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1. Introduction. Back in early stages of the general topology M. Frèchet [3] introduced dimensional types of topological spaces. If a space X is homeomorphic with some subspace of a space Y, then X has a smaller or equal dimensional type than Y, in symbols $dX \leqq dY$. Dimensional types of metric separable spaces were investigated by S. Banach [1], K. Kuratowski [4], [6], W. Sierpiński [6], [9] and others. One can find some information about this in Sierpiński's *General Topology* [10] p. 130 or in Kuratowski's *Topology* I [5] p. 112 and 433.

We generalize dimensional types in the following manner. If a space X is the image of some subspace of a space Y under a closed mapping, then X has a smaller or equal closed-map type than Y, in symbols $ctX \leqq ctY$. If $ctX \leqq ctY$ and doesn't hold $ctY \leqq ctX$, then X has a smaller closed-map type than Y, in symbols $ctX < ctY$. If there doesn't hold $ctX \leqq ctY$ or $ctY \leqq ctX$, then spaces X and Y are imcomparable.

Obviously $dX \leqq dY$ implies $ctX \leqq ctY$. The inverse implication is not true. It is a well-known fact that the Cantor set C can be mapped continuously onto a non-void closed interval J. So $ctC \leqq ctJ$, $ctJ \leqq ctC$ and $dC \leqq dJ$, but $dJ \leqq dC$ doesn't hold.

In this paper we investigate closed-map types of first countable spaces, exactly hereditarily normal and hereditarily separable ones. Nice non-metrizable examples are the Songenfrey line and its uncountable subspaces. Theorem 1 is a main tool for proving other results. Other theorems are applications of Theorem 1 and in few cases their proofs are similar to those of S. Banach [1], K. Kuratowski [4], [6] and W. Sierpiński [6], [9] on dimensional types.

All undefined notations are as in Engelking's *General Topology* [2]. The cardinality of a set X is denoted by $|X|$. The cardinal number assigned to the set of real numbers is denoted by $\mathfrak{c}$ and is called *continuum*. By $\mathfrak{c}^+$ we denote the least cardinal number greater than $\mathfrak{c}$. If $|X| = \lambda$, then 2^λ denotes the cardinality of the family of all subsets of X. By $f \upharpoonright M$ we denote the restriction of a function f to the set M. A mapping means a continuous function. A mapping $f: X \to Y$ is a closed mapping if it is closed as a function from X onto $f(X)$. If X is a topological space and $A \subset X$, then $\bar{A}$ denotes the closure of A.

2. The main theorem. The proof of the below lemma is straightforward, so we omit it.

Lemma 1. *Suppose X and Y are first countable spaces with X a T_1-space. A mapping $f: X \to Y$ is closed iff each sequence $\{x_n: n = 1, 2, \ldots\} \subset X$, such that $\lim f(x_n) = y$ and $y \neq f(x_n)$ for all n, has a limit point in $f^{-1}(y)$.* □

Lemma 2. *Let X and Y be first countable Hausdorff spaces. If $f: A \to f(A) \subset Y$ and $g: B \to g(B) \subset Y$ are closed mappings, $D \subset A \subset \bar{D} \subset X$, $D \subset B \subset \bar{D} \subset X$, $f \upharpoonright D = g \upharpoonright D$, $t \in f(A) \cap g(B)$ and $f^{-1}(t)$ is a nowhere dense, relative $\bar{D}$, subset, then $f^{-1}(t) = g^{-1}(t)$.*

Proof. Suppose $t \in f(A) \cap g(B)$ and $f^{-1}(t)$ is a nowhere dense, relative $\bar{D}$, subset. Assume to the contrary that $b \in g^{-1}(t) \setminus f^{-1}(t)$. There exists a sequence $E = \{a_1, a_2, \ldots\} \subset D \setminus f^{-1}(t)$ such that $\lim a_n = b$, since spaces X and Y are Hausdorff $b \notin A$ and the set E is closed relative A. Whence the set $f(E)$ is closed relative $f(A)$ and $t \notin f(E)$. On the other hand $t \in \overline{g(E)}$, since $f(E) = g(E)$, $\lim g(a_n) = g(b) = t$ and $t \in f(A)$. So we have $t \in f(A) \cap \overline{f(E)} = f(E)$, a contradiction. Thus we have $g^{-1}(t) \subset f^{-1}(t)$. The inclusion $f^{-1}(t) \subset g^{-1}(t)$ one can prove in a similar way. □

Let X and Y be topological spaces and $f: A \to Y$ be a function. A function $h: B \to Y$ is a X, Y-expansion of f if $h \upharpoonright A = f$, $A \subset B \subset \bar{A} \subset X$ and $t \in f(A)$ implies $f^{-1}(t) = h^{-1}(t)$. Any X, Y-expansion of f is a closed X, Y-expansion of f if it is a closed mapping.

