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COMMUTATIVE SEMIGROUPS WHOSE LATTICE OF TOLERANCES IS BOOLEAN

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Let S be a semigroup. A reflexive and symmetric relation A on S is called a *tolerance* [*stable tolerance*] if $(a, b) \in A$ and $(c, d) \in A$ imply $(a, b)(c, d) = (ac, bd) \in A$ [$(a, b) \in A$ implies $x(a, b) = (xa, xb) \in A$ and $(a, b)x = (ax, bx) \in A$ for all $x \in S$]. By $\mathscr{C}(S)$ [$\mathscr{T}(S), \mathscr{U}(S)$] we denote the lattice of all congruences [tolerances, stable tolerances] on S. It is easy to show that $\mathscr{C}(S) \subseteqq \mathscr{T}(S) \subseteqq \mathscr{U}(S)$. All commutative semigroups S with boolean lattices $\mathscr{C}(S)$ have been found by Hamilton-Nordahl[1]. Sitnikov [2] gave a description of all commutative semigroups S with boolean lattices $\mathscr{U}(S)$. The aim of this paper consists in a characterization of all commutative semigroups S whose tolerance lattice $\mathscr{T}(S)$ is complemented or boolean.

Let S be a commutative semigroup. Clearly $A \in \mathscr{T}(S)$ if and only if A is a reflexive and symmetric subsemigroup of the direct product $S \times S$. By $\vee$ and $\wedge$ we denote the join or meet in the lattice $\mathscr{T}(S)$. Let $M \subseteqq S \times S$. By $T(M)$ we denote the least tolerance on S containing M. The symbol S^1 stands for S if S has an identity, otherwise it stands for with an identity adjoined.

In the sequel we will make use also of the following properties of tolerances, which may be easily verified.

(1) $A \wedge B = A \cap B$ for all $A, B \in \mathscr{T}(S)$.

(2) $A \vee B = T(A \cup B) = A \cup B \cup AB$ for all $A, B \in \mathscr{T}(S)$.

(3) Let $a, b, x, y \in S$ and $a \neq b$, $x \neq y$. Then $(x, y) \in T(a, b)$ if and only if there exist $z \in S^1$ and a positive integer m such that either $(x, y) = (a, b)^m z$ or $(x, y) = (b, a)^m z$.

By $E(S)$ we denote the set of all idempotents of a commutative semigroup S. It is well known that $E(S)$ is partially ordered by: $e \leqq f$ if $ef = e$. We write $e < f$ for $e \leqq f$ and $e \neq f$. By $e \parallel f$ we denote the fact that idempotents e, f are incomparable. G_e denotes the maximal subgroup of S containing an idempotent $e \in E(S)$ and by x^{-1} we denote the inverse element of $x \in G_e$ in G_e. Terminology and notation not defined here may be found in [3].

Theorem 1. *Let S be a commutative regular semigroup. If the lattice $\mathscr{T}(S)$ is complemented, then S is either a group or a group with zero.*

Proof. Suppose that S is a commutative regular semigroup and the lattice $\mathscr{T}(S)$ is complemented.

I. We first shall show that the semilattice $E(S)$ is a chain. By way of contradiction, assume that there exist $f, g \in E(S)$ such that $f \parallel g$. Put $e = fg$. Then $e < f$ and $e < g$. Let $A = T(e, f)$. By hypothesis, there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$. Then $(f, g) \in A \vee \bar{A}$. If $(f, g) \in \bar{A}$, then $(e, f) = (g, f) f \in A \wedge \bar{A}$, which contradicts (1). Hence we have $(f, g) \notin \bar{A}$. It follows from (2) and (3) that $(f, g) = (e, f)(u, v)$ or $(f, g) = (f, e)(u, v)$ for some $(u, v) \in \bar{A}$. If $(f, g) = (e, f)(u, v)$, then $f = eu$ and so $f = ef = e$, a contradiction. If $(f, g) = (f, e)(u, v)$, then $g = ev$ and so $g = eg = e$, a contradiction.

II. Now we shall prove that card $E(S) \leqq 2$. By way of contradiction, assume that card $E(S) \geqq 3$. It follows from the part I of the proof that there exist $e, f, g \in E(S)$ such that $e < f < g$. Let $A = T(e, f)$. By hypothesis, there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$. Then $(e, g) \in A \vee \bar{A}$. We have $(e, g) \notin \bar{A}$. Indeed, if $(e, g) \in \bar{A}$, then $(e, f) = (e, g) f \in A \wedge \bar{A}$, a contradiction. According to (2) and (3), we have $(e, g) = (e, f)(u, v)$ or $(f, e)(u, v)$ for some $(u, v) \in \bar{A}$. If $(e, g) = (e, f)(u, v)$, then $g = fv$ and so $g \leqq f$, a contradiction. If $(e, g) = (f, e)(u, v)$, then $g = ev$ and so $g \leqq e$, a contradiction.

