$ is uniformly continuous, $f(a) = x$, $f(b) = y$. Any $J \subset \mathrm{R}$ will be considered to be endowed with the usual metric.

**1.9. Lemma.** *Let $d$ be a pseudometric on a set $X$. Let $J \subset \mathrm{R}$, $\bar{J} = [\![a, b]\!]$, $a \in J$, $b \in J$, let $f: J \to (X, d)$ be uniformly continuous, $\operatorname{var} f < \infty$. Put $v(t) = \operatorname{var} f \upharpoonright (J \cap [\![a, t]\!])$ for $t \in J$. Then $v[J]$ is dense in the interval $[\![0, \operatorname{var} f]\!]$.*

Proof. Suppose $v[J]$ is not dense in $[\![0, \operatorname{var} f]\!]$. The function $v$ is non-decreasing, hence there exist $p \in [\![a, b]\!]$ and $\alpha$, $\beta$, $0 < \alpha < \beta < \operatorname{var} f$ such that $v(t) \leqq \alpha$ for any $t \in J$, $t < p$ and $v(t) \geqq \beta$ for any $t \in J$, $t > p$. Assume $a < p < b$. Choose $\varepsilon > 0$, $\varepsilon < \frac{1}{2}(\beta - \alpha)$ and $\delta > 0$ such that $s, t \in J$, $s < t < s + \delta$ imply $d(f(s), f(t)) < \varepsilon$. Let $t_i \in J$, $a = t_0 < t_1 < \ldots < t_n = b$. Let $h$ be the greatest index with $t_h < p$ and $k$ the smallest index with $t_k > p$. In view of the density of $J$, there are $s_1, s_3 \in J$ such that $t_h < s_1 < p < s_3 < t_k$ and $s_3 - s_1 < \delta$. Put $s_2 = p = t_{h+1}$ if $k = h + 2$

and $s_2 = s_1$ if $k = h + 1$. Now we have

$$\begin{aligned}\Sigma(d(f(t_{i-1}), f(t_i)); \ i = 1, \ldots, n) &\leqq \Sigma(d(f(t_{i-1}), f(t_i)); \ 1 \leqq i \leqq h) + \\ &+ d(f(t_h), f(s_1)) + d(f(s_1), f(s_2)) + d(f(s_2), f(s_3)) + \\ &+ d(f(s_3), f(t_k)) + \Sigma(d(f(t_{i-1}), f(t_i); \ k < i \leqq n) \leqq \\ &\leqq v(s_1) + d(f(s_1), f(s_2)) + d(f(s_2), f(s_3)) + v(b) - v(s_3) < \\ &< \alpha + 2\varepsilon + \operatorname{var} f - \beta < \operatorname{var} f,\end{aligned}$$

which is a contradiction. The cases $p = a$ and $p = b$ are similar but simpler.

**1.10. Proposition.** *Let $d$ be a pseudometric on a set $X$, $c > 0$. For $x, y$ in $X$ put $\tilde{d}(x, y) = \inf\{\operatorname{var} f; f \in \mathbf{P}(x, y)\}$, $\tilde{d}_c(x, y) = \min\{\tilde{d}(x, y), c\}$. Then, for any $x, y$, $d(x, y) \leqq \tilde{d}(x, y)$, $d(x, y) = 0$ implies $\tilde{d}(x, y) = 0$ and $d(x, y) \leqq c$ implies $d(x, y) \leqq \tilde{d}_c(x, y)$. The function $\tilde{d}_c$ is a pseudometric on $X$ preconvex for distances less than $c$. If for any $x, y$ in $X$ there is $f$ in $\mathbf{P}(x, y)$ with $\operatorname{var} f < \infty$ then $\tilde{d}$ is a globally preconvex peudometric on $X$.*

Proof. Let $x, y \in X$, $r > 0$, $s > 0$, $\tilde{d}(x, y) < r + s$. Choose $f$ in $\mathbf{P}(x, y)$ with $\operatorname{var} f < r + s$. By 1.9, there are $z \in X$ and $g \in \mathbf{P}(x, z)$, $h \in \mathbf{P}(z, y)$ with $\operatorname{var} g < r$, $\operatorname{var} h < s$. Then $\tilde{d}(x, z) < r$, $\tilde{d}(z, y) < s$. The rest of the proof is easy.

**1.11. Proposition.** *Let $d, e$ be pseudometrics on a set $X$, $d(x, y) \leqq e(x, y)$ for all $x, y$, let $c > 0$. If $e$ is preconvex for distances less than $c$, then $\tilde{d}_c(x, y) \leqq e(x, y)$ and if $e$ is globally preconvex, then $\tilde{d}(x, y) \leqq e(x, y)$ for all $x, y$.*

Proof. Suppose $f$ is preconvex for distances less than $c$. Let $e(x, y) < c$. Let $k > e(x, y)$. Take the mapping $f: \mathrm{Q}_2 \to (X, e)$ from 1.7. Clearly, $f: \mathrm{Q}_2 \to (X, d)$ is Lipschitz with constant $k$, $f \in \mathbf{P}(x, y)$, $\operatorname{var} f \leqq k$. Hence $\tilde{d}(x, y) \leqq e(x, y)$ and both assertions follow.

As an easy consequence we obtain the following interesting characterization of preconvexity.

**1.12. Corollary.** *Let $d$ be a pseudometric on a set $X$, $c > 0$. Then $d$ is preconvex for distances less than $c$ if and only if $\min\{d(x, y), c\} = \tilde{d}_c(x, y)$ for all $x, y$, and $d$ is globally preconvex if and only if $d = \tilde{d}$.*

It is evident that on a linear space, every pseudometric generated by a seminorm is preconvex. The following example shows that, for metric linear spaces, there is no good relation between local convexity and preconvexity of the metric.