Lemma 3. *Suppose X is a first countable hereditarily normal space, and Y is a first countable regular space. If A is a dense subset of X and $f: A \to Y$ is a closed mapping, then there exists the maximal X, Y-expansion of f.*

Proof. Take $F(x) = f(x)$ for $x \in A$ and $F(x) = h(x)$ whenever $x \in h^{-1}(t)$, $t \notin f(A)$ and h is a closed X, Y-expansion of f. The function F is well defined because of Lemma 2.

Suppose to the contrary that F is not continuous. Thus there has to exist a sequence $x_1, x_2, \ldots$ such that $\lim x_n = b$ and the sequence $F(x_1), F(x_2), \ldots$ isn't convergent to $F(b)$. Since the space Y is regular we can assume that there is an open neighbourhood W of the point $F(b)$ such that $F(x_n) \notin \bar{W}$ for all n, if need be we can take a suitable subsequence. Let h_n be a closed X, Y-expansion of f such that $h_n(x_n)$ exists and let $x_n^m \in h_n^{-1}(Y \setminus \bar{W}) \cap A$ be such that $\lim x_n^m = x_n$. Take $E = \{x_n^m: n, m = 1, 2, \ldots\}$. Since $b \in \bar{E}$ there exists a sequence $\{y_1, y_2, \ldots\} \subset E$ such that $\lim y_n = b$. Let h be a closed X, Y-expansion of f such that $h(b)$ exists. We have $h(y_n) = f(y_n) = F(y_n) \notin \bar{W}$ and $\lim h(y_n) = h(b) = F(b) \in W$ because of h is continuous, a contradiction. Thus the function F is continuous.

Suppose $F^{-1}(b) \neq \emptyset$, $\lim a_n = b$ and $F(y_n) = a_n \neq b$ for all n. Assume to the contrary that the sequence $y_1, y_2, \ldots$ has no limit points in $F^{-1}(Y)$, if a limit point of this sequence exists, then it has to belong to $F^{-1}(b)$ because of F is continuous, but this doesn't contrary the fact that F is a closed mapping. Sets $F^{-1}(b)$ and $\{y_1, y_2, \ldots\}$ are disjoint and closed relative $F^{-1}(Y)$. Since X is hereditarily normal there exists an open set V such that $F^{-1}(b) \subset V$ and $\{y_1, y_2, \ldots\} \cap \bar{V} = \emptyset$. Let h be a closed X, Y-expansion of f such that $h^{-1}(b)$ is non-empty and let $y_n^m \in A \setminus \bar{V}$

be such that $\lim y_n^m = y_n$. The set $D = \{y_n^m : n, m = 1, 2, \ldots\}$ has no limit points in the set $F^{-1}(b)$. On the other hand $b \in \overline{h(D)}$ and therefore there exists a sequence $\{q_1, q_2, \ldots\} \subset h(D) = F(D)$ such that $\lim q_n = b$. Let $p_n \in F^{-1}(q_n) \cap D$ for all n. Since Lemma 1 the set $\{p_1, p_2, \ldots\} \subset D$ has a limit point in $h^{-1}(b) = F^{-1}(b)$ because of h is a closed mapping, a contradiction. Thus F is a closed mapping. □

Theorem 1. *Suppose X is a first countable hereditarily normal and hereditarily separable space, Y is a regular space and $|Y| \leqq \mathfrak{c}$. There exists a family S satisfying the following:*

