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# A GENERAL FORM OF THE PRODUCT INTEGRAL AND LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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## 0. INTRODUCTION

Let $f: [a, b] \to R$, $q \in R$. It is well known (see e.g. [1], [2]) that the following two conditions are equivalent:

(0.1) $f$ is Perron integrable, $q = \int_a^b f(t)\,dt$;

(0.2) for every $\varepsilon > 0$ there exists such a $\delta: [a, b] \to (0, \infty)$ that

$$\left|q - \sum_{i=1}^{k} f(t_i)(x_i - x_{i-1})\right| < \varepsilon$$

provided

$$a = x_0 \leqq t_1 \leqq x_1 \leqq \ldots \leqq x_{k-1} \leqq t_k \leqq x_k = b\,,$$
$$[x_{i-1}, x_i] \subset (t_i - \delta(t_i),\ t_i + \delta(t_i))\,.$$

Denote by $M_n$ the set of the $n \times n$ matrices. Let $A: [a, b] \to M_n$ be continuous and let $U: [a, b] \to M_n$ be the fundamental matrix of

(0.3) $$\dot{x} = A(t)\,x\,,$$

$U(a) = I$. It is well known that

(0.4) for every $\varepsilon > 0$ there exists such a $\delta > 0$ that

$$\|U(t) - [I + A(s_k)(y_k - y_{k-1})] \ldots [I + A(s_1)(y_1 - y_0)]\| < \varepsilon$$

provided

$$a = y_0 \leqq s_1 \leqq y_1 \leqq \ldots \leqq y_{k-1} \leqq s_k \leqq y_k = t \leqq b\,,$$
$$[y_{i-1}, y_i] \subset (s_i - \delta,\ s_i + \delta)$$

(cf. [3], [4]). An approach analogous to the Lebesgue and Lebesgue-Stieltjes integral can be found in [5].

Let $A: [a, b] \to M_n$, $Q \in M_n$ regular. The purpose of this paper is to examine the consequence of the following assumption:

(0.5) for every $\varepsilon > 0$ there exists $\delta: [a, b] \to (0, \infty)$ such that

$$\|Q - [I + A(t_k)(x_k - x_{k-1})] \ldots [I + A(t_1)(x_1 - x_0)]\| < \varepsilon$$

provided $x_0, t_0, x_1, \ldots, t_k, x_k$ satisfy the same conditions as in (0.2).

Under these conditions $Q$ is called the *product integral* and denoted by (PP) $\int_a^b (I + A(t)\,\mathrm{d}t)$ (PP stands for Perron product). Main result:

*The integral* (PP) $\int_a^b (I + A(t)\,\mathrm{d}t)$ *exists if and only if there exists a matrix function $U$ which is* $\mathrm{ACG}_*$ (see [6]) *on $[a, b]$, $U(a) = I$, $U(t)$ regular for $t \in [a, b]$, such that $\dot{U}(t) = A(t)\,U(t)$* a.e.; *in this case the integral exists for $s \in [a, b]$ and $U(s) =$ = (PP) $\int_a^s (I + A(t)\,\mathrm{d}t)$. In other words, the problem*

$$\dot{u} = A(t)\,u\,, \quad u(s) = y$$

*has a unique* $\mathrm{ACG}_*$ *solution on $[a, b]$ for every $s \in [a, b]$ and $y \in R^n$ if and only if* (PP) $\int_a^b (I + A(t)\,\mathrm{d}t)$ *exists.*

Incidentally, a new characterization of $\mathrm{ACG}_*$ functions was obtained (cf. (3.19)).

The same concept of the product integral results by replacing the product $[I + A(t_k)(x_k - x_{k-1})] \dots [I + A(t_1)(x_1 - x_0)]$ in (0.5) by $\exp[A(t_k)(x_k - x_{k-1})] \dots \exp[A(t_1)(x_1 - x_0)]$. In order to prove this fact, a more general setting is introduced in Section 1, namely, $I + A(t)(x - y)$ is replaced by a function $V$ of a point variable and an interval variable.

In Section 1 the values of $A$ may be e.g. linear bounded mappings of a Banach space, but in Section 2 and after it is essential that the underlying space is finite dimensional (cf. Lemma 2.2). In Section 2 an analogue of the Saks-Henstock Lemma (see e.g. [2]) is derived (cf. Theorem 2.4), which is then used repeatedly, especially in the examination of the differentiation properties of $U$ in Section 3. Applications to linear differential equations and some examples can be found in Section 4.

## 1. PRODUCT INTEGRAL: DEFINITION AND PROPERTIES

Let $n \in N$ and let $M_n$ be the set of all $n \times n$ matrices equipped with a norm $\|\cdot\|$. Denote $\mathscr{J} = \{[x, y];\ x \leqq y\}$ and, if $[a, b] \subset R$ is a compact interval, $\mathscr{J}_{ab} =$ $= \{[x, y] \in \mathscr{J};\ x, y \in [a, b]\}$. A partition of the interval $[a, b]$ is any ordered $k$-tuple of pairs of the form $\Delta = \{(t_i, J_i);\ t_i \in J_i = [x_{i-1}, x_i] \in \mathscr{J}_{ab}\}$, $i = 1, 2, \dots, k$, $x_0 = a$, $x_k = b$. Given a function $V\colon [a, b] \times \mathscr{J}_{ab} \to M_n$, we denote

$$P(V, \Delta) = V(t_k, J_k)\, V(t_{k-1}, J_{k-1}) \dots V(t_1, J_1)\,.$$

If $\delta\colon [a, b] \to (0, \infty)$ is a positive function (called *gauge*) then a partition $\Delta$ is said to be *$\delta$-fine* if

$$J_i \subset (t_i - \delta(t_i),\, t_i + \delta(t_i))\,.$$

**1.1. Definition.** A function $V\colon [a, b] \times \mathscr{J}_{ab} \to M_n$ is called *PP-integrable* (*Perron product integrable*) if there is a regular matrix $Q \in M_n$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ such that

$$\|Q - P(V, \Delta)\| \leqq \varepsilon \tag{1.0}$$

for every $\delta$-fine partition $\Delta$ of $[a, b]$. Then $Q$ is called the *PP-integral of $V$* and we write

$$Q = (\mathrm{PP}) \int_a^b V(t, \mathrm{d}t) .$$

Let us introduce the following notation for certain conditions concerning $V$:

(1.1) $V$ is PP-integrable and $(\mathrm{PP}) \int_a^b V(t, \mathrm{d}t) = Q$;

(1.2) for every $t \in [a, b]$ and $\zeta > 0$ there is $\sigma > 0$ such that

$$\|V(t, [x, y]) - I\| \leqq \zeta$$

for all $x, y \in [a, b]$, $t - \sigma < x \leqq t \leqq y < t + \sigma$.

($I$ stands for the unit matrix.)