**1.13. Example.** Let $0 < p < 1$. The space $L^p([\![0, 1]\!])$ endowed with the standard metric $d$ defined by $d(x, y) = \int_0^1 |x(t) - y(t)|^p \, dt$ is not locally convex. Given any $x, y$, then by the continuity of the integral there exists a number $s$ such that, if $z$ is defined by $z(t) = y(t)$ for $t \leqq s$ and $z(t) = x(t)$ for $t > s$, then $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$. Therefore, by 1.2 (3), $d$ is globally preconvex. On the other hand the standard metric $e$ on $l^p$ is not preconvex. Indeed, choose $a > 0$ and put $x = (0, 0, \ldots)$, $y = (a, 0, \ldots)$. Then $2.2^{-p} a^p > a^p = e(x, y)$. Now $e(x, z) < 2^{-p}a^p$ implies $z_0 < \frac{1}{2}a$, $e(z, y) < 2^{-p}a^p$ implies $z_0 > \frac{1}{2}a$, which is impossible.

Now we will present some propositions which will illustrate how the classical results on convex sets in normed linear spaces may extend to the case of spaces endowed with preconvex pseudometrics.

**1.14. Proposition.** *Let $d$ be a pseudometric on a set $X$, preconvex for distances less than $c$. If $e$ is any pseudometric on $X$ which is uniformly continuous on $(X, d)$, then there exists a constant $k$, $0 < k < \infty$ such that $d(x, y) < c$ implies $e(x, y) < k$, specially* $\sup\{e\text{-diam } B_d(x, c);\ x \in X\} < \infty$.

Proof. There exists a natural number $m$ such that $d(u, v) < c/m$ implies $e(u, v) < 1$. By 1.2 (2) for $x, y$ with $r_i = c/m$ and the triangle inequality for $e$, we get $e(x, y) < m$.

The following extends a result of Corson and Klee [2].

**1.15. Proposition.** *Let $(X, d), (Y, e)$ be pseudometric spaces, $f: (X, d) \to (Y, e)$ uniformly continuous. Then for each $\delta > 0$ there exists a constant $k$ such that if $d$ is preconvex for distances less than $c$, then $\delta \leqq d(x, y) < c$ implies $e(f(x), f(y)) \leqq k \,.\, d(x, y)$.*

Proof. Choose $\eta > 0$ such that $d(u, v) < \eta$ implies $e(f(u), f(v)) < \frac{1}{2}$. Put $k = 1/\min\{\eta, \delta\}$. Let $\delta \leqq d(x, y) < c$. Let $m$ be the least integer such that $k \,.\, d(x, y) < m$. By 1.2, there exist $x_0 = x, x_1, \ldots, x_m = y$ with $d(x_{i-1}, x_i) < 1/k$ for $i = 1, \ldots, m$. But $1/k \leqq \eta$, hence $e(f(x), f(y)) \leqq \Sigma(e(f(x_{i-1}), f(x_i));\ 1 \leqq i \leqq m) < \frac{1}{2}m \leqq \frac{1}{2}(k \,.\, d(x, y) + 1) \leqq \frac{1}{2}k(d(x, y) + \delta) \leqq k \,.\, d(x, y)$.

**1.16. Proposition.** *Let $(X, d)$, $(Y, e)$ be pseudometric spaces, $f: (X, d) \to (Y, e)$ uniformly continuous. Let $d$ be preconvex for distances less than $c$. Then the module of continuity of $f$, i.e. the function $\psi_f$ defined for $r > 0$ by*

$$\psi_f(r) = \sup\{e(f(x), f(y));\ d(x, y) < r\}$$

*is subaditive on the interval* $]0, c[$.

Proof. Let $r, s > 0$, $r + s < c$. Given $\delta > 0$, there exist $x, y$ such that $d(x, y) < r + s$ and $\psi_f(r + s) < e(f(x), f(y)) + \delta$. Take $z$ with $d(x, z) < r$, $d(z, y) < s$. Now $\psi_f(r + s) < e(f(x), f(z)) + e(f(z), f(y)) + \delta \leqq \psi_f(r) + \psi_f(s) + \delta$. Since $\delta$ was arbitrary, $\psi_f$ is subadditive.

Observe that the subadditivity of $\psi_f$ implies that the values of $\psi_f$ are finite. The uniform continuity of $f$ means $\lim_{r \to 0} \psi_f(r) = 0$ and, $\psi_f$ being non-decreasing, this fact with the subadditivity implies the uniform continuity of $\psi_f$. As a consequence of 1.16 and a theorem of Aronszajn and Panitchpakdi [1] we obtain

**1.17. Proposition.** *Let $(X, d_X)$ be a subspace of a pseudometric space $(Y, d)$, $H$ the space $C(K)$ of continuous functions on an extremally disconnected compact space $K$ endowed with the supremum norm. If $d_X$ is globally preconvex, preconvex, then every uniformly continuous mapping of $X$ into $H$ can be extended over $Y$, over some uniform neighbourhood of $X$ in $Y$, respectively, with preservation of the module of continuity.*

In the literature, a concept of convexity of metrics has been introduced and studied (see e.g. [6], [9]). Recall (in a little more general fashion) that a pseudometric $d$ on a set $X$ is said to be convex for distances less than $c > 0$, if for any $x$, $y$ in $X$ with $d(x, y) < c$ there exists $z$ in $X$ such that $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$. If the condition holds for at least one $c$, then $d$ is said to be convex.

