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ALGEBRAIC SPECTRAL SUBSPACES

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One of the basic problems in the theory of automatic continuity is the following. Suppose X and Y are two Banach spaces and $\vartheta: X \to Y$ a linear (not necessarily continuous) mapping which intertwines two continuous linear mappings T on X and S on Y

$$\vartheta T = S \vartheta .$$

The problem is to find conditions on S and T which imply the continuity of every such ϑ. It turned out that the following condition is necessary: If Z is a subspace of Y such that

$$(\lambda - S)\, Z = Z$$

for all complex λ then $Z = (0)$.

Spaces with this property are called *divisible subspaces* in [4]. There is a related concept which turns out to be useful in automatic continuity problems. To describe it, consider first a normal operator T on a Hilbert space H. If F is a closed subset of the complex plane consider the corresponding spectral projection $E(F)$; since $\sigma(T \mid E(F)) \subset F$ we have

$$(\lambda - T)\, E(F) = E(F)$$

for all $\lambda \notin F$. The same equality holds if we identify $E(F)$ with its range; it is possible to show that this relation actually characterizes the spaces $E(F)$. It is obvious that such subspaces may be considered in a more general situation. Clearly it may be expected that these subspaces will turn out to be useful in the study of decomposable operators and their relations to automatic continuity. This is indeed the case: the fact that the spaces E are defined in purely algebraic terms makes it possible to obtain the system of inclusions $\vartheta E_T(F) \subset E_S(F)$ to which the automatic continuity methods are then applied. A systematic study of such subspaces can be found in [2]. In the present note we pursue further the investigation of this concept and of its relation to a class of operators for which there exists a spectral decomposition into subspaces of this type so that the decompositions have a purely algebraic description.

The starting point of the work presented in this note was a series of highly interesting lectures given by K. B. Laursen at the 17th Functional Analysis Seminar held in Jilemnice in May 1986. In the discussions which followed these lectures the present authors succeeded in strengthening some of the known results, clarifying the connections between some of the concepts and simplifying the proofs. In particular, there is a series of new results concerning duality.

1. PRELIMINARIES

Let X be a linear space, E a subspace of X. Denote by X^* the algebraic dual of X, A^0 will denote the annihilator of $A \subset X$. If T is a linear operator on X then the following pairs of assertions are equivalent:

1° *$TE \subset E$ iff $T^*E^0 \subset E^0$*

2° *$TE \supset E$ iff $T^{*-1}E^0 \subset E^0$*

3° *$T^{-1}E \subset E$ iff $E^0 \subset T^*E^0$*

Proof. The proof of 1° is straightforward. To prove 2°, assume first that $TE \supset E$ and suppose that $T^*f \in E^0$ for some f. Then $\langle E, f\rangle \subset \langle TE, f\rangle = \langle E, T^*f\rangle = 0$. On the other hand, assume the inclusion $T^{*-1}E^0 \subset E^0$ and consider an $e \in E$. Suppose that $e \notin TE$; it follows that there exists an $f \in X^*$ such that $\langle TE, f\rangle = 0$ and $\langle e, f\rangle \neq 0$. Thus $T^*f \in E^0$ and $f \notin E^0$, a contradiction. Hence $E \subset TE$.

The proof of 3° is analogous. Assume $T^{-1}E \subset E$ and consider an $f \in E^0$. Define a linear form g on the subspace $TX + E$ by the formula

$$g(Tx + e) = f(x)\,.$$

To see that this definition is legitimate it suffices to show that $Tx + e = 0$ implies $f(x) = 0$. If $Tx + e = 0$ we have $Tx \in E$, $x \in T^{-1}E$ so that $x \in E$. Since $f \in E^0$ it follows that $f(x) = 0$. If $\hat{g}$ is an arbitrary extension of g to the whole of X, we have $\hat{g} \in E^0$ and $f = T^*\hat{g}$.

On the other hand, assume $E^0 \subset T^*E^0$ and consider an $x \in T^{-1}E$. Suppose that $x \notin E$; it follows that there exists an f such that $\langle E, f\rangle = 0$ and $\langle x, f\rangle \neq 0$. Thus $f \in E^0 \subset T^*E^0$ so that $f = T^*g$ for some $g \in E^0$. Since $Tx \in E$ we have $\langle x, f\rangle = \langle x, T^*g\rangle = \langle Tx, g\rangle = 0$, a contradiction.