If $A: [a, b] \to M_n$, then the functions

$$V_1(t, [x, y]) = I + A(t)(y - x) , \tag{1.3}$$

$$V_2(t, [x, y]) = \exp[A(t)(y - x)] \tag{1.4}$$

satisfy (1.2). It will be shown later (Theorem 2.12) that if one of the functions $V_1$, $V_2$ is PP-integrable then so is the other, and both the integrals coincide. If this is the case, then the function $A: [a, b] \to M_n$ will be said to be *PP-integrable*.

**1.2. Theorem.** *Let* (1.1), (1.2) *be fulfilled. Then there exists such* $K > 0$ *that for* $s \in (a, b)$ *the integrals on the left-hand side of*

$$(\mathrm{PP}) \int_s^b V(t, \mathrm{d}t)\, (\mathrm{PP}) \int_a^s V(t, \mathrm{d}t) = (\mathrm{PP}) \int_a^b V(t, \mathrm{d}t) \tag{1.5}$$

*exist, the equality holds and*

$$\|(\mathrm{PP}) \int_a^s V(t, \mathrm{d}t)\| \leqq K , \quad \|((\mathrm{PP}) \int_a^s V(t, \mathrm{d}t))^{-1}\| \leqq K .$$

Proof. Put $\varepsilon_0 = \frac{1}{2}\|Q^{-1}\|^{-1}$ and find a gauge $\delta_0$ such that

$$\|Q - P(V, \Delta)\| \leqq \varepsilon_0 \tag{1.6}$$

holds for every $\delta_0$-fine partition $\Delta$, and

$$\|V(t, [x, y]) - I\| \leqq \tfrac{1}{2} \tag{1.7}$$

for $t, x, y \in [a, b]$, $t - \delta_0(t) < x \leqq t \leqq y < t + \delta_0(t)$. As the first step we shall prove

(1.8) For every $t \in [a, b]$ there is $K_1(t) > 0$ such that

(i) if $s \in (t - \delta_0(t), t] \cap (a, b]$ and $\Delta_1$ is a $\delta_0$-fine partition of $[a, s]$ then $\max\{\|P(V, \Delta_1)\|, \|P(V, \Delta_1)^{-1}\|\} \leqq K_1(t)$;

(ii) if $s \in [t, t + \delta_0(t)) \cap [a, b)$ and $\Delta_2$ is a $\delta_0$-fine partition of $[s, b]$ then $\max\{\|P(V, \Delta_2)\|, \|P(V, \Delta_2)^{-1}\|\} \leqq K_1(t)$.

In order to prove (i) denote by $\Delta_3$ a $\delta_0$-fine partition of $[t, b]$. Set

$$\Delta = \Delta_1 \circ (t, [s, t]) \circ \Delta_3$$

where $\circ$ denotes the union of ordered finite sets in which the ordering of the resulting set is given by the order of factors on the right-hand side. Then $\Delta$ is a $\delta_0$-fine partition of $[a, b]$ and consequently, (1.6) holds. Since $P(V, \Delta) = P(V, \Delta_3)\, V(t, [s, t])\, P(V, \Delta_1)$ we have

$$\|V(t, [s, t])^{-1} P(V, \Delta_3)^{-1} Q - P(V, \Delta_1)\| \leqq$$
$$\leqq \varepsilon_0 \|V(t, [s, t])^{-1}\| \, \|P(V, \Delta_3)^{-1}\| ,$$
$$\|P(V, \Delta_1)^{-1} - Q^{-1} P(V, \Delta_3)\, V(t, [s, t])\| \leqq$$
$$\leqq \varepsilon_0 \|Q^{-1}\| \, \|P(V, \Delta_1)^{-1}\| = \tfrac{1}{2} \|P(V, \Delta_1)^{-1}\| .$$

Taking into account (1.7), we find that (i) follows. (ii) is proved analogously.

As the second step we will prove

(1.9) For every $t \in [a, b]$ there is $K_2(t)$ such that $\max \{\|P(V, \Delta_1)\|, \|P(V, \Delta_1)^{-1}\|, \|P(V, \Delta_2)\|, \|P(V, \Delta_2)^{-1}\|\} \leqq K_2(t)$ provided $s \in (t - \delta_0(t),\, t + \delta_0(t)) \cap [a, b]$ and $\Delta_1, \Delta_2$ are $\delta_0$-fine partitions of $[a, s]$ and $[s, b]$, respectively.

In order to prove that the desired inequality holds for the first two norms in the case $s \in [t, t + \delta_0(t))$ set $\Delta = \Delta_1 \circ \Delta_2$. Since $P(V, \Delta) = P(V, \Delta_2)\, P(V, \Delta_1)$ and $\Delta$ is a $\delta_0$-fine partition of $[a, b]$, (1.6) implies the inequalities

$$\|P(V, \Delta_2)^{-1} Q - P(V, \Delta_1)\| \leqq \varepsilon_0 \|P(V, \Delta_2)^{-1}\| ,$$
$$\|P(V, \Delta_1)^{-1} - Q^{-1} P(V, \Delta_2)\| \leqq \varepsilon_0 \|Q^{-1}\| \, \|P(V, \Delta_1)^{-1}\| = \tfrac{1}{2} \|P(V, \Delta_1)^{-1}\| .$$

Since by (1.8) (ii) both $\|P(V, \Delta_2)\|$, $\|P(V, \Delta_2)^{-1}\|$ are bounded independently of $\Delta_2$, we conclude that $\|P(V, \Delta_1)\|$, $\|P(V, \Delta_1)^{-1}\|$ are bounded. The rest of (1.9) follows by an analogous argument.

As the third step, (1.10) is obtained from (1.9) by a compactness argument:

(1.10) There is a constant $K \geqq 1$ such that

(i) if $s \in (a, b]$ and $\Delta_1$ is a $\delta_0$-fine partition of $[a, s]$, then
$$\max \{\|P(V, \Delta_1)\|, \|P(V, \Delta_1)^{-1}\|\} \leqq K;$$
(ii) if $s \in [a, b)$ and $\Delta_2$ is a $\delta_0$-fine partition of $[s, b]$, then
$$\max \{\|P(V, \Delta_2)\|, \|P(V, \Delta_2)^{-1}\|\} \leqq K.$$

As the fourth step we will prove

(1.11) Let $0 < \varepsilon \leqq \varepsilon_0$ and let $\delta$ correspond to $\varepsilon$ so that (1.0) holds for every $\delta$-fine partition $\Delta$ of $[a, b]$ and that $\delta(t) \leqq \delta_0(t)$ for $t \in [a, b]$. Let $s \in (a, b]$ and let $\Delta_1, \Delta_3$ be $\delta$-fine partitions of $[a, s]$. Then

(*) $\|P(V, \Delta_1) - P(V, \Delta_3)\| \leqq 2K\varepsilon$

with the constant $K$ from (1.10).