It is known that, if $(X, d)$ is a complete pseudometric space and $c > 0$, then the following properties are equivalent:

(1) $d$ is convex for distances less than $c$.

(2) If $x, y \in X$, $0 < d(x, y) < c$, then there exists $z \in X$ such that $d(x, z) > 0$, $d(z, y) > 0$ and $d(x, z) + d(z, y) = d(x, y)$.

(3) If $x, y \in X$, $0 < d(x, y) < c$, then there exists an isometric mapping $f: [\![0, d(x, y)]\!] \to (X, d)$ such that $f(0) = x$, $f(d(x, y)) = y$.

Clearly (see 1.2), convexity implies preconvexity. We are going to show that the reverse implication holds in case $(X, d)$ is compact and need not hold if $(X, d)$ is complete.

**1.18. Proposition.** *Let $d$ be a pseudometric on a set $X$ preconvex for distances less than $c$. If $(X, d)$ is compact, then $d$ is convex for distances less than $c$.*

Proof. Let $x, y \in X$, $d(x, y) = 2r < c$. For each $n \in \mathrm{N}$ with $2/n < c - 2r$, choose $z_n$ such that $d(x, z_n) < r + 1/n$, $d(z_n, y) < r + 1/n$. The sequence $(z_n)$ has a convergent subsequence, let $z$ be its limit. Then clearly $d(x, z) \leqq r$, $d(z, y) \leqq r$, hence $d(x, z) = d(z, y) = r$.

**1.19. Example.** Let $X$ be the subset of the plane $\mathrm{R}^2$ consisting of all points $(x, y)$ where $0 \leqq y = (1 - |x|)/m$ and $m = 1, 2, \ldots$. Let $d$ be the usual Euclidean metric on $X$, $\tilde{d}$ the metric from 1.10. Put $u = (-1, 0)$, $v = (1, 0)$. Let us prove that the metric space $(X, \tilde{d})$ is complete. Suppose $(z_n)$ is a Cauchy sequence in $(X, \tilde{d})$, both $u, v$ are not cluster points of $(z_n)$. Then there is $n_0$ such that $(z_n; n > n_0)$ ranges in the set $\{(x, y); \varepsilon \leqq y = (1 - |x|)/m\}$ with some fixed integer $m$ and $\varepsilon > 0$. If $(z_n)$ does not converge to $(0, 1/m)$ then there is $n_1$ such that $(z_n; n > n_1)$ ranges in a compact segment where $d$ and $\tilde{d}$ coincide. By 1.10, $\tilde{d}$ is globally preconvex. Clearly, $\tilde{d}(u, v) = d(u, v) = 2$. But there is no point $w$ in $X$ such that $d(u, w) \leqq 1$, $d(w, v) \leqq 1$. Thus, as $\tilde{d} \geqq d$, $d$ is not convex for distances less than 3. Put $X_n = \{(x, y) \in \mathrm{R}^2; (n(x - 2n), y) \in X\}$ for $n = 1, 2, \ldots$. Let $Y$ be the union of all $X_n$'s with all segments in $\mathrm{R}^2$ with end points $(2n + 1/n, 0)$, $(2n + 2 - 1/(n + 1), 0)$, where $n = 1, 2, \ldots$. Then the corresponding metric $\tilde{d}$ on $Y$ is again globally preconvex, $(Y, \tilde{d})$ is complete, but $\tilde{d}$ is not convex.

**1.20. Remark.** The reader might observe that the concept of preconvexity is meaningful for very general functions and the proofs of most propositions used only these properties of pseudometrics: $d(x, x) = 0 \leqq d(x, y) \leqq d(x, z) + d(z, y)$ for any $x, y, z$, which are axioms for quasi-pseudometric. E.g., the function $\sigma$ defined for any reals $s, t$ by $\sigma(s, t) = t - s$ for $s \leqq t$, $\sigma(s, t) = 1$ for $s > t$ is a quasi-metric on $\mathrm{R}$ inducing the Sorgenfrey topology. Thus the following propositions also hold for quasi-pseudometrics: 1.2 (1)–(3), 1.3, 1.6, 1.9 with the uniform continuity understood with respect to the restriction of $\sigma$ to $J$, similarly 1.10, 1.11, 1.12; further 1.14 without the conclusion on diameters, 1.15, 1.16. Also the "Metrization Lemma" (2.2 below) without the asumption of symmetry of entourages provides for the existence and the uniqueness of a quasi-pseudometric. On the other hand, 1.4 is not true in any direction for quasi-pseudometrics as shows the following example.

Let $X$ be the set of all pairs $x = (x_1, x_2)$ where $x_i \in \mathsf{R}$, $x_2 \geqq 0$. The function $d$ defined by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + \sigma(x_2, y_2)$ is a quasi-metric on $X$. Put $Z = \{(x_1, x_2) \in X;\ x_2 > 0\}$, $Y = Z \cup \{(2^{-n}, 0);\ n \in \mathsf{N}\}$. Clearly $Z$ is dense in $(X, d)$, hence $Y$ is dense in $(X, d)$, $Z$ is dense in $(Y, d_Y)$. However, $d$, $d_Z$ are preconvex for distances less than 1 and $d_Y$ is not preconvex.