Suppose X_1, X_2 are two subspaces of X; then

$$(X_1 \cap X_2)^0 = X_1^0 + X_2^0\,.$$

Proof. The inclusion $X_1^0 + X_2^0 \subset (X_1 \cap X_2)^0$ is obvious. If $f \in (X_1 \cap X_2)^0$ define an $f_1 \in X^*$ by the requirement that its restriction to $X_1 + X_2$ satisfies

$$f_1(x_1 + x_2) = f(x_2)$$

for $x_1 \in X_1$, $x_2 \in X_2$. To see that this definition is legitimate suppose that $x_1 + x_2$ is also represented in the form $x_1' + x_2'$ with $x_1' \in X_1$, $x_2' \in X_2$. Then

$$x_2' - x_2 = x_1 - x_1' \in X_1 \cap X_2$$

so that $f(x_2' - x_2) = 0$. Clearly $f_1 \in X_1^0$. If f_2 stands for $f - f_1$ we have, for $x_2 \in X_2$,

$$f_2(x_2) = f(x_2) - f_1(x_2) = 0\,.$$

Thus $f_2 \in X_2^0$ and $f = f_1 + f_2$.

2. ALGEBRAIC SPECTRAL SUBSPACES

This section presents the definition and the most important properties of the basic concept of our investigation. Some of the propositions which follow are already contained in the paper [2]; they are included here since our approach yields considerable simplifications of the proofs.

We begin by restating the definition of the basic notion as it appears in [2].

Let E be a vector space, $T: E \to E$ be a linear mapping and F a subset of the complex plane $\mathbb{C}$.

(2.1) **Definition.** Consider the class of all linear subspaces Y in E which satisfy $(T - \lambda) Y = Y$ for all $\lambda \in \mathbb{C} \setminus F$ and set

$$E_T(F) := \operatorname{span} Y .$$

It is obvious that $(T - \lambda) E_T(F) = E_T(F)$ for $\lambda \in \mathbb{C} \setminus F$ as well so that the class which we consider has a maximal element (if ordered by inclusion).

Our starting point will be the following simple observation which will turn out to be advantageous in our further considerations.

(2.2) **Proposition.** *The space $E_T(F)$ is the union of all sets $M \subset X$ such that*

$$(T - \lambda) M \supset M$$

for all $\lambda \in \mathbb{C} \setminus F$.

Proof. Denote by Z the above union. Clearly Z is a linear subspace invariant with respect to T and $Z \subset (T - \lambda) Z$ for all $\lambda \in \mathbb{C} \setminus F$. Thus $(T - \lambda) Z \subset Z \subset (T - \lambda) Z$ so that $Z = (T - \lambda) Z$ for all $\lambda \in \mathbb{C} \setminus F$ whence $Z \subset E_T(F)$. The inclusion $E_T(F) \subset Z$ is immediate.

The following three propositions are known [2]; we are including them since the idea used in our approach provides a different and simpler proof.

(2.3) **Proposition.** *If $\lambda_0 \in F$ then*

$$(\lambda_0 - T)^{-1} E_T(F) \subset E_T(F) .$$

Proof. Write M for $(\lambda_0 - T)^{-1} E_T(F)$. To prove our assertion it will be sufficient to show that

$$M \subset (\lambda - T) M$$

for all $\lambda \in \mathbb{C} \setminus F$. To see that, take an arbitrary $x \in M$ and a $\lambda \in \mathbb{C} \setminus F$. Since $(\lambda_0 - T) x \in E_T(F)$ it follows that $(\lambda_0 - T) x = (\lambda - T) e$ for a suitable $e \in E_T(F)$. Thus $(\lambda - T) e = (\lambda_0 - T) x = (\lambda_0 - \lambda) x + (\lambda - T) x$ whence

$$x = (\lambda - T) \frac{x - e}{\lambda - \lambda_0}$$

$$\text{and} \quad (\lambda_0 - T) \frac{x - e}{\lambda - \lambda_0} \in E_T(F) , \quad \text{so that} \quad \frac{x - e}{\lambda - \lambda_0} \in M .$$

Remark. In the preceding lemma, we can replace the inclusion $(\lambda_0 - T)^{-1} E_T(F) \subset E_T(F)$ by equality. Indeed, since the E_T are invariant with respect to T, we have $E_T(F) \subset (\xi - T)^{-1} E_T(F)$ for arbitrary ξ.