In order to prove (1.11) denote by $\Delta_2$ a $\delta$-fine partition of $[s, b]$ and put $\Delta_4 = \Delta_1 \circ \Delta_2$, $\Delta_5 = \Delta_3 \circ \Delta_2$. Then $\Delta_4, \Delta_5$ are $\delta$-fine partitions of $[a, b]$, hence (1.0) holds with $\Delta$ replaced by $\Delta_4$ or $\Delta_5$, which yields

$$\|P(V, \Delta_2)\, P(V, \Delta_1) - P(V, \Delta_2)\, P(V, \Delta_3)\| \leqq 2\varepsilon$$

and (*) holds by (1.10) (ii).

Fifth step: Theorem 1.2 follows from (1.11), from an analogous assertion for partitions of $[s, b]$, and from (1.10).

**1.3. Remark.** The converse of Theorem 1.2 also holds. In detail: If (1.2) holds and if the integrals on the left-hand side of (1.5) exist for some $s \in (a, b)$, then the integral on the right-hand side exists as well and (1.5) holds.

Indication of proof. Let $\varepsilon > 0$ and let $\delta_1(\delta_2)$ correspond to $\varepsilon$ and (PP) $\int_a^s V(t, \mathrm{d}t)$ ((PP) $\int_s^b V(t, \mathrm{d}t)$) according to (1.0). Let $\delta$ be a gauge on $[a\ b]$, satisfying $\delta(u) \leqq \delta_1(u)$, $u + \delta(u) < s$ for $u < s$, $\delta(s) \leqq \min(\delta_1(s), \delta_2(s))$, $\delta(u) \leqq \delta_2(u)$, $u - \delta(u) > s$ for $u > s$. Let $\Delta = \{(t_i, [x_{i-1}, x_i]);\ i = 1, 2, \ldots, k\}$ be a $\delta$-fine partition of $[a, b]$. Then there exists such a $j$ that $t_j = s$. If $x_{j-1} < t_j < x_j$, put

$$\Delta_1 = \{(t_1, [x_0, x_1]), (t_2, [x_1, x_2]), \ldots, (t_j, [x_{j-1}, t_j])\},$$
$$\Delta_2 = \{(t_j, [t_j, x_j]), (t_{j+1}, [x_j, x_{j+1}]), \ldots, (t_k, [x_{k-1}, x_k])\}.$$

If $\delta(s)$ is small, then $\|P(V, \Delta) - P(V, \Delta_2)\, P(V, \Delta_1)\|$ is small and moreover, $P(V, \Delta_1)$ is close to (PP) $\int_a^s V(t, \mathrm{d}t)$, $P(V, \Delta_2)$ is close to (PP) $\int_s^b V(t, \mathrm{d}t)$.

## 2. EQUIVALENT INTEGRABLE FUNCTIONS

Let $L \geqq 1$ satisfy the following condition:

$$L^{-1} \max_{l,m} |Z_{lm}| \leqq \|Z\| \leqq L \max_{l,m} |Z_{lm}|$$

for every $Z \in M_n$, $Z = (Z_{lm})$.

We first establish two lemmas concerning products of matrices.

**2.1. Lemma.** *Let* $Y_1, Y_2, \ldots, Y_k \in M_n$, $\sum_{i=1}^{k} \|Y_i\| \leqq 1$, $X = (I + Y_k)(I + Y_{k-1}) \ldots \ldots (I + Y_1) - I - \sum_{i=1}^{k} Y_i$.

*Then*

$$\|X\| \leqq \left(\sum_{i=1}^{k} \|Y_i\|\right)^2.$$

Proof. Put $\lambda_i = \|Y_i\|$, $\lambda = \sum_{i=1}^{k} \lambda_i$. Since $0 \leqq \lambda \leqq 1$, we have

$$(1 + \lambda_k)(1 + \lambda_{k-1}) \ldots (1 + \lambda_1) - 1 - \lambda_1 - \ldots - \lambda_k \leqq \mathrm{e}^\lambda - 1 - \lambda \leqq \lambda^2$$

and writing

$$X = \sum_{j_2 > j_1} Y_{j_2} Y_{j_1} + \sum_{j_3 > j_2 > j_1} Y_{j_3} Y_{j_2} Y_{j_1} + \ldots + Y_k Y_{k-1} \ldots Y_2 Y_1$$

we see that

$$\|X\| \leqq \sum_{j_2 > j_1} \lambda_{j_2} \lambda_{j_1} + \sum_{j_3 > j_2 > j_1} \lambda_{j_3} \lambda_{j_2} \lambda_{j_1} + \ldots + \lambda_k \lambda_{k-1} \ldots \lambda_2 \lambda_1 \leqq \lambda^2.$$

**2.2. Lemma.** *Let* $0 < \vartheta < \frac{1}{9} L^{-4}$, $Z_1, Z_2, \ldots, Z_r \in M_n$. *For every p-tuple of numbers*

$\{j_1, j_2, \ldots, j_p\} \subset \{1, 2, \ldots, r\}$, $j_1 < j_2 < \ldots < j_p$ *let the inequality*

(2.1) $$\|(I + Z_{j_p})(I + Z_{j_{p-1}}) \ldots (I + Z_{j_1}) - I\| \leqq \vartheta$$

*hold. Then*

(2.2) $$\sum_{j=1}^{r} \|Z_j\| \leqq 4n^2 L^2 \vartheta .$$

Proof. (2.1) evidently yields

$$\|Z_j\| \leqq \vartheta \quad \text{for} \quad j = 1, 2, \ldots, r .$$

Let us write $Z_j = (Z_{j;l,m})$ nad denote by $\varphi(j)$, $\psi(j)$ such numbers from $\{1, 2, \ldots, n\}$ that

$$|Z_{j;\varphi(j),\psi(j)}| = \max_{l,m} |Z_{j;l,m}| ,$$

and by $J(l, m)$ the set

$$J(l, m) = \{j \in \{1, 2, \ldots, r\};\ \varphi(j) = l,\ \psi(j) = m\} .$$

Assume that (2.2) is not valid. Then there exist numbers $l, m$ such that (writing $J(l, m) = J$) we have

$$\sum_{j \in J} \|Z_j\| > 4L^2 \vartheta$$

and consequently,

$$\sum_{j \in J} |Z_{j;l,m}| > 4L\vartheta .$$

Denote $J_1 = \{j \in J = J(l, m);\ Z_{j;l,m} \geqq 0\}$, $J_2 = J \setminus J_1$. Then we have at least one of the inequalities

$$\sum_{j \in J_1} Z_{j;l,m} > 2L\vartheta , \quad -\sum_{j \in J_2} Z_{j;l,m} > 2L\vartheta .$$