## 2. PRECONVEX UNIFORMITIES

As we have seen in the preceeding section, preconvex pseudometrics have nice uniform properties. This leads us to the study of uniformities defined by a collection of preconvex pseudometrics. Recall that a collection $\mathscr{D}$ of uniformly continuous pseudometrics on a uniform space $X$ is a basis for uniformly continuous pseudometrics on $X$, shortly a basis of the uniformity of $X$, if each uniform cover of $X$ can be refined by some $\mathscr{B}_d(r)$ with $d \in \mathscr{D}$ and $r > 0$ or, equivalently, the collection of all sets $\{(x, y);\ d(x, y) < r\}$ with $d \in \mathscr{D}$ and $r > 0$ is a base of the filter of uniform entourages.

**2.1. Definition.** A uniformity, and the corresponding uniform space, will be called *preconvex*, if the uniformity has a basis consisting of preconvex pseudometrics.

Before we present the fundamental theorem characterizing preconvex uniform spaces in terms of uniform covers and entourages, we prove the following metrization lemma. If $V$ is an entourage, we define $V^n$ for $n = 1, 2, \ldots$ by $V^1 = V$ and $V^{n+1} = V \circ V^n$.

**2.2. Lemma.** *Let $(V_n;\ n \in \mathsf{N})$ be a sequence of symmetric entourages on a set $X$ such that*

(i) $$V_{n-1} \circ V_{n-1} = V_n$$

*for each $n \in \mathsf{N}$. Then there exists a pseudometric $d$ on $X$ such that it is preconvex for distances less than 1 and*

(ii) $$\{(x, y);\ d(x, y) < 2^{-n}\} \subset V_n \subset \{(x, y);\ d(x, y) \leqq 2^{-n}\}$$

*for each $n \in \mathsf{N}$. If $e$ is another pseudometric with the same properties then $e(x, y) = d(x, y)$ for $(x, y) \in V_0$.*

Proof. Clearly, $V_n \supset V_{n-1}$ for each $n$. Put

$$f(x, y) = 2^{-n} \quad \text{if} \quad (x, y) \in V_n \setminus V_{n+1}\,,$$
$$f(x, y) = 0 \quad \text{if} \quad (x, y) \in V_n \quad \text{for each} \quad n\,,$$
$$f(x, y) = 1 \quad \text{if} \quad (x, y) \in (X \times X) \setminus V_0\,.$$

Now, for any $x, y \in X$, put

$$d(x, y) = \inf \Sigma(f(x_{j-1}, x_j); j = 1, \ldots, k)$$

where the infimum is taken over all finite sequences $(x_j; j = 0, \ldots, k)$ such that $x_j \in X$, $x_0 = x$, $x_k = y$ and $k > 0$. Clearly, $d(x, y) \leqq f(x, y) \leqq 1$ for any $x, y$, which proves the second inclusion in (ii), and $d$ is a pseudometric. Now suppose $0 < b \leqq 1$, $d(x, y) < b$. Then there are $k$ and $x_0 = x, x_1, \ldots, x_k = y$ such that $\Sigma(f(x_{j-1}, x_j); j = 1, \ldots, k) < b$. If $f(x_{j-1}, x_j) = 2^{-a_j}$ then $(x_{j-1}, x_j) \in V_{a_j}$, if $f(x_{j-1}, x_j) = 0$ choose $a_j$ arbitrary but so large that $\Sigma(2^{-a_j}; j = 1, \ldots, k) < b$; thus again $(x_{j-1}, x_j) \in V_{a_j}$. Put $a = \max(a_j; j = 1, \ldots, k)$. By (i), $V_{a_j} = V_a^{m_j}$ where $m_j = 2^{a-a_j}$. Now $(x, y) \in V_{a_k} \circ \ldots \circ V_{a_1} = V_a^{m_k} \circ \ldots \circ V_a^{m_1} = V_a^m$ where $m = m_1 + \ldots \ldots + m_k < 2^a b$. In case $b = 2^{-n}$ we have $m < 2^{a-n}$, hence $(x, y) \in V_n$ and thus (ii) is completely proved. Moreover, for arbitrary $b$, $(x, y) \in V_{a+1}^{2m}$, hence there is $z$ such that $(x, z) \in V_{a+1}^m$, $(z, y) \in V_{a+1}^m$, this implies $d(x, z) \leqq m \,.\, 2^{-a-1} < 2^a b \,.\, 2^{-a-1} = b/2$ and $d(z, y) < b/2$, therefore by 1.2, $d$ is preconvex for distances less than 1.

Now, let $e$ be another pseudometric on $X$ that fulfills (ii) and is preconvex for distances less than 1. Let $(x, y) \in V_0$. If $(x', y') \in V_n$, then $e(x', y') \leqq 2^{-n}$ thus $e(x', y') \leqq f(x', y')$ and by the definition of $d$, $e(x, y) \leqq d(x, y)$. Suppose that $e(x, y) < d(x, y)$. Let $h$, $p$ be positive integers, $h < 2^p$, $e(x, y) < h \,.\, 2^{-p} < d(x, y)$. By the preconvexity of $e$ and 1.2 (2), there exist $z_0 = x, z_1, \ldots, z_h = y$ such that $e(z_{i-1}, z_i) < 2^{-p}$ for $i = 1, \ldots, h$. By (ii), $d(z_{i-1}, z_i) \leqq 2^{-p}$ hence $d(x, y) \leqq h \,.\, 2^{-p}$ which is a contradiction. Therefore $e(x, y) = d(x, y)$.

The reader is familiar with the classical metrization lemma where the condition (i) is replaced by

$$V_{n+1} \circ V_{n+1} \circ V_{n+1} \subset V_n \,.$$

As shown in [3], $V_{n+1} \circ V_{n+1} \subset V_n$ is not sufficient, therefore the equality in (i) is essential.