(2.4) **Proposition.** $E_T(F) = \bigcap_{\lambda \notin F} E_T(\mathbb{C} \setminus \{\lambda\})$.

Proof. Write P for the intersection on the right hand side. To prove the Proposition it will be sufficient to show that $P \subset (\lambda - T) P$ for all $\lambda \notin F$. To see that, consider a $p \in P$ and a $\mu \notin F$. Since $p \in P \subset E_T(\mathbb{C} \setminus \{\mu\})$ we have $p = (\mu - T) p'$ for a suitable $p' \in E_T(\mathbb{C} \setminus \{\mu\})$. Now take an arbitrary $\lambda \notin F$ and $\lambda \neq \mu$. Since $\mu \in \mathbb{C} \setminus \{\lambda\}$ and $(\mu - T) p' = p \in E_T(\mathbb{C} \setminus \{\lambda\})$ we have $p' \in E_T(\mathbb{C} \setminus \{\lambda\})$ by the preceding proposition. Thus $p' \in P$ and the proof is complete.

(2.5) **Proposition.** $E_T(\bigcap_\alpha F_\alpha) = \bigcap_\alpha E_T(F_\alpha)$.

Proof. Using the preceding proposition, we have

$$\bigcap_\alpha E_T(F_\alpha) = \bigcap_\alpha \bigcap_{\lambda \notin F_\alpha} E_T(\mathbb{C} \setminus \{\lambda\}) = \bigcap_{\lambda \in \bigcup_\alpha (\mathbb{C} \setminus F_\alpha)} E_T(\mathbb{C} \setminus \{\lambda\}) =$$

$$= \bigcap_{\lambda \in \mathbb{C} \setminus \bigcap F_\alpha} E_T(\mathbb{C} \setminus \{\lambda\}) = E_T(\bigcap_\alpha F_\alpha) .$$

The following lemma could be obtained as a direct consequence of (2.3) but we prefer to give a direct proof.

(2.6) **Proposition.** $\operatorname{Ker}(\lambda - T) \subset E(\{\lambda\})$.

Proof. If $x \in \operatorname{Ker}(\lambda - T)$ and $\mu \neq \lambda$ we have

$$x = (\mu - T) \frac{x}{\mu - \lambda} .$$

Thus

$$\operatorname{Ker}(\lambda - T) \subset (\mu - T) \operatorname{Ker}(\lambda - T) \quad \text{for all} \quad \mu \in \mathbb{C} \setminus \{\lambda\}$$

whence $\operatorname{Ker}(\lambda - T) \subset E(\{\lambda\})$.

It seems that, in the context of the present investigation, another notion of spectrum is more appropriate, a spectrum smaller in general than the spectrum in the usual sense of the word: a number λ is declared to be an element of this smaller spectrum if the equation $(\lambda - T) x = y$ does not possess solutions for all y.

(2.7) **Definition.** Denote by $\sigma_S(T)$ the set of all complex numbers λ for which $\lambda - T$ is not surjective.

Similarly, $\sigma(T)$ will be the set of those λ for which $\lambda - T$ is not bijective.

Clearly, $\sigma_S(T) \subset \sigma(T)$.

(2.8) **Proposition.** *Let F be a subset of the complex plane. Then*

1° $\sigma_S(T \mid E_T(F)) \subset F$. *Moreover, if M is invariant with respect to T and $\sigma_S(T \mid M) \subset H$ then*

$$M \subset E_T(H)$$

2° $E_T(F) = E_T(\sigma_S(T \mid E_T(F)))$

3° *if* $E_T(\emptyset) = (0)$ *then* $\sigma(T \mid E_T(F)) \subset F$.

Proof. Assertion 1° is a direct consequence of the definition.