For definiteness assume that the first of these inequalities holds. Since $\|Z_j\| \leqq \vartheta$, we have

$$Z_{j;l,m} \leqq L\vartheta \quad \text{for} \quad j \in J_1$$

and there is a set $J_1^* \subset J_1$ such that

(2.3) $$2L\vartheta < \sum_{j \in J_1^*} Z_{j;l,m} \leqq 3L\vartheta .$$

Hence

$$\sum_{j \in J_1^*} \|Z_j\| \leqq 3L^2 \vartheta \leqq 1$$

and the family $\{Z_j;\ j \in J_1^*\}$ satisfies the assumption of Lemma 2.1. Hence

(2.4) $$\prod_{j \in J_1^*} (I + Z_j) - I = \sum_{j \in J_1^*} Z_j + X ,$$

$$\|X\| \leqq \Big(\sum_{j \in J_1^*} \|Z_j\|\Big)^2 \leqq 9L^4 \vartheta^2 .$$

Moreover, by (2.1) the norm of the left-hand side in (2.4) does not exceed $\vartheta$, which yields

$$\Big\|\sum_{j \in J_1^*} Z_j\Big\| \leqq \vartheta + 9L^4 \vartheta^2 .$$

However, (2.3) together with the inequality for the norm yields

$$\|\sum_{j\in J_1^*} Z_j\| \geqq L^{-1} \max_{l,m} \sum_{j\in J_1^*} Z_{j;l,m} > 2\vartheta\,,$$

which combined with the previous inequality yields

$$2\vartheta < \vartheta + 9L^4\vartheta^2\,,$$

that is,

$$\vartheta > \tfrac{1}{9}L^{-4}$$

which contradicts the assumption. Hence (2.2) holds.

**2.3. Lemma.** *Let* (1.1), (1.2) *be fulfilled and let* $U\colon [a, b] \to M_n$ *be defined by*

$$U(t) = (\mathrm{PP}) \int_a^t V(s, \mathrm{d}s)\,, \quad U(a) = I\,.$$

*Let* $\varepsilon > 0$ *and find a gauge* $\delta$ *as in Definition* 1.1 *so that* (1.0) *holds for any* $\delta$*-fine partition* $\Delta$ *of* $[a, b]$. *Let*

$$(2.5)\qquad a = \eta_0 \leqq \xi_1 \leqq \tau_1 \leqq \eta_1 \leqq \xi_2 \leqq \ldots \leqq \eta_{r-1} \leqq \xi_r \leqq \tau_r \leqq \eta_r \leqq \xi_{r+1} = b\,,$$

$$[\xi_j, \eta_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j))$$

*and put*

$$U^{-1}(\eta_j)\, V(\tau_j, [\xi_j, \eta_j])\, U(\xi_j) = I + Z_j\,.$$

*Then*

$$(2.6)\qquad \|(I + Z_r)(I + Z_{r-1}) \ldots (I + Z_1) - I\| \leqq \|Q^{-1}\|\,\varepsilon\,.$$

Proof. Choose $\omega > 0$ and find gauges $\delta_j$ on $[\eta_j, \xi_{j+1}]$, $j = 0, 1, \ldots, r$, such that $\delta_j(t) \leqq \delta(t)$ and that

$$(2.7)\qquad \|P(V, \Delta_j) - U(\xi_{j+1})\, U^{-1}(\eta_j)\| < \omega$$

for every $\delta_j$-fine partition $\Delta_j$ of $[\eta_j, \xi_{j+1}]$, $j = 0, 1, \ldots, r$.
(Notice that $U(\xi_{j+1})\, U^{-1}(\eta_j) = (\mathrm{PP}) \int_{\eta_j}^{\xi_{j+1}} V(t, \mathrm{d}t)$.)

If $\Delta_j$ are $\delta_j$-fine partitions of $\eta_j, \xi_{j+1}$ then $\Delta = \Delta_0 \circ (\tau_1, [\xi_1, \eta_1]) \circ \Delta_1 \circ \ldots \ldots \circ \Delta_{r-1} \circ (\tau_r, [\xi_r, \eta_r]) \circ \Delta_r$ is a $\delta$-fine partition of $[a, b]$. Consequently

$$\|Q - P(V, \Delta_r)\, V(\tau_r, [\xi_r, \eta_r])\, P(V, \Delta_{r-1}) \ldots P(V, \Delta_1)\, V(\tau_1, [\xi_1, \eta_1])\, P(V, \Delta_0)\| < \varepsilon\,,$$

that is

$$\begin{aligned}(2.8)\qquad &\|I - U^{-1}(b)\, P(V, \Delta_r)\, U(\eta_r)\, U^{-1}(\eta_r)\, V(\tau_r, [\xi_r, \eta_r])\,.\\ &\quad .\, U(\xi_r)\, U^{-1}(\xi_r)\, P(V, \Delta_{r-1})\, U(\eta_{r-1})\, U^{-1}(\eta_{r-1}) \ldots\\ &\ldots U(\xi_2)\, U^{-1}(\xi_2)\, P(V, \Delta_1)\, U(\eta_1)\, U^{-1}(\eta_1)\, V(\tau_1, [\xi_1, \eta_1])\,.\\ &\quad .\, U(\xi_1)\, U^{-1}(\xi_1)\, P(V, \Delta_0)\| \leqq \|Q^{-1}\|\,\varepsilon\,.\end{aligned}$$

Put

$$U^{-1}(\xi_{j+1})\, P(V, \Delta_j)\, U(\eta_j) = I + W_j\,, \quad j = 0, 1, \ldots, r\,.$$

Then writing (2.7) in the form

$$\|U^{-1}(\xi_{j+1})\, P(V, \Delta_j)\, U(\eta_j) - I\| \leqq \|U^{-1}(\xi_{j+1})\|\, \|U(\eta_j)\|\, \omega$$

and applying Theorem 1.2 we obtain

$$(2.9) \qquad \|W_j\| \leqq K^2\omega .$$

Using the notation introduced above we rewrite (2.8) as

$$\|I - (I + W_r)(I + Z_r)(I + W_{r-1}) \dots (I + W_1)(I + Z_1)(I + W_0)\| \leqq \|Q^{-1}\| \varepsilon$$

and since $\omega > 0$ was arbitrary, (2.9) implies (2.6).