(i) *S consists of closed mappings from X into Y,*

(ii) $|S| \leqq \mathfrak{c}$,

(iii) *for each $B \subset X$ and for each closed mapping $h: B \to h(B) \subset Y$ there exists $f \in S$ and a countable set $E \subset h(B)$ such that $f \restriction B \setminus h^{-1}(E) = h \restriction B \setminus h^{-1}(E)$ and $f^{-1}(t) = h^{-1}(t)$ for each $t \in h(B) \setminus E$.*

Proof. Let P be a countable subset of X and let $f: P \to Y$ be a continuous function. If there exists a closed mapping $h: A \to Y$ such that $P \subset A \subset \bar{P} \subset X$ and $h \restriction P = f$, then let F be the maximal closed X, Y-expansion of $h \restriction h^{-1}(h(P))$, it exists because of Lemma 3. Let us fix a particular F for a function f. Let S be the family of all such mappings F. Obviously the family S has the cardinality of at most continuum, i.e. $|S| \leqq \mathfrak{c}$.

Suppose $B \subset X$ and $f: B \to f(B) \subset Y$ is a closed mapping. Let $P \subset B$ be a dense in B and countable subset. Take $F \in S$, which is assigned to $f \restriction P$, and let $E = f(P)$. Thus we have $f \restriction B \setminus f^{-1}(E) = F \restriction B \setminus f^{-1}(E)$ and if $t \in f(B) \setminus E$, then $F^{-1}(t) = f^{-1}(t)$ because of Lemma 2. □

3. A generalization of Kuratowski's theorem [4].

Lemma 4. ([5] p. 425). *If X is an infinite set of the cardinality λ and S is a family of the cardinality of at most λ of functions from subsets of X onto subsets of the cardinality λ of X, then there exists a family F of the cardinality 2^λ of subsets of X such that conditions $Y, Z \in F$ and $Y \neq Z$ imply $|f(Z) \setminus Y| = \lambda$, for each function $f \in S$.* □

Theorem 2. *In any hereditarily separable, hereditarily normal and first countable space of the cardinality continuum there is a family of $2^{\mathfrak{c}}$ subspaces whose closed-map types are imcomparable.*

Proof. Let S be the family of closed mappings obtained by virtue of Theorem 1 in the case $X = Y$, where X is as in hypotheses. If $f \in S$, then let f_* be some fixed one-to-one function such that $f(f_*(x)) = x$, for each $x \in f(X)$. Take the family $S_* = \{f_* : f \in S$ and $|f(X)| = \mathfrak{c}\}$ and make use of Lemma 4. The family F of subspaces of X, which we obtain, is that we required. Indeed, let $A, B \in F$, $A \neq B$ and $D \subset A$. Suppose to the contrary that $h: D \to B$ is a closed mapping onto B. Let $P \subset D$ be a countable set such that $D \subset \bar{P}$ and let $f \in S$ be the closed mapping assigned to $h \restriction P$. We have $|f_*(B) \setminus A| = \mathfrak{c}$ and $f_*(B) \setminus A \subset f_*(B) \setminus D$, a contradiction, because the set $f_*(B) \setminus D$ is countable. □

Theorem 2 is a generalization of the similar result on dimensional types, see K. Kuratowski [4] and [5] p. 433.

4. A generalization of Banach's result [1]. The below lemma is a modification of Banach's Lemme from [1]. One can prove it by making some changes in the proof of Banach's Lemme.

Lemma 5. *Suppose λ is an infinite cardinal number and $\tau < \lambda$. Let E be a set of the cardinality λ and let S be a family of functions from subsets of E onto subsets of E. If $|S| \leqq \lambda$ and $|f^{-1}(t)| \leqq \tau$ for each $f \in S$ and $t \in E$, then there exists a family $\{H_\alpha: \alpha < \lambda\}$ of subsets of E such that the following holds:*

(i) *$\{H_\alpha: \alpha < \lambda\}$ is a partition of E,*

(ii) *if $\gamma < \lambda$, then $|\bigcup\{H_\alpha: \alpha < \gamma\}| < \lambda$,*

(iii) *if $f \in S$, then there is $\gamma < \lambda$ such that $f(H_\alpha) \subset H_\alpha$ for each $\alpha > \gamma$.* □

A space X is *unperfect* if its compact subspaces are countable, e.g. a metric space is unperfect if it lacks any copy of the Cantor set. We will need the following Michael's result [8]. If X is a paracompact space, Y is a first countable space and $f: X \to Y$ is a closed mapping, then the boundary of $f^{-1}(t)$ is compact for each $t \in Y$.