To prove 2°, write H for $\sigma_S(T \mid E_T(F))$ so that $H \subset F$. Accordingly, $E_T(H) \subset \subset E_T(F)$. On the other hand, setting $M = E_T(F)$, M is invariant with respect to T and we have $\sigma_S(T \mid M) \subset H$ so that $M \subset E_T(H)$. Hence $E_T(F) \subset E_T(H)$.

Further, suppose that $E_T(\emptyset) = (0)$ and consider an $x \in E_T(F)$ such that $(\lambda - T)\,x = = 0$ for some $\lambda \notin F$. Since $\operatorname{Ker}(\lambda - T) \subset E_T(\{\lambda\})$ we have $x \in E_T(F) \cap E_T(\{\lambda\}) = = E_T(F \cap \{\lambda\}) = E_T(\emptyset) = (0)$ so that $\lambda - T \mid E_T(F)$ is injective. The surjectivity is obvious.

(2.9) **Proposition.**

$$E_T(F) = E_T(F \cap \sigma_S(T)) = E_T(F \cap \sigma(T)) = $$
$$= E_T(\sigma_S(T \mid E_T(F)) = E_T(F \cap \sigma(T \mid E_T(F)) .$$

Proof. Since $E_T(\sigma_S(T)) = X$ we have

$$E_T(F) \subset E_T(F) \cap E_T(\sigma_S(T)) = E_T(F \cap \sigma_S(T)) \subset E_T(F) .$$

Since $\sigma_S(T \mid E_T(F)) \subset F$ and $E_T(F) = E_T(\sigma_S(T \mid E_T(F)))$ we have

$$E_T(F) = E_T(\sigma_S(T \mid E_T(F)) = E_T(F \cap \sigma_S(T \mid E_T(F))) \subset$$
$$\subset E_T(F \cap \sigma(T \mid E_T(F))) \subset E_T(F) .$$

(2.10) **Definition.** The operator T is said to be *algebraically decomposable* if, for an arbitrary covering of the complex plane $\mathbb{C}$ by two open sets G_1, G_2 there exist linear manifolds X_1, X_2 in X invariant with respect to T such that

1° $X = X_1 + X_2$

2° $\sigma(T \mid X_i) \subset G_i$ for $i = 1, 2$.

If T is algebraically decomposable it follows from Proposition (2.8) that $X_i \subset \subset E_T(G_i)$ for $i = 1, 2$ so that $X = E_T(G_1) + E_T(G_2)$.

It would seem more natural to replace, in condition 2°, the spectrum by σ_S. In this case we could use, in condition 1° instead of X_i the concrete spaces $E_T(G_i)$. It follows from 3° of Proposition (2.8) that, under the assumption $E_T(\emptyset) = (0)$ both notions of decomposability coincide.

(2.11) **Proposition.** *Let F be a subset of $\mathbb{C}$. Then the space $E_T(F)^0$ is invariant with respect to T^*. Furthermore,*

1° $\sigma_S(T^* \mid E_T(\mathbb{C} \setminus F)^0) \subset F$
so that

$$E_T(\mathbb{C} \setminus F)^0 \subset E_{T^*}(F)$$

2° *if* $E_{T^*}(\emptyset) = (0)$ *then*

$$\sigma(T^* \mid E_T(\mathbb{C} \setminus F)^0) \subset F$$

3° *If T is algebraically decomposable then*

$$\sigma(T^* \mid E(\mathbb{C} \setminus F)^0) \subset F^- .$$

Proof. Suppose $f \in E_T(\mathbb{C} \setminus F)^0$ and consider a fixed $\lambda \in \mathbb{C} \setminus F$. Define a linear form on X by the requirement that

$$g((\lambda - T)x + m) = f(x)$$

for $x \in X$ and $m \in E(\mathbb{C} \setminus F)$. This definition of g is unambiguous: indeed, if $(\lambda - T)x + m = 0$ for some $x \in X$ and $m \in E_T(\mathbb{C} \setminus F)$ then $(\lambda - T)x \in E_T(\mathbb{C} \setminus F)$, $\lambda \in \mathbb{C} \setminus F$ and this implies $x \in E_T(\mathbb{C} \setminus F)$ by (2.3) so that $f(x) = 0$. It follows from the definition of g that $g \in E_T(\mathbb{C} \setminus F)^0$. Since $f = (\lambda - T^*)g$ we have shown that

$$E_T(\mathbb{C} \setminus F)^0 \subset (\lambda - T^*)E_T(\mathbb{C} \setminus F)^0$$

for all $\lambda \in \mathbb{C} \setminus F$ and this implies $\sigma_S(T^* \mid E_T(\mathbb{C} \setminus F)^0) \subset F$ and $E_T(\mathbb{C} \setminus F)^0 \subset E_{T^*}(F)$.