**2.4. Theorem.** *Let* (1.1), (1.2) *be fulfilled, let* $\varepsilon > 0$ *satisfy* $\|Q^{-1}\| \varepsilon < \frac{1}{9}L^{-4}$. *Let* $\xi_j, \tau_j, \eta_j, Z_j, K$ *be the same as in Lemma* 2.3. *Then*

$$(2.10) \qquad \sum_{j=1}^{r} \|Z_j\| \leqq K_3\varepsilon , \quad \textit{where} \quad K_3 = 4n^2L^2\|Q^{-1}\| ,$$

*and*

$$(2.11) \quad \sum_{j=1}^{r} \|V(\tau_j, [\xi_j, \eta_j]) - (\mathrm{PP}) \textstyle\int_{\xi_j}^{\eta_j} V(t, \mathrm{d}t)\| \leqq K_4\varepsilon , \quad \textit{where} \quad K_4 = K^2K_3 .$$

Proof. (2.10) follows from Lemmas 2.3 and 2.2; (2.11) follows from (2.10) by Theorem 1.2 since

$$(\mathrm{PP}) \int_{\xi_j}^{\eta_j} V(t, \mathrm{d}t) = U(\eta_j)\, U^{-1}(\xi_j) .$$

**2.5.** Remark. Theorem 2.4 is the product-integral version of the Saks-Henstock lemma, cf. e.g. [2].

**2.6. Theorem.** *Let* (1.1), (1.2) *be fulfilled, let* $U$ *be defined as in Lemma* 2.3. *Then* $U$ *is continuous.*

Proof. Let $\varepsilon_0$, $\delta_0$ be the same as in Theorem 1.2. Given $\varepsilon$, $0 < \varepsilon < \varepsilon_0$, find the corresponding gauge $\delta$ so that (1.0) holds for every $\delta$-fine partition $\Delta$ of $[a, b]$. Let $s \in [a, b]$. By (1.2) there is $\sigma > 0$ such that

$$\|V(s, [x, y]) - I\| < \varepsilon$$

provided $s - \sigma < x \leqq s \leqq y < s + \sigma$. Let $t \in (a, b)$ satisfy $s - \min(\sigma, \delta(s)) < t < s$ and let $\Delta_1$ be a $\delta$-fine partition of $[a, t]$, $\Delta_2 = \Delta_1 \circ (s, [t, s])$. Then $\|U(t) - P(V, \Delta_1)\| \leqq 2K\varepsilon$, $\|U(s) - P(V, \Delta_2)\| \leqq 2K\varepsilon$ (cf. Theorem 1.2). On the other hand, $P(V, \Delta_2) = V(s, [t, s])\, P(V, \Delta_1)$, hence

$$\|P(V, \Delta_2) - P(V, \Delta_1)\| \leqq \|V(s, [t, s]) - I\| \, \|P(V, \Delta_1)\| \leqq K\varepsilon$$

and finally,

$$\|U(t) - U(s)\| \leqq 5K\varepsilon .$$

The proof for $s < t < s + \min(\sigma, \delta(s))$ is analogous.

**2.7. Theorem.** *Let* $K_5 > 0$ *and let a function* $W\colon [a, b] \to M$ *satisfy*

$$(2.12) \qquad \max\{\|W(t)\|, \|W^{-1}(t)\|\} \leqq K_5$$

*for* $t \in [a, b]$. *Let* $V\colon [a, b] \times J_{ab} \to M$. *For every* $\vartheta > 0$ *let there exist a gauge* $\delta$

*on* $[a, b]$ *such that*

$$\sum_{j=1}^{k} \|V(t_j, [x_{j-1}, x_j]) - W(x_j) W^{-1}(x_{j-1})\| \leqq \vartheta \tag{2.13}$$

*for every* $\delta$*-fine partition* $\Delta$ *of* $[a, b]$.

*Then the integral* (PP) $\int_a^b V(t, \mathrm{d}t)$ *exists and is equal to* $W(b)\, W^{-1}(a)$.

Proof. Let $0 < \vartheta \leqq K_5^{-2}$. In Lemma 2.1 set $Y_j = W^{-1}(x_j)\, V(t_j, [x_{j-1}, x_j])\,.$ $.\, W(x_{j-1}) - I$. Then using (2.12) and (2.13) we see that the assumption of the lemma is fulfilled and its assertion reads

$$\|W^{-1}(x_k)\, V(t_k, [x_{k-1}, x_k])\, W(x_{k-1})\, W^{-1}(x_{k-1})\, V(t_{k-1}, [x_{k-2}, x_{k-1}]) \ldots .$$
$$\ldots V(t_1, [x_0, x_1])\, W(x_0) - I\| \leqq \sum_{i=1}^{k} \|Y_i\| + \Big(\sum_{i=1}^{k} \|Y_i\|\Big)^2 .$$

Recalling that $\sum_{i=1}^{k} \|Y_i\| \leqq K_5^2 \vartheta$, $x_k = b$, $x_0 = a$ and using (2.12) we conclude

$$\|V(t_k, [x_{k-1}, x_k])\, V(t_{k-1}, [x_{k-2}, x_{k-1}]) \ldots$$
$$\ldots V(t_1, [x_0, x_1] - W(b)\, W^{-1}(a)\| \leqq K_5^4 \vartheta + K_5^6 \vartheta^2$$

which proves the theorem.

**2.8. Definition.** Functions $V_i\colon [a, b] \times J_{a,b} \to M$, $i = 1, 2$, are called *equivalent*, notation $V_1 \sim V_2$, if for every $\varepsilon > 0$ there is a gauge $\delta$ such that

$$\sum_{j=1}^{k} \|V_1(t_j, [x_{j-1}, x_j]) - V_2(t_j, [x_{j-1}, x_j])\| < \varepsilon \tag{2.14}$$

for every $\delta$-fine partition $\Delta$ of $[a, b]$.

**2.9. Theorem.** *Let* (1.1), (1.2) *be fulfilled and let* $V \sim V_2$. *Then the integral on the left-hand side of*

$$\text{(PP)} \int_a^b V_2(t, \mathrm{d}t) = \text{(PP)} \int_a^b V(t, \mathrm{d}t) \tag{2.15}$$

*exists and* (2.15) *holds.*

Proof. Let $U$ be defined as in Lemma 2.3. Chose $\varepsilon > 0$ and find a gauge $\delta$ on $[a, b]$ such that (1.0) and (2.14) (with $V_1$ replaced by $V$) hold for every $\delta$-fine partition of $[a, b]$. By Theorem 2.4, (2.11) holds, and combining (2.11) and (2.14) we conclude

$$\sum_{j=1}^{k} \|V_2(t_j, [x_{j-1}, x_j]) - U(x_j)\, U^{-1}(x_{j-1})\| \leqq (K_4 + 1)\, \varepsilon .$$

The proof is now completed by Theorem 2.7 in which we put $W(t) = U(t)$.

**2.10.** Remark. The converse of Theorem 2.9 holds in the following sense: If (PP) $\int_a^t V(s, \mathrm{d}s) =$ (PP) $\int_a^t V_2(s, \mathrm{d}s)$ holds for $t \in [a, b]$, then $V \sim V_2$. This follows immediately from Theorem 2.7.