Theorem 3. *If X is an unperfect, hereditarily separable, hereditarily paracompact and first countable space with $|X| = \mathfrak{c}$ and Y is a regular and first countable space with $|Y| \leqq \mathfrak{c}$, then there exists a family $\{A_\alpha: \alpha < \mathfrak{c}\}$ such that the following holds:*

(i) *for each $\alpha < \mathfrak{c}$, there is $A_\alpha \subset X$ such that $|A_\alpha| = \mathfrak{c}$,*

(ii) *for each $B \subset Y$, if there are distinct $\alpha, \beta < \mathfrak{c}$ with $ctB \leqq ctA$ and $ctB \leqq ctA$, then $|B| < \mathfrak{c}$.*

Proof. Let S be a family of closed mappings as in Theorem 1. If $f \in S$, then let $Z = \bigcup\{f^{-1}(t): t \in Y$ and $f^{-1}(t)$ is nowhere dense relative $f^{-1}(Y)\}$. Take $f_* = f \restriction Z$ and $S_* = \{f_*: f \in S\}$. Now we make use of Lemma 5. Let us observe that $f^{-1}(t)$ is always countable because of Michael's result [8]. We obtain a family $\{H_\alpha: \alpha < \mathfrak{c}\}$ as in Lemma 5. Let $\{P_\alpha: \alpha < \mathfrak{c}\}$ be a family of pairwise disjoint subsets of $\{\alpha: \alpha < \mathfrak{c}\}$ such that $|P_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$. Sets $A_\alpha = \bigcap\{H_\beta: \beta \in P_\alpha\}$ are that we required. □

Theorem 3 is a generalization of the result on dimensional types obtained in S. Banach [1].

5. Metric cases. It is well-known facts that any metric separable space is embeddable in the Hilbert cube Q and that any metric compact space is the image of the Cantor set C under a closed mapping, among others Q is the image of C under some closed mapping. Therefore any metric separable space is the image of some subspace of C under a closed mapping. Thus if we consider the closed-map type of a space X, then an interesting case is, when X lacks any copy of the Cantor set, i.e. X is unperfect.

Lemma 6. *Suppose X is a metric separable space. If there is a closed mapping f from X onto the Cantor set C, then X contains some copy of C.*

Proof. Let $A \subset C$ be a copy of the Cantor set such that if $t \in A$, then $f^{-1}(t)$ is a nowhere dense subset of X, whence $f^{-1}(t)$ is compact by virtue of Vaĭnšteĭn's Lemma [2] p. 356. The set $f^{-1}(A)$ being an uncountable compact metric space, because of Lubben's result [2] p. 236, contains some copy of the Cantor set. □

From the above lemma we obtain immediately the following.

Theorem 4. *If a metric separable space X is unperfect, then its closed-map type is smaller than the closed-map type of the Cantor set C, i.e. $ctX < ctC$.* □

If we use Theorem 1 and make appropriate changes in proofs in [6], then we obtain the following:

Theorem 5. *If X is a metric separable space and the closed-map type of X is smaller than the closed-map type of the Cantor set C, then there exists a metric space Y such that $ctX < ctY < ctC$.* □

Theorem 6. *There exists a family $\{A_\alpha: \alpha < \mathfrak{c}^+\}$ of metric separable spaces such that $\alpha < \beta < \mathfrak{c}^+$ implies $ctA_\alpha < ctA_\beta$.* □

We don't know for which metric separable space X one can assume that the family as in Theorem 6 is contained in X. Can it be any metric separable space X with $|X| = \mathfrak{c}$?

Theorem 7. *Suppose X is a metric separable space with $|X| = \mathfrak{c}$. There exists a family $\{A_\alpha: \alpha < \mathfrak{c}\}$ of subspaces of X such that $|A_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$ and if $ctB \leqq ctA_\alpha$, $ctB \leqq ctA_\beta$ and $\alpha \neq \beta$, then $|B| < \mathfrak{c}$.*

Proof. We will need the following Lanšev's result [7]. If X is a metric space and $f: X \to Y$ is a closed mapping, then $Y = Y_0 \cup Y_1 \cup Y_2 \cup \ldots$, where $f^{-1}(y)$ is compact for each $y \in Y_0$ and Y_n are discrete for all n. If X is, additionally, separable, then Y_0 is a metrizable and separable subspace, cf. the Hanai-Morita-Stone theorem [2] p. 356, and $\bigcup\{Y_n: n = 1, 2, \ldots\}$ is a coutable subspace.