To prove 2°, suppose that $E_{T^*}(\emptyset) = (0)$. In view of 1°, it suffices to prove that $\lambda - T^*$ is injective on $E_T(\mathbb{C} \setminus F)^0$ for $\lambda \notin F$. Indeed, if $\lambda \notin F$, we have

$$\operatorname{Ker}(\lambda - T^*) \cap E_T(\mathbb{C} \setminus F)^0 \subset E_{T^*}(\{\lambda\}) \cap E_{T^*}(F) = E_{T^*}(\emptyset) = (0).$$

Suppose now that T is algebraically decomposable; it remains to show that, for $\lambda \in \mathbb{C} \setminus F$, the operator $T^* - \lambda$ is injective on $E_T(\mathbb{C} \setminus F)^0$. The sets $\mathbb{C} \setminus \{\lambda\}$ and $\mathbb{C} \setminus F^-$ form an open covering of $\mathbb{C}$ so that $X = E_T(\mathbb{C} \setminus \{\lambda\}) + E_T(\mathbb{C} \setminus F)$.

Suppose $(\lambda - T^*)f = 0$ for some $f \in E_T(\mathbb{C} \setminus F)^0$. Since $E_T(\mathbb{C} \setminus F^-) \subset E_T(\mathbb{C} \setminus F)$ we have $\langle E_T(\mathbb{C} \setminus F^-), f\rangle = 0$. Furthermore,

$$\begin{aligned}\langle E_T(\mathbb{C} \setminus \{\lambda\}), f\rangle &\subset \langle(\lambda - T)E_T(\mathbb{C} \setminus \{\lambda\}), f\rangle = \\ &= \langle E_T(\mathbb{C} \setminus \{\lambda\}), (\lambda - T^*)f\rangle = 0.\end{aligned}$$

Since $X = E_T(\mathbb{C} \setminus \bar{F}) + E_T(\mathbb{C} \setminus \{\lambda\})$ it follows that $f = 0$.

(2.12) **Proposition.** *Suppose that $E_T(\emptyset) = (0)$. Then, for every F,*

$$E_{T^*}(F) = E(\mathbb{C} \setminus F)^0 + E_{T^*}(\emptyset).$$

Proof. Consider an $f \in E_{T^*}(F)$. Define a linear form $g \in X^*$ by the requirement that $g(x + y) = f(x)$ for $x \in E_T(F)$, $y \in E_T(\mathbb{C} \setminus F)$. This definition is legitimate since

$$x + y = x' + y'$$

implies $x - x' = y' - y \in E_T(F) \cap E_T(\mathbb{C} \setminus F) = (0)$ so that $x = x'$ and $f(x) = f(x')$. Furthermore $h = f - g$ is annihilated by $E_T(F)$. Thus $h \in E_T(F)^0 \subset E_{T^*}(\mathbb{C} \setminus F)$. At the same time $h = f - g$, $f \in E_{T^*}(F)$ and $g \in E_T(\mathbb{C} \setminus F)^0 \subset E_{T^*}(F)$. Thus $h \in E_{T^*}(\emptyset)$ and $f = g + h$.

(2.13) **Theorem.** *Let T be algebraically decomposable. Then the restriction of T^* to $E_T(\emptyset)^0$ is algebraically decomposable as well.*

In particular, if $E_T(\emptyset) = (0)$ then T^* is algebraically decomposable.

Proof. Let G_1, G_2 be an open covering of the complex plane. Let U_1, U_2 be open sets such that $U_i^- \subset G_i$ and $U_1 \cup U_2 = \mathbb{C}$. Since $(\mathbb{C} \setminus U_1) \cap (\mathbb{C} \setminus U_2) = \emptyset$ we have $E_T(\emptyset) = E_T(\mathbb{C} \setminus U_1) \cap E_T(\mathbb{C} \setminus U_2)$ whence $E_T(\emptyset)^0 = E_T(\mathbb{C} \setminus U_1)^0 + E_T(\mathbb{C} \setminus U_2)^0$ and, according to (2.11),

$$\sigma(T^* \mid E_T(\mathbb{C} \setminus U_i)^0) \subset U_i^- \subset G_i$$

for $i = 1, 2$.