**2.11. Lemma.** *Let* $V_i\colon [a, b] \times \mathscr{J}_{a,b} \to M$, $i = 1, 2$, *satisfy the following con-*

*ditions*:

(i) *there is a set* $E \in [a, b]$, $m(E) = 0$, *and functions* $\delta_0, \varrho: E \to (0, \infty)$ *such that*
$$\|V_1(t, [x, y]) - V_2(t, [x, y])\| \leqq \varrho(t)(y - x)$$
*for* $t \in E$, $t - \delta_0(t) < x \leqq t \leqq y < t + \delta_0(t)$;

(ii) *for every* $\varepsilon > 0$ *there is a gauge* $\tilde{\delta}$ *on* $[a, b]$ *such that*
$$\|V_1(t, [x, y]) - V_2(t, [x, y])\| \leqq \varepsilon(y - x)$$
*for* $t \in [a, b] \setminus E$, $t - \tilde{\delta}(t) < x \leqq t \leqq y < t + \tilde{\delta}(t)$.

*Then* $V_1 \sim V_2$.

Proof. Choose $\varepsilon > 0$ and let $\tilde{\delta}$ be the function from (ii). Put $E_i = \{t \in E; i - 1 < \varrho(t) \leqq i\}$, $i = 1, 2, \ldots$. Since $m(E_i) = 0$, we find $\delta_i: E_i \to (0, \infty)$ such that $\delta_i(t) \leqq \delta_0(t)$,
$$m\Big(\bigcup_{t \in E_i} (t - \delta_i(t), t + \delta_i(t))\Big) < \frac{\varepsilon}{i\, 2^i}.$$
Put
$$\delta(t) = \tilde{\delta}(t) \quad \text{for} \quad t \in [a, b] \setminus E\,,$$
$$\delta(t) = \delta_i(t) \quad \text{for} \quad t \in E_i\,, \quad i = 1, 2, \ldots.$$
Then for any $\delta$-fine partition $\Delta$ of $[a, b]$ the sum in (2.14) splits into two parts, one with $t_j \in E_i$ for some $i$ and the other with $t_j \notin E$. The former is estimated by
$$\sum_{i=1}^{\infty} \sum_{t_j \in E_i} \varrho(t_j)(x_j - x_{j-1}) \leqq \sum_{i=1}^{\infty} i \frac{\varepsilon}{i\, 2^i} = \varepsilon$$
while the latter does not exceed $\sum_{t_j \notin E} \varepsilon(x_j - x_{j-1}) \leqq (b - a)\varepsilon$. Hence (2.14) holds with $(1 + b - a)\varepsilon$ on the right-hand side, which completes the proof.

**2.12. Theorem.** *Let* $A: [a, b] \to M_n$ *and put*
$$V_1(t, [x, y]) = I + A(t)(y - x)\,,$$
$$V_2(t, [x, y]) = \exp[A(t)(y - x)]$$
*for* $t \in [a, b]$, $a \leqq x \leqq y \leqq b$. *If one of the integrals in*

$$\text{(PP)} \int_a^b (I + A(t)\, \mathrm{d}t) = \text{(PP)} \int_a^b \exp(A(t)\, \mathrm{d}t) \tag{2.16}$$

*exists then the other exists as well and* (2.16) *holds*.

Proof. The functions $V_1$, $V_2$ are equivalent by Lemma 2.11 ($E = \emptyset$). Since (1.2) is fulfilled for both $V_1$, $V_2$, the assertion follows via Theorem 2.9.

**2.13. Theorem.** *Let* $A_i: [a, b] \to M_n$, $i = 1, 2$, $A_1(t) = A_2(t)$ *for almost all* $t \in [a, b]$. *If one of the integrals in*

$$\text{(PP)} \int_a^b (I + A_1(t)\, \mathrm{d}t) = \text{(PP)} \int_a^b (I + A_2(t)\, \mathrm{d}t) \tag{2.17}$$

*exists then the other exists as well and* (2.17) *holds*.

Proof is analogous to that of Theorem 2.12.

## 3. DERIVATIVE OF THE PRODUCT INTEGRAL WITH RESPECT TO THE UPPER LIMIT

**3.1. Theorem.** *Let* (1.1), (1.2) *be fulfilled and let* $U$ *have the meaning from Lemma* 2.3. *Then there exists a set* $T \subset [a, b]$ *with* $m(T) = b - a$ *and for every* $\varepsilon > 0$, $t \in T$ *there is* $\vartheta > 0$ *such that*

$$\|V(t, [x, y]) - U(y)\, U^{-1}(x)\| \leqq \varepsilon(y - x) \tag{3.1}$$

*provided* $t - \vartheta < x \leqq t \leqq y < t + \vartheta$, $x, y \in [a, b]$.

Proof. Let $T$ be the set of $t \in [a, b]$ that (3.1) holds and denote $E = [a, b] \setminus T$. Denote by $E_r$ the set of such $t \in [a, b]$ that there exists a sequence $x_l = x_l(t)$, $y_l = y_l(t)$ satisfying $x_l \leqq t \leqq y_l$, $x_l < y_l$, $y_l - x_l \to 0$ as $l \to \infty$ and

$$\|V(t, [x_l, y_l]) - U(y_l)\, U^{-1}(x_l)\| \geqq r^{-1}(y - x)\,. \tag{3.2}$$

Then evidently $E = \bigcup_{r=1}^{\infty} E_r$. Assume that $m_e(E) > 0$ ($m_e$ denotes the exterior measure). Then there is $r$ such that $m_e(E_r) > 0$. Let $\varepsilon > 0$ be such that $\|Q^{-1}\|\, \varepsilon < \frac{1}{9}L^{-4}$, $K_4\varepsilon < \frac{1}{2}r^{-1}\, m_e(E_r)$ ($L, K_4$ were introduced in Theorem 2.4), and find $\delta$ according to Definition 1.1 so that (1.0) holds for every $\delta$-fine partition $\Delta$ of $[a, b]$. For every $t$ find $l_0 = l_0(t)$ such that

$$t - x_l(t) < \delta(t)\,, \quad y_l(t) - t < \delta(t)$$

for all $l \geqq l_0$.

The system $\{[x_1(t), y_1(t)];\ t \in E,\ l \geqq l_0(t)\}$ covers the set $E$ in the sense of Vitali. Therefore there is a finite subsystem

$$\{[\xi_j, \eta_j];\ j = 1, 2, \ldots, s\}$$

such that $\tau_j - \delta(\tau_j) < \xi_j \leqq \tau_j \leqq \eta_j < \tau_j + \delta(\tau_j)$, $\eta_j \leqq \xi_{j+1}$, so that

$$m_e(E \setminus \bigcup_{j=1}^{s} [\xi_j, \eta_j]) < \tfrac{1}{2}\, m_e(E_r)\,.$$

Consequently,

$$\sum_{j=1}^{s} (\eta_j - \xi_j) \geqq m_e(E \cap \bigcup_{j=1}^{s} [\xi_j, \eta_j]) > \tfrac{1}{2}\, m_e(E_r)$$

and (3.2) yields

$$\sum_{j=1}^{s} \|V(\tau_j, [\xi_j, \eta_j]) - U(\eta_j)\, U^{-1}(\xi_j)\| > r^{-1} \sum_{j=1}^{s} (\eta_j - \xi_j) >$$
$$> \tfrac{1}{2}r^{-1}\, m_e(E_r) > K_4\varepsilon$$

which contradicts (2.11) from Theorem 2.4.