**2.3. Theorem.** *Let $X$ be a uniform space. Then the following properties are equivalent*:

(1) *$X$ is preconvex.*

(2) *There is a basis $\mathfrak{B}$ for uniform covers of $X$ such that for each $\mathscr{G}$ in $\mathfrak{B}$ there is $\mathscr{H}$ in $\mathfrak{B}$ such that* St $\mathscr{H} = \mathscr{G}$.

(2′) *There is a basis $\mathfrak{B}$ for uniform covers of $X$ such that for each $\mathscr{G}$ in $\mathfrak{B}$ there is $\mathscr{H}$ in $\mathfrak{B}$ such that* St $\mathscr{H}$ *refines $\mathscr{G}$ and $\mathscr{G}$ refines* St $\mathscr{H}$.

(3) *There is a basis $\mathscr{V}$ for uniform entourages on $X$ such that for each $V$ in $\mathscr{V}$ there exists $W$ in $V$ with $W \circ W = V$ and each $V$ is symmetric.*

Proof. Suppose (1), let $\mathscr{D}$ be the collection of preconvex pseudometrics that is a basis of the uniformity. Then the collection of all $\mathscr{B}_d(r)$ where $d \in \mathscr{D}$, $r > 0$ and $d$ is preconvex for distances less than $r$ is a basis for uniform covers. By 1.2 (4), St $\mathscr{B}_d(r/2) = \mathscr{B}_d(r)$, hence (2) holds. (2) $\Rightarrow$ (2′) is trivial.

Suppose (2′). If $\mathscr{G} \in \mathfrak{B}$ put $V_{\mathscr{G}} = \bigcup\{G \times G; G \in \mathscr{G}\}$. Clearly, $V_{\mathscr{G}}$ is a symmetric uniform entourage and the collection of all $V_{\mathscr{G}}$ is a basis for uniform entourages.

Given $\mathscr{G}$, let $\mathscr{H}$ be the cover described in (2'). Let us prove that $V_{\mathscr{H}} \circ V_{\mathscr{H}} \approx V_{\mathscr{G}}$. Suppose $(x, z) \in V_{\mathscr{H}}$, $(z, y) \in V_{\mathscr{H}}$. There are $H, K$ in $\mathscr{H}$ such that $x \in H$, $z \in H \cap K$, $y \in K$. Thus $x, y$ belong to $\mathrm{St}(z, \mathscr{H})$, this is a subset of some $G$ in $\mathscr{G}$, hence $(x, y) \in V_{\mathscr{G}}$. If $(x, y) \in V_{\mathscr{G}}$, then $x \in G$, $y \in G$ for some $G$ in $\mathscr{G}$. But $G \subset \mathrm{St}(z, \mathscr{H})$ for some $z$. This implies $(x, z) \in V_{\mathscr{H}}$, $(z, y) \in V_{\mathscr{H}}$. Thus (3) follows.

Suppose (3). Given any uniform entourage $U$, choose $V_0$ in $\mathscr{V}$ such that $V_0 \subset U$. There is $V_1 \in \mathscr{V}$ such that $V_1 \circ V_1 = V_0$. Proceeding by induction we get a sequence $(V_n;\ n \in \mathsf{N})$ in $\mathscr{V}$ such that $V_{n+1} \circ V_{n+1} = V_n$. Let $d$ be the pseudometric from Lemma 2.2. Then $d$ is uniformly continuous, preconvex for distances less than 1 and $\{(x, y);\ d(x, y) < 1\} \subset U$, which proves (1).

The main task of this section is to find some general sufficient conditions for a uniform space to be preconvex.

**2.4. Proposition.** *If $X$ is a uniform space and $\Delta d\, X = 0$ then $X$ is preconvex.*

Proof. Given a uniform cover of $X$, refine it by a uniform partition $\mathscr{P}$. For $x, y \in X$ put $d(x, y) = 0$ if there is $P$ in $\mathscr{P}$ such that $x \in P$, $y \in P$ and $d(x, y) = 1$ otherwise. Then $d$ is a uniformly continuous pseudometric preconvex for distances less than 1, $\mathscr{B}_d(1)$ refines $\mathscr{P}$.

**2.5. Proposition.** *The product and the sum of arbitrary family of preconvex uniform spaces are preconvex, too.*

Proof is straightforward and may be left to the reader.

**2.6. Example.** Let $X$ be the subset of the Euclidean plane $\mathsf{R}^2$ consisting of all points $(n, k \cdot 2^{-n})$ where $n \in \mathsf{N}$ and $k = 0, 1, \ldots, 2^n$. We take on $X$ the uniformity defined by the usual metric $d$. Let $e$ be an arbitrary uniformly continuous pseudometric on $X$ such that $e(x, y) < 1$ implies $d(x, y) < 1$. Then for any $x \in X$ the set $B_e(x, 1)$ is finite, hence $e$ is quasidiscrete. For any $\varepsilon$ with $0 < \varepsilon < 1$, since $e$ is uniformly continuous, there is $\delta > 0$ such that $d(x, y) < \delta$ implies $e(x, y) < \varepsilon$. If $2^{-n} < \delta$ then the numbers $e((n, 0), (n, 2^{-n})), \ldots, e((n, 1 - 2^{-n}), (n, 1))$ are smaller than $\varepsilon$ but not all are equal to zero, as $e((n, 0), (n, 1)) \geqq 1$. Thus $e$ is not uniformly quasidiscrete and, by 1.3, $e$ is not preconvex. Hence $X$ is not preconvex.