III. We shall show that S is a group or a group with zero. If card $E(S) = 1$, then it is well known that the regular semigroup S is a group. Suppose that card $E(S) = 2$. Then S is a semilattice of two groups G_e and G_f, where $e < f$ and $e, f \in E(S)$.

A. First we shall show that $xy = x$ for all $x \in G_e$ and $y \in G_f$. By way of contradiction, assume that $ab \neq a$ for some $a \in G_e$ and some $b \in G_f$. We have $eb \neq e$. Indeed, if $eb = e$, then $a = ae = aeb = ab$, a contradiction. It is clear that $b \neq f$. Let $A = T(eb, e)$. By hypothesis, there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$. Then $(b, f) \in A \vee \bar{A}$. If $(b, f) \in \bar{A}$, then $(eb, e) = (b, f) e \in \bar{A}$, a contradiction. Hence we have $(b, f) \notin \bar{A}$. According to (2) and (3), there exist $(u, v) \in \bar{A}$ and a positive integer m such that $(b, f) = (eb, e)^m (u, v)$ or $(e, eb)^m (u, v)$. This gives in both cases that $f \in eS$. Consequently $f \leqq e$, which is a contradiction.

B. Finally we shall prove that card $G_e = 1$. By way of contradiction, assume that card $G_e > 1$. Put $A = T(e, f)$. By hypothesis, there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$.

We shall show that $(x, f) \in \bar{A}$ for all $x \in G_e$, $x \neq e$. We have $(x, f) \in A \vee \bar{A}$. Assume that $(x, f) \notin \bar{A}$ for some $x \in G_e$, $x \neq e$. Since $f \notin eS$, then according to (2) and (3), there exists $(u, v) \in \bar{A}$ such that $(x, f) = (e, f)(u, v)$. If $u \in G_f$, then, by the part IIIA of the proof, we have $x = eu = e$, which is a contradiction. Therefore we have $u \in G_e$. Consequently we obtain $x = eu = efu = fu$ and so $(x, f) = f(u, v) \in \bar{A}$, a contradiction.

Now, we can choose $x \in G_e$, $x \neq e$. It follows from the preceding consideration that $(x, f) \in \bar{A}$ and $(x^{-1}, f) \in \bar{A}$. Then $(e, f) = (x, f)(x^{-1}, f) \in A \wedge \bar{A}$, which is a contradiction. Therefore card $G_e = 1$ and so S is a group with zero.

Theorem 2. *Let S be a commutative non-regular semigroup. If the lattice $\mathscr{T}(S)$ is complemented, then S is a zero semigroup.*

Proof. Suppose that S is a commutative non-regular semigroup and the lattice $\mathscr{T}(S)$ is complemented.

I. We first shall prove that $a^2 = a^3$ for every non-regular element of S. Let $a \in S \setminus a^2S$. Then $a \neq a^2$. Assume that $a^2 \neq a^3$ and put $A = T(a^2, a^3)$. By hypothesis, there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$. Then $(a, a^2) \in A \vee \bar{A}$. We have $(a, a^2) \notin \bar{A}$. Indeed, if $(a, a^2) \in \bar{A}$, then $(a^2, a^3) = (a, a^2)\, a \in A \wedge \bar{A}$, a contradiction. It follows from (2) and (3) that $(a, a^2) \in a^2S^1 \times a^2S^1$, which is a contradiction. Consequently we have $a^2 = a^3$.

II. We shall show that card $E(S) = 1$. Let us choose an element a of S such that $a \notin a^2S$. Then, by the part I of the proof, we have $a^2 \in E(S)$ and so card $E(S) \geqq 1$. Assume that there exists $e \in E(S)$ such that $e \neq a^2$. It is clear that $e \neq a$. Let $A = T(a^2, e)$. By hypothesis, there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$. Then $(a, e) \in A \vee \bar{A}$. We have $(a^2, e) = (a, e)^2$ and so $(a, e) \notin \bar{A}$. According to (2) and (3), we obtain $(a, e) \in eS \times a^2S$ and so there exist $u, v \in S$ such that $a = eu$ and $e = a^2v$. Thus we have $a = a^2uv$, which is a contradiction.