We can assume that X is an unperfect space, if need be we can take a suitable subspace. By virtue of Theorem 3 there exists a family $\{A_\alpha: \alpha < \mathfrak{c}\}$ of subspace of X such that $|A_\alpha| = \mathfrak{c}$ and if B is a subspace of the Hilbert cube, $ctB \leqq ctA_\alpha$, $ctB \leqq ctA_\beta$ and $\alpha \neq \beta$, then $|B| < \mathfrak{c}$. This family is that we required. Suppose to the contrary that there are $\alpha \neq \beta$ and a space B such that $|B| = \mathfrak{c}$, $ctB \leqq ctA_\alpha$ and $ctB \leqq ctA_\beta$. Thus there are $H \subset A_\alpha$, $G \subset A_\beta$ and closed mappings $h: H \to B = h(H)$, $g: G \to B = g(G)$. Let $B_* = \{y \in B: h^{-1}(y)$ and $g^{-1}(y)$ are compact$\}$. The space B_* is metrizable, separable and $|B_*| = \mathfrak{c}$ because of Lašnev result, a contradiction, since B_* is embeddable in the Hilbert cube, $ctB_* \leqq ctA_\alpha$ and $ctB_* \leqq ctA_\beta$. □

In next two theorems we assume that the cardinal number $\mathfrak{c}$ is regular. We don't know if this assumption can be omitted.

Theorem 8. *If X is a metric separable space with $|X| = \mathfrak{c}$, then there is a family $\{B_\alpha: \alpha < \mathfrak{c}\}$ of subspaces of X such that $\alpha < \beta$ implies $ctB_\alpha < ctB_\beta$.*

Proof. Let $\{A_\alpha: \alpha < \mathfrak{c}\}$ be a family as in Theorem 7. Take $B_\alpha = \bigcup\{A_\beta: \beta \leqq \alpha\}$. If $\alpha < \beta < \mathfrak{c}$, then $ctB_\alpha \leqq ctB_\beta$ since $B_\alpha \subset B_\beta$. We will show that $ctB_\beta \leqq ctB_\alpha$ doesn't hold. Suppose to the contrary that there are $H \subset B_\alpha$ and a closed mapping $h: H \to A_\beta = h(H) \subset B_\beta$. We have $|A_\beta \cap h(A_\gamma)| < \mathfrak{c}$ for each $\gamma \leqq \alpha$. Whence $|A_\beta| = |\bigcup\{A_\beta \cap h(A_\gamma): \gamma \leqq \alpha\}| < \mathfrak{c}$, a contradiction, because of $\mathfrak{c}$ is regular. □

Theorem 9. *If X is a metric separable space with $|X| = \mathfrak{c}$, then there exists a family $\{B_\alpha: \alpha < \lambda\}$ of subspaces of X such that if $\alpha < \beta$, then $ctB_\beta < ctB_\alpha$.*

Proof. Let $\{A_\alpha: \alpha < \mathfrak{c}\}$ be a family as in Theorem 7. Take $B_\alpha = \bigcap\{A_\beta: \alpha \leqq \beta < \lambda\}$. If $\alpha < \beta < \lambda$, then $ctB_\beta \leqq ctB_\alpha$ because of $B_\beta \subset B_\alpha$. We will show that $ctB_\alpha \leqq ctB_\beta$ doesn't hold. Suppose to the contrary that there are $H \subset B_\beta$ and a closed mapping $h: H \to A_\alpha = h(H) \subset B_\alpha$. We have $|A_\alpha \cap h(A_\gamma)| < \mathfrak{c}$ for each γ such that $\beta \leqq \gamma < \lambda$. Whence $|A_\alpha| = |\bigcup\{A_\alpha \cap h(A_\gamma): \beta \leqq \gamma < \lambda\}| < \mathfrak{c}$, a contradiction. □

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