If $x, y \in X$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, put $d_1(x, y) = |x_1 - y_1|$, $d_2(x, y) = |x_2 - y_2|$. Clearly, $d_1, d_2$ are uniformly continuous pseudometrics preconvex for distances less than 1. Since $d(x, y) \leqq d_1(x, y) + d_2(x, y)$, the pseudometric $d_1 + d_2$ is not (as just proved) preconvex. Moreover we see: (a) A collection, e.g. $\{d_1, d_2\}$, of preconvex pseudometrics is a subbasis of a non-preconvex uniformity, (b) the collection of all preconvex uniformly continuous pseudometrics need not be a basis of some uniformity.

**2.7. Example.** Let $Y$ be the subset of the Euclidean plane $\mathsf{R}^2$ consisting of all points $(2^{-m}, i \cdot 2^{-m})$ where $m \in \mathsf{N}$, $i = 0, 1, \ldots, 2^m$. Let $Y$ be endowed again with the uniformity inherited from $\mathsf{R}^2$. Since $Y$ is discrete, it contains neither a continuous one-to-one image of $\mathsf{Q}_2$ (see 1.7). The Euclidean metric on $Y$ is not preconvex.

However, put for $k \in \mathsf{N}$ and any $m, n, i, j$

$$d_k((2^{-m}, i \,.\, 2^{-m}), (2^{-n}, j \,.\, 2^{-n})) = |i \,.\, 2^{-m} - j \,.\, 2^{-n}| \quad \text{for} \quad m, n > k\,,$$
$$= 1 \quad \text{for} \quad m \leqq k \quad \text{or} \quad n \leqq k\,.$$

then each $d_k$ is a uniformly continuous pseudometric on $Y$ preconvex for distances less than 1. Further, the collection $\{d_k;\ k \in \mathsf{N}\}$ is a basis of the uniformity of $Y$, thus $Y$ is preconvex. This shows that 1.3 and 1.6 do not hold either for preconvex metrizable uniform spaces.

In the sequel, we are going to show that the class of preconvex uniform spaces is surprisingly large. For our purpose, we need a slightly strenghtened concept of essentiality (see [4], chapter VIII).

**2.8. Definition.** Let $\mathscr{G} = (G_\alpha;\ \alpha \in A)$ be a uniform cover of a uniform space $X$. We say that $\mathscr{G}$ is *strongly essential* if for any uniform cover $(H_\alpha;\ \alpha \in A)$ of $X$ with $H_\alpha \subset G_\alpha$ for all $\alpha \in A$ and any $B \subset A$ we have $\bigcap(H_\alpha\ \alpha \in B) \neq \emptyset$ whenever $\bigcap(G_\alpha;\ \alpha \in B) \neq \emptyset$. A uniform space $X$ will be called *strongly essential* if the uniformity of $X$ has a basis consisting of strongly essential uniform covers.

The strong essentiality implies essentiality as shows the following simple proposition (cf. [4], VIII.1, IV.14).

**2.9. Proposition.** *A uniform cover $\mathscr{G} = (G_\alpha;\ \alpha \in A)$ of a uniform space $X$ is essential if and only if for any uniform cover $(H_\alpha;\ \alpha \in A)$ of $X$ with $H_\alpha \subset G_\alpha$ for each $\alpha \in A$ and any finite $B \subset A$ we have $\bigcap(H_\alpha;\ \alpha \in B) \neq \emptyset$ whenever $\bigcap(G_\alpha;\ \alpha \in B) \neq \emptyset$. If the cover $\mathscr{G}$ is essential, $\gamma \in A$, $G_\gamma \neq \emptyset$ then $(G_\alpha;\ \alpha \in A \setminus \{\gamma\})$ is not a uniform cover of $X$.*

By a general abstract complex over a set $A$ we will understand every set $K$ of subsets of $A$ such that if $B \in K$, $C \in K$, $B \cap C \neq \emptyset$ then $B \cap C \in K$. Further, we denote by $\mathbf{M}(A)$ the set of all families $(u_\alpha;\ \alpha \in A)$ where $u_\alpha \in \mathsf{R}$, $0 \leqq u_\alpha \leqq 1$ for each $\alpha$ and $u_\alpha = 1$ for at least one $\alpha$. Consider $\mathbf{M}(A)$ as the subspace of $l^\infty(A)$, i.e. endowed with the metric $d$ defined by $d(u, v) = \sup\,(|u_\alpha - v_\alpha|;\ \alpha \in A)$. For $u \in \mathbf{M}(A)$, put $\operatorname{supp} u = \{\alpha \in A;\ u_\alpha > 0\}$. If $K$ is a general abstract complex over $A$, let $\mathbf{M}_K(A)$ be the subspace $\{u \in \mathbf{M}(A);\ \operatorname{supp} u \in K\}$ (thus $\mathbf{M}_K(A)$ is a certain "geometric realization" of $K$).