As an easy consequence of Theorem 3.1 we obtain

**3.2. Theorem.** *Let* $A: [a, b] \to M_n$,

$$V(t, [x, y]) = I + A(t)\,(y - x) \tag{3.3}$$

*and let* (1.1) *hold. Then the derivative* $\dot{U}$ *exists and*

$$\dot{U}(t)\, U^{-1}(t) = A(t) \quad \textit{for a.e.} \quad t \in [a, b]\,, \tag{3.4}$$

*where* $U$ *has the meaning from Lemma* 2.3.

**3.3. Theorem.** *Let* (3.3), (1.1) *be fulfilled and let* $U$ *have the meaning from Lemma* 2.3.

*Then* $U$ *satisfies the following condition*:

(3.5) *Let* $\sigma > 0$, $C \subset [a, b]$, $m(C) = 0$. *Then there exists* $\delta^*: C \to (0, \infty)$ *such that*

$$\sum_{j=1}^{r} \|U(\eta_j) - U(\xi_j)\| \leqq \sigma$$

*provided* $\tau_j \in C$, $\tau_j - \delta^*(\tau_j) < \xi_j \leqq \tau_j \leqq \eta_j < \tau_j + \delta^*(\tau_j)$, $[\xi_j, \eta_j] \subset [a, b]$ *for* $j = 1, 2, \ldots, r$, $\eta_j \leqq \xi_{j+1}$ *for* $j = 1, 2, \ldots, r - 1$.

Proof. Put $C_i = \{t \in C;\ i - 1 \leqq \|A(t)\| < i\}$, $i = 1, 2, \ldots$. There exist functions $\delta_i: C_i \to (0, \infty)$ such that

$$m\big(\bigcup_{t \in C_i} (t - \delta_i(t), t + \delta_i(t))\big) \leqq \frac{\sigma}{i\, 2^{i+1} K}\,. \tag{3.6}$$

Let $\|Q^{-1}\|\, \varepsilon < \frac{1}{9} L^{-4}$, $2KK_4\varepsilon < \sigma$ (for $K$ see Lemma 2.3, for $L$, $K_4$ see Theorem 2.4). Find $\delta$ according to Definition 1.1 so that (1.0) holds for every $\delta$-fine partition $\Delta$ of $[a, b]$. Without loss of generality we may suppose $\delta(t) \leqq \delta_i(t)$ for $t \in C_i$, $i = 1, 2, \ldots$. Then we can write the inequality (2.11) from Theorem 2.4 in the form

$$\sum_{j=1}^{r} \|I + A(t)\,(\eta_j - \xi_j) - U(\eta_j)\, U^{-1}(\xi_j)\| \leqq K_4 \varepsilon\,. \tag{3.7}$$

By (3.6) and the definition of $C_i$ we have

$$\sum_{j=1}^{r} \|A(\tau_j)\,(\eta_j - \xi_j)\| =$$

$$= \sum_{i=1}^{\infty} \sum_{\tau_j \in C_i} \|A(\tau_j)\,(\eta_j - \xi_j)\| \leqq \sum_{i=1}^{\infty} i\, \frac{\sigma}{i\, 2^{i+1} K} = \frac{\sigma}{2K}\,,$$

and (3.7) yields

$$\sum_{j=1}^{r} \|I - U(\eta_j)\, U^{-1}(\xi_j)\| \leqq K_4 \varepsilon + \frac{\sigma}{2K}\,,$$

hence

$$\sum_{j=1}^{r} \|U(\eta_j) - U(\xi_j)\| \leqq KK_4 \varepsilon + \tfrac{1}{2}\sigma < \sigma\,.$$

**3.4. Theorem.** *Let* $U: [a, b] \to M_n$ *satisfy* $U(a) = I$, *let* $U$ *be regular with* $\|U^{-1}(t)\| \leqq K$ *for* $t \in [a, b]$. *Further, let* $U$ *satisfy* (3.5) *and the condition*

(3.8) *the derivative* $\dot{U}(t)$ *exists for almost all* $t \in [a, b]$.

*Set* $A(t) = \dot{U}(t)\, U^{-1}(t)$ *if* $\dot{U}(t)$ *exists,* $A(t)$ *arbitrary elsewhere. Then the integral in*

(3.9) $(\mathrm{PP}) \int_a^b (I + A(t)\,\mathrm{d}t) = U(b)$

*exists and* (3.9) *holds.*

Proof. Let $\vartheta > 0$ and let $T$ be the set of $t \in [a, b]$ for which $\dot{U}(t)$ exists. Put $C = [a, b] \setminus T$, $\sigma = \frac{1}{4}\vartheta K^{-1}$. Find $\delta^*$ from (3.5). Since $m(C) = 0$, there is $\delta^0 \colon C \to (0, \infty)$, $\delta^0(t) \leqq \delta^*(t)$, such that

$$\sum_{j=1}^{r} \|A(\tau_j)\| (\eta_j - \xi_j) \leqq \sigma \tag{3.10}$$

provided $\tau_j, \xi_j, \eta_j$ satisfy the conditions from (3.5) with $\delta^0$ instead of $\delta^*$ (this can be shown in the same way as the analogous inequality in the proof of Theorem 3.3).

Further, there is $\tilde{\delta} \colon [a, b] \to (0, \infty)$ such that

$$\|A(t)\, U(t)\, (y - x) - U(y) + U(x)\| \leqq \frac{y - x}{b - a}\, \sigma\,, \tag{3.11}$$

$$\|U(t) - U(x)\| \leqq \frac{\sigma}{(b - a)\,(\|A(t)\| + 1)} \tag{3.12}$$

for $t \in T$, $t - \tilde{\delta}(t) < x \leqq t \leqq y < t + \tilde{\delta}(t)$, $[x, y] \subset [a, b]$, $x < y$. Let $\delta \colon [a, b] \to (0, \infty)$ satisfy

$$\delta(t) \leqq \tilde{\delta}(t) \quad \text{for} \quad t \in [a, b]\,,$$
$$\delta(t) \leqq \min\,(\delta^*(t), \delta^0(t)) \quad \text{for} \quad t \in C\,,$$

and let $\Delta$ be a $\delta$-fine partition of $[a, b]$. Then

$$\sum_{j=1}^{r} \|I + A(t_j)\,(x_j - x_{j-1}) - U(x_j)\, U^{-1}(x_{j-1})\| \leqq$$
$$\leqq \sum_{t_j \in C} \|A(t_j)\|\,(x_j - x_{j-1}) + \sum_{t_j \in C} \|U^{-1}(x_{j-1})\|\,\|U(x_j) - U(x_{j-1})\| +$$
$$+ \sum_{t_j \in T} \|U^{-1}(x_{j-1})\|\,\|A(t_j)\, U(t_j)\,(x_j - x_{j-1}) - [U(x_j) - U(x_{j-1})]\| +$$
$$+ \sum_{t_j \in T} \|U^{-1}(x_{j-1})\|\,\|A(t_j)\,[U(t_j) - U(x_{j-1})]\|\,(x_j - x_{j-1}) \leqq$$
$$\leqq \sigma + K\sigma + K\sigma + K\sigma = \vartheta$$

by (3.10), (3.5), (3.11), (3.12), Lemma 2.3 and the definition of $\sigma$ at the beginning of the proof. The proof is completed by applying Theorem 2.7.