III. Now, we shall prove that S contains only one regular element. Let $a \in S \setminus a^2S$. Then $a \neq a^2 = a^3 = h$ and $E(S) = \{h\}$. By way of contradiction, assume that there exists a regular element b of S such that $b \neq h$. Clearly $a \neq b$, $bh = b$ and $ah = h$. Put $A = T(h, b)$. By hypothesis, there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$. Then $(a, b) \in A \vee \bar{A}$. We have $(h, b) = (a, b)\, h$ and so, by (1), $(a, b) \notin \bar{A}$. It follows from (2) and (3) that $(a, b) \in hS \times bS$ or $(a, b) \in bS \times hS$. If $a \in hS$, then $a = hu$ for some $u \in S$ and so $h = ah = h(hu) = hu = a$, a contradiction. If $a \in bS$, then $a \in hbS$, which is analogously impossible. Consequently every element $x \in S$, $x \neq h$, is not regular.

IV. Finally, we shall show that S is a zero semigroup. It follows from the preceding considerations that S is a semigroup with the zero 0 and $x^2 = 0$ for every $b \in S$. Assume that there exist $a, b \in S$ such that $ab \neq 0$. Then $a \neq 0 \neq b$ and $a^2 = 0 = b^2$. Let $A = T(a, 0)$. By hypothesis there exists $\bar{A} \in \mathscr{T}(S)$ such that $A \wedge \bar{A} = \mathrm{id}_S$ and $A \vee \bar{A} = S \times S$. Therefore $(b, 0) \in A \vee \bar{A}$. If $(b, 0) \in \bar{A}$, then $(ab, 0) = (a, 0)\, b = (b, 0)\, a \in A \wedge \bar{A}$, which contradicts (1). Consequently we have $(b, 0) \notin \bar{A}$. (2) and (3) imply that $(b, 0) = (a, 0)(u, v)$ for some $(u, v) \in \bar{A}$. Thus we have $b = au$ and $ab = a^2u = 0$, a contradiction. Hence S is a zero semigroup. The proof is complete.

It is well known that $\mathscr{C}(G) = \mathscr{T}(G)$ for every group G. Let S be a group G with a zero 0. Put $E = E(S)$ and $Z = \{0\}$. Evidently $Z \subsetneqq E$ and card $E = 2$. For every $A \in \mathscr{T}(G)$ we put $\varphi(A, \mathrm{id}_E) = A \cup (Z \times Z)$ and $\varphi(A, E \times E) = A \cup (S \times Z) \cup (Z \times S)$. It is easy to show that φ is a lattice-isomorphism of $\mathscr{T}(G) \times \mathscr{T}(E)$ onto $\mathscr{T}(S)$.

It is clear that there holds

Lemma. *Let S be a group G with a zero 0. Then $\mathscr{C}(G) = \mathscr{T}(G)$ and the lattices $\mathscr{C}(G) \times \mathscr{T}(E(S))$, $\mathscr{T}(S)$ are isomorphic.*

Theorem 3. *Let S be a commutative semigroup, then the lattice $\mathscr{T}(S)$ is complemented if and only if S is one of the following:*

(i) *a zero semigroup;*

(ii) *a group G satisfying the following condition:*

(*) *G is a restricted direct product of cyclic groups of prime order;*

(iii) *a group G with zero and G satisfies the condition (*).*

Proof. By Theorem 4.4.7 of [4] a commutative group G satisfies the condition (*) if and only if $\mathscr{C}(G)$ is complemented. The rest of the proof follows from Theorem 1, Theorem 2, Lemma and from the fact that tolerance lattices of zero semigroups are boolean (see Corolary of [5]).

Corollary 1. *Let S be a commutative non-zero semigroup containing at least three elements. Then S is a group G satisfying the condition (*) if and only if the lattices $\mathscr{T}(S)$ and $\mathscr{C}(S)$ are complemented.*

Proof. This follows from Theorem 3 and from the fact that card $G = 1$ for a group G with a zero 0, whenever the lattice $\mathscr{C}(S)$ is complemented, where $S = G \cup Z$ and $Z = \{0\}$. Indeed, it is easy to show that for every proper congruence A on S we have $A = (G \times G) \cup (Z \times Z) \neq S \times S$.

Theorem 4. *Let S be a commutative semigroup, then the lattice $\mathscr{T}(S)$ is boolean if and only if S is one of the following:*

(i) *a zero semigroup;*

(ii) *a group G satisfying the following condition:*

(**) *G is a restricted direct product of cyclic groups of prime order such that no two different factors have the same order;*

(iii) *a group G with zero and G satisfies the condition (**).*

Proof. Using [6], p. 89, this can be proved analogously as in the proof of Theorem 3.

Corollary 2. *Let S be a commutative semigroup containing at least three elements. Then S is a group G satisfying the condition (**) if and only if the lattices $\mathscr{T}(S)$ and $\mathscr{C}(S)$ are boolean.*

The proof is analogous to the proof of Corollary 1. Note that if S is a zero semigroup with the boolean lattice $\mathscr{C}(S)$, then it follows from Theorem 19 of [1] that card $S \leqq 2$.

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