**2.10. Lemma.** *Let $K$ be a general abstract complex over a set $A$. Let $\tilde{d}_1$ be the metric on $\mathbf{M}_K(A)$ defined in* 1.10. *Then $\tilde{d}_1(u, v) \leqq 2d(u, v)$ for any $u, v \in \mathbf{M}_K(A)$.*

Proof. Let $d(u, v) < 1$. Put $B = \operatorname{supp} u \cap \operatorname{supp} v$. Then $B \neq \emptyset$, $B \in K$ and if $u_\alpha = 1$ or $v_\alpha = 1$ then $\alpha \in B$. Let us define $f\colon [\![-1, 1]\!] \to \mathbf{M}_K(A)$. Put $w_\alpha = \max\{u_\alpha, v_\alpha\}$ if $\alpha \in B$ and $w_\alpha = 0$ if $\alpha \in A \setminus B$. For $\alpha \in A$, put $f_\alpha(t) = tv_\alpha + (1 - t)\,w_\alpha$ for $0 \leqq t \leqq 1$, $f_\alpha(t) = -tu_\alpha + (1 + t)\,w_\alpha$ for $-1 \leqq t \leqq 0$. Finally, put $f(t) = (f_\alpha(t);\ \alpha \in A)$. A simple calculation shows that $f(t) \in \mathbf{M}_K(A)$ for any $t$, $f(-1) = u$, $f(0) = w$, $f(1) = v$ and $d(f(s), f(t)) \leqq d(u, v)\,.\,|s - t|$ for any $s, t$. Thus $\operatorname{var} f \leqq 2d(u, v)$, hence $\tilde{d}_1(u, v) \leqq \tilde{d}(u, v) \leqq 2d(u, v)$.

**2.11. Theorem.** *Every strongly essential uniform space is preconvex.*

Proof. Given a uniform cover of a strongly essential uniform space $X$, refine it by a strongly essential uniform cover $\mathscr{G} = (G_\alpha;\ \alpha \in A)$. We are going to search for a uniformly continuous pseudometric $e$ on $X$ such that it will be preconvex for distances less than 1 and $\mathscr{B}_e(1)$ will refine $\mathscr{G}$. We may suppose $X \neq \emptyset$. Let $c$ be an arbitrary uniformly continuous pseudometric on $X$ such that $\mathscr{B}_c(1)$ refines $\mathscr{G}$. For $\alpha \in A$, $x \in X$, put $f_\alpha(x) = \min\{1, c\text{-dist}(x, X \setminus G_\alpha)\}$, $f(x) = (f_\alpha(x);\ \alpha \in A)$. Let $K = \{C \subset A;\ \bigcap(G_\alpha;\ \alpha \in C) \neq \emptyset\}$. It is easy to see that $f(x) \in \mathbf{M}_K(A)$ and $d(f(x), f(y)) \leqq c(x, y)$ for any $x, y$, hence $f: X \to \mathbf{M}_K(A)$ is uniformly continuous.

Let us prove that $f[X]$ is dense in $\mathbf{M}_K(A)$. Assuming the contrary let $a' \in \mathbf{M}_K(A)$ and $r > 0$ such that $B_d(a', 5r) \subset \mathbf{M}_K(A) \setminus f[X]$. Clearly $r \leqq 1/5$. Then there exist $a \in \mathbf{M}_K(A)$ and $\gamma \in A$ such that $d(a, a') \leqq 2r$, $a_\gamma = 1$ and $r \leqq a_\alpha \leqq 1 - 2r$ for $\alpha \in \operatorname{supp} a$, $\alpha \neq \gamma$. Put $H = \operatorname{supp} a$, $H' = H \setminus \{\gamma\}$. Put $P = \{u \in \mathbf{M}_K(A);\ d(u, a) > 2r\}$, clearly $f[X] \subset P$. Define a mapping $p: P \to \mathbf{M}_K(A)$ by $p(u) = (p_\alpha(u);\ \alpha \in A)$ and $p_\alpha(u)$ for $u \in \mathbf{M}_K(A)$ by the following formulas:

if $\alpha \in H'$ put

$$\begin{aligned} p_\alpha(u) &= 0 && \text{if } u_\alpha \leqq a_\alpha - r\,, \\ &= (u_\alpha - a_\alpha + r)/r && \text{if } a_\alpha - r \leqq u_\alpha \leqq a_\alpha\,, \\ &= 1 && \text{if } a_\alpha \leqq u_\alpha\,; \end{aligned}$$

if $\alpha \in A \setminus H$ put

$$\begin{aligned} p_\alpha(u) &= u_\alpha/r && \text{if } 0 \leqq u_\alpha \leqq r\,, \\ &= 1 && \text{if } r \leqq u_\alpha\,; \end{aligned}$$

and finally

$$\begin{aligned} p_\gamma(u) = \max\{0, \min\{&\inf\{(a_\alpha + r - u_\alpha)/r;\ \alpha \in H'\}\,, \\ &\inf\{(2r - u_\alpha)/r;\ \alpha \in A \setminus H\}, u_\gamma\}\}\,. \end{aligned}$$