Thus far we have investigated the notions of spectrum and decomposability defined purely in algebraic terms. In what follows we intend to compare them with the corresponding notions for bounded operators on a Banach space.

From now on let X be a Banach space and T be a continuous linear operator on X.

Since the resolvent of a bounded operator is a holomorphic function it is natural to define, for an $x \in X$, the spectrum $\gamma_T(x)$ as the complement of the set of those complex numbers λ for which there exists a holomorphic function f (not necessarily unique) defined on a neighbourhood $\mathcal{U}_\lambda$ of λ so that $(\mu - T) f(\mu) = 0$ for $\mu \in \mathcal{U}_\lambda$.

The concept which corresponds to the E spaces in the continuous case is clearly the following one: given a subset $F \subset \mathbb{C}$ set

$$X_T(F) = \{x \in X;\ \gamma_T(x) \subset F\}\,.$$

Using a standard argument from the theory of decomposable operators (see, e.g. [5]) it is easy to prove the inclusion

$$X_T(F) \subset E_T(F)$$

for all $F \subset \mathbb{C}$. In particular,

$$X_T(\emptyset) \subset E_T(\emptyset)\,.$$

It is less known that the same argument yields the following proposition (although not formulated in these terms, it is essentially contained already in Remark 1 of [6]).

(2.14) **Proposition.** $X_T(\emptyset) = (0)$ *if and only if T has the so called single valued extension property* (SVEP): *if f is a function holomorphic on an open set G which satisfies $(\lambda - T) f(\lambda) = 0$ for all $\lambda \in G$ then f is identically zero.*

Proof. Consider a nonzero $x \in X_T(\emptyset)$. Then there exists an open subset $G \subset \mathbb{C}$, and two distinct functions f_1, f_2 holomorphic on G such that

$$x = (\lambda - T) f_1(\lambda) = (\lambda - T) f_2(\lambda) \quad \text{for} \quad \lambda \in G\,.$$

Indeed, otherwise an application of the Liouville theorem would yield $x = 0$, a contradiction. Set $g = f_1 - f_2$. The function g is not identically zero and satisfies $(\lambda - T)\, g(\lambda) = 0$ in G so that SVEP does not hold.

On the other hand, suppose that there exists a nonzero holomorphic h on an open set G such that

$$(\lambda - T)\, h(\lambda) = 0$$

for all $\lambda \in G$. If we show that each $h(\lambda)$, $\lambda \in G$ belongs to $X_T(\emptyset)$ the proof will be complete.

Indeed, we have

$$h(\lambda) = (\mu - T) \frac{h(\mu) - h(\lambda)}{\mu - \lambda} \quad \text{for} \quad \mu \in \mathbb{C} \setminus \{\lambda\}$$

and

$$h(\lambda) = (\mu - T)\,\tilde{h}(\mu) \quad \text{for} \quad \mu \in G\,,$$

where the analytic function $\tilde{h}$ is defined as follows

$$\tilde{h}(\mu) = \begin{cases} \dfrac{h(\mu) - h(\lambda)}{\mu - \lambda} & \text{for} \quad \mu \in G \setminus \{\lambda\}\,, \\ h'(\lambda) & \text{for} \quad \mu = \lambda\,. \end{cases}$$

(2.15) **Corollary.** *If $E_T(\emptyset) = (0)$ then T has the* SVEP.

Recall now that a continuous linear operator T is said to be *decomposable* if, for each open covering G_1, G_2 of $\mathbb{C}$ there exist two closed linear subspaces X_1, X_2 of X invariant with respect to T such that

1° $X = X_1 + X_2$

2° $\sigma(T \mid X_i) \subset G_i$ for $i = 1, 2$.

Every decomposable operator is known to possess the SVEP. Also, all subspaces $X_T(F)$ are closed for closed F, T^* is decomposable as well and $X_{T^*}(F) = X_T(\mathbb{C} \setminus F)^0$ for closed F.