**3.5.** Remark. Let $(\mathrm{PP}) \int_a^b V(t, \mathrm{d}t)$ exist, let $\varphi \colon [a, b] \to [0, \infty]$ be such a function that

$$\|V(t, [x, y])\| \leqq \varphi(t)\,|y - x| \tag{$*$}$$

for $a \leqq x \leqq t \leqq y \leqq b$. Set again $U(t) = (\mathrm{PP}) \int_a^t V(s, \mathrm{d}s)$, $U(a) = I$ and assume that the derivative $\dot{U}(t)$ exists a.e. in $[a, b]$. Define $A$ as in Theorem 3.4. It follows from $(*)$ that $U$ satisfies (3.5) so that $(\mathrm{PP}) \int_a^t V(s, \mathrm{d}s) = (\mathrm{PP}) \int_a^t (I + A(s)\,\mathrm{d}s)$ by Theorem 3.4. Putting $V_3(t, [x, y]) = A(t)\,(y - x)$, we have $V \sim V_3$ by Remark 2.10.

**3.6. Definition.** A function $A: [a, b] \to M_n$ is said to be *PS-integrable* (*Perron summation integrable*) nad a matrix $R$ is its integral, notation $R = (\text{PS}) \int_a^b A(t)\, dt$, if for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ such that

$$\|R - S(A, \Delta)\| < \varepsilon$$

holds for every $\delta$-fine partition $\Delta$ of $[a, b]$, where

$$S(A, \Delta) = \sum_{j=1}^{k} A(t_j)(x_j - x_{j-1}).$$

Notice that if $n = 1$ then $A: [a, b] \to R$ is integrable in the sense of Definition 3.6 iff it is Perron integrable, and both integrals coincide.

**3.7. Theorem.** *Let*

$$A(t_1) A(t_2) = A(t_2) A(t_1) \quad \textit{for} \quad t_1, t_2 \in [a, b]. \tag{3.13}$$

*If one of the integrals in the formula*

$$(\text{PP}) \prod_a^b (I + A(t)\, dt) = \exp\left[(\text{PS}) \int_a^b A(t)\, dt\right] \tag{3.14}$$

*exists then the other exists as well and* (3.14) *holds.*

Proof. In virtue of (3.13) we have $(\text{PP}) \prod_a^b \exp(A(t)\, dt) = \exp\left[(\text{PS}) \int_a^b A(t)\, dt\right]$ provided one of the integrals exists, and (3.14) follows from Theorem 2.12.

In what follows we will use Theorem 3.7 to transfer some of the results established above for the product integral to the sum integral of a real function. As a consequence of Theorems 1.2, 3.2, 3.3 we obtain

**3.8. Theorem.** *Let $f: [a, b] \to R$ be PS-integrable. Then the integral* $(\text{PS}) \int_a^t f(s)\, ds$ *exists for $t \in (a, b]$. Put $F(a) = 0$, $F(t) = (\text{PS}) \int_a^t f(s)\, ds$. Then $F$ satisfies the conditions*

(3.15) $\dot{F}(t) = f(t)$ *for almost all* $t \in [a, b]$.

(3.16) *Let $C \subset [a, b]$, $m(C) = 0$. For every $\varepsilon > 0$ there exists $\delta: C \to (0, \infty)$ such that*

$$\sum_{j=1}^{r} |F(\eta_j) - F(\xi_j)| < \varepsilon$$

*provided $\tau_j \in C$, $\tau_j - \delta(\tau_j) < \xi_j \leqq \tau_j \leqq \eta_j < \tau_j + \delta(\tau_j)$, $\xi_j, \eta_j \in [a, b]$ for $j = 1, 2, \ldots, r$ and $\eta_j \leqq \xi_{j+1}$ for $j = 1, 2, \ldots, r - 1$.*

Theorem 3.4 yields

**3.9. Theorem.** *Let $F: [a, b] \to R$ have a derivative almost everywhere in $[a, b]$ and let* (3.16) *be fulfilled. Put $f(t) = \dot{F}(t)$ for $t \in [a, b]$ at which $\dot{F}(t)$ exists, $f(t)$ arbitrary elsewhere. Then the integral* $(\text{PS}) \int_a^b f(t)\, dt$ *exists and equals $F(b) - F(a)$.*

**3.10.** The notion of an absolutely continuous function was extended in a natural way to the notion of an $\text{ACG}_*$ function (generalized absolutely continuous), see e.g. [6]. Let us recall the following results concerning $\text{ACG}_*$ functions:

(3.17) If $F: [a, b] \to R$ is $\mathrm{ACG}_*$, then the derivative $\dot{F}(t)$ exists for a.e. $t \in [a, b]$. If we put $f(t) = \dot{F}(t)$ at such $t$ where the derivative exists, $f(t)$ arbitrary elsewhere, then

$$F(t) - F(a) = (\mathrm{PS}) \int_a^t f(s)\,\mathrm{d}s \quad \text{for} \quad t \in (a, b]$$

(i.e., the integral exists and the formula holds). (In particular, if $\dot{F}(t) = 0$ for a.e. $t \in [a, b]$, then $F$ is a constant function.)

(3.18) Let $(\mathrm{PS}) \int_a^b f(t)\,\mathrm{d}t$ exist. Put $F(a) = 0$, $F(t) = (\mathrm{PS}) \int_a^t f(s)\,\mathrm{d}s$ for $t \in (a, b]$. Then $F$ is an $\mathrm{ACG}_*$ function and $f(t) = \dot{F}(t)$ a.e.

These assertions together with Theorems 3.8, 3.9 imply

(3.19) $F: [a, b] \to R$ is $\mathrm{ACG}_*$ if and only if it has a derivative a.e. and satisfies (3.16).

The notion of generalized absolute continuity can be extended to functions $H$ with values in $M_n$, $R^n$ or $C^n$ (either by modifying in the natural way the definition of real $\mathrm{ACG}_*$ functions or by requiring that all entries of $H$ – or, as the case may be, their real and imaginary parts – be $\mathrm{ACG}_*$ functions). Then it is not difficult to prove that (3.17)–(3.19) hold also for such functions.

Theorems 3