Obviously $\operatorname{supp} p(u) \subset \operatorname{supp} u$ for any $u$. If $p_\alpha(u) < 1$ for each $\alpha \neq \gamma$, then $u_\alpha < a_\alpha$ for $\alpha \in H'$ and $u_\alpha < r$ for $\alpha \in A \setminus H$, hence $u_\gamma = 1$ and $p_\gamma(u) = 1$. Therefore $\operatorname{supp} p(u) \neq \emptyset$, $p(u) \in \mathbf{M}_K(A)$. A simple calculation shows that, for any $u, v$ in $P$, $|p_\alpha(u) - p_\alpha(v)| \leqq |u_\alpha - v_\alpha|/r$ if $\alpha \in A$, $\alpha \neq \gamma$ and $|p_\gamma(u) - p_\gamma(v)| \leqq d(u, v)/r$, hence $p$ is Lipschitz with constant $1/r$. Let us show that, for any $u \in P$ there exists $\alpha \in H$ with $p_\alpha(u) = 0$. If $u \in \mathbf{M}_K(A)$ and $|u_\alpha - a_\alpha| < 2r$ for each $\alpha \in A \setminus \{\gamma\}$ then $u_\alpha < 1$ for each $\alpha \in A \setminus \{\gamma\}$, thus $u_\gamma = 1 = a_\gamma$ and $d(u, a) \leqq 2r$. Let $u \in P$. Then $d(u, a) > 2r$ and hence $|u_\alpha - a_\alpha| \geqq 2r$ for some $\alpha \in A \setminus \{\gamma\}$. Suppose $p_\alpha(u) > 0$ for each $\alpha \in H'$. Then $u_\alpha > a_\alpha - r$ for $\alpha \in H'$, hence there exists either $\alpha \in H'$ with $u_\alpha > a_\alpha + r$ or $\alpha \in A \setminus H$ with $u_\alpha \geqq 2r$. But this implies $p_\gamma(u) = 0$. Now let $T = \{u \in \mathbf{M}_K(A);\ H \not\subset \operatorname{supp} u\}$. Clearly $p[P] \subset T$. For $\alpha \in A$ put $S_\alpha = \{u \in \mathbf{M}_K(A);\ \alpha \in \operatorname{supp} u\}$. Given $v \in \mathbf{M}_K(A)$, there is $\alpha \in A$ with $v_\alpha = 1$ and then $B_d(v, 1) \subset S_\alpha$, hence $(S_\alpha;\ \alpha \in A)$ is a uniform cover of $\mathbf{M}_K(A)$. Further, $(S_\alpha \cap T;\ \alpha \in A)$ is a uniform cover of $T$, $p^{-1}[S_\alpha] \subset S_\alpha$, $f^{-1}[S_\alpha] \subset G_\alpha$. Thus $((p \circ f)^{-1}[S_\alpha];\ \alpha \in A)$ is a uniform cover of $X$,

$(p \circ f)^{-1} [S_\alpha] \subset G_\alpha$. But $\bigcap((p \circ f)^{-1} [S_\alpha]; \alpha \in H) = \emptyset$ and $\bigcap(G_\alpha; \alpha \in H) \neq \emptyset$. It contradicts the strong essentiality of $X$ and proves that $f[X]$ is dense in $\boldsymbol{M}_K(A)$.

The metric $\tilde{d}_1$ is by 2.10 uniformly continuous and by 1.10 preconvex for distances less than 1. The restriction $d'$ of $\tilde{d}_1$ to $f[X]$ is by 1.5 preconvex for distances less than 1 too. Since $d(v, w) \leqq d'(v, w)$ for any $v, w$ in $f[X]$ (see 1.10) we have for each $u \in f[X]$, $B_{d'}(u, 1) \subset B_d(u, 1) \subset S_\alpha$ for some $\alpha \in A$. Finally put, for any $x, y$ in $X$, $e(x, y) = d'(f(x), f(y))$. Then clearly $e$ is a uniformly continuous pseudometric on $X$, $e$ is preconvex for distances less than 1 and given $x \in X$, $B_e(x, 1) = f^{-1}[B_{d'}(f(x), 1)] \subset \subset f^{-1}[S_\alpha] \subset G_\alpha$ for some $\alpha \in A$.

Observe that Proposition 2.4 is a simple corollary of Theorem 2.11. Isbell ([4], Theorem VIII.4) proved that every locally fine uniform space admits a basis consisting of point-finite essential covers, so it is strongly essential (Notice that locally fine spaces have an equivalent description as subspaces of (topologically) fine spaces [7]). Thus as an immediate consequence of Theorem 2.11, we have

**2.12. Corollary.** *Every locally fine (therefore every fine or every totally bounded) uniform space is preconvex.*

Immediately from 2.12 and 2.3 (2) we get

**2.13. Corollary.** *In a paracompact topological space there is a basis $\mathfrak{B}$ of open covers such that for each $\mathscr{G}$ in $\mathfrak{B}$ there is $\mathscr{H}$ in $\mathfrak{B}$ with* St $\mathscr{H} = \mathscr{G}$.

We are able to extend 2.12 directly even to the class of so-called sub-metric-fine spaces. Recall that a uniform space $X$ is sub-metric-fine if every uniformly continuous mapping $f: X \to Y$ where $Y$ is a complete metric space remains uniformly continuous into the fine uniformity of $Y$ (for details and other descriptions see e.g. [8]). Note that every locally fine space is sub-metric-fine.

**2.14. Proposition.** *Every sub-metric-fine uniform space is preconvex.*

Proof. Take a uniform cover $\mathscr{G}$ of a sub-metric fine space $X$. Then there is (see e.g. [4], I.14) a uniformly continuous mapping $f$ of $X$ onto a dense subset of a complete metric space $Y$ and a uniform cover $\mathscr{H}$ of $Y$ such that $\{f^{-1}[H]; H \in \mathscr{H}\}$ refines $\mathscr{G}$. Since $X$ is sub-metric-fine, $f$ remains uniformly continuous into the fine uniformity of $Y$. By 2.12, we can find a preconvex continuous pseudometric $d$ on $Y$ such that $\mathscr{B}_d(1)$ refines $H$. The restriction of $d$ to $f[X]$ is by 1.5 preconvex too. The pseudometric $e$ on $X$ defined by $e(x, y) = d(f(x), f(y))$ is uniformly continuous, preconvex and $\mathscr{B}_e(1)$ refines $\mathscr{G}$.

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## EQUADIFF 7

Czechoslovak Conference on Differential Equations and Their Applications EQUADIFF 7 will be held in Prague, 21—25 August 1989. Chairman of the Organizing Committee is Jaroslav Kurzweil, Secretary is Jiří Jarník. Mailing address:

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