The result of the present note reveal a striking similarity between the continuous and the algebraic case: the X_T spaces are replaced by E_T spaces. There is one difference, however: while decomposability already implies $X_T(\emptyset) = (0)$ the algebraic decomposability does not imply $E_T(\emptyset) = (0)$ in general.

3. CONCLUDING REMARKS

In this section we collect some propositions concerning the spaces $E_T(F)$ which improve results known from the literature: either in the sense that a stronger statement is obtained or the proof simplified.

We first examine the behaviour of the E_T spaces if the operator T is replaced by its restriction to an invariant subspace M or by the quotient operator acting on the space X/M.

(3.1) **Proposition.**

$$E_{T|E_T(G)}(F) = E_T(G) \cap E_T(F) = E_T(G \cap F)\,.$$

Proof. The inclusion $E_{T|E_T(G)}(F) \subset E_T(G) \cap E_T(F)$ is obvious.

On the other hand, $E_T(G \cap F)$ is a subspace of $E_T(G)$ and, for $\lambda \in \mathbb{C} \setminus F \subset \mathbb{C} \setminus (F \cap G)$, we have $E_T(G \cap F) \subset (T - \lambda)\, E_T(G \cap F)$. By maximality, $E_T(G \cap F) \subset E_{T|E_T(G)}(F)$.

(3.2) **Proposition.** *Let X, Y be two linear spaces and Q a linear mapping $Q\colon X \to Y$. Suppose the linear mappings T on X and S on Y satisfy the intertwining relation*

$$QT = SQ\,.$$

Then $QE_T(G) \subset E_S(G)$ for all G.

Furthermore, under the additional hypothesis $E_S(G) \subset \operatorname{Range} Q$ *the following two conditions are equivalent*

$$\operatorname{Ker} Q \subset E_T(G)\,,$$

$$Q^{-1}\,E_S(G) \subset E_T(G)\,.$$

In particular, if $\operatorname{Ker} Q \subset E_T(G)$ *and* $E_S(G) \subset \operatorname{Range} Q$ *then*

$$E_S(G) = Q\,E_T(G)\,.$$

Proof. If $\lambda \notin G$ we have

$$Q\,E_T(G) \subset Q(\lambda - T)\,E_T(G) = (\lambda - S)\,Q\,E_T(G)\,;$$

by maximality, this implies

$$Q\,E_T(G) \subset E_S(G)\,.$$

Now consider the second statement of our proposition, the proof of the equivalence of the two conditions. Since the second condition trivially implies the first one, it will be sufficient to assume the inclusions $E_S(G) \subset \operatorname{Range} Q$, $\operatorname{Ker} Q \subset E_T(G)$ and prove the inclusion $M \subset (\lambda - T)\,M$ for all $\lambda \notin G$ where $M = Q^{-1}\,E_S(G)$. Let us remark in passing that it is easy to see that M is invariant with respect to T so that the inclusion to be proved may also be restated in the form

$$\sigma_S(T \mid Q^{-1}\,E_S(G)) \subset G\,.$$

Consider an $x \in M$; then $Qx \in E_S(G) \subset (\lambda - S)\,E_S(G)$. Since $E_S(G)$ is contained in the range of Q there exists a y such that $Qy \in E_S(G)$ and

$$Qx = (\lambda - S)\,Qy = Q(\lambda - T)\,y\,.$$

Thus $x - (\lambda - T)\,y \in \operatorname{Ker} Q \subset E_T(G) \subset (\lambda - T)\,E_T(G)$ so that there exists a $z \in E_T(G)$ for which $x - (\lambda - T)\,y = (\lambda - T)\,z$. Now $Q(y + z) \in E_S(G) + Q\,E_T(G) = E_S(G)$ so that $y + z \in M$. Since $x = (\lambda - T)\,(y + z)$ the proof is complete.

To prove the last assertion it suffices to use the fact just proved and to observe that the inclusion $E_S(G) \subset \operatorname{Range} Q$ implies $E_S(G) \subset QQ^{-1}\,E_S(G)$. Hence

$$E_S(G) \subset QQ^{-1}\,E_S(G) \subset Q\,E_T(G)\,.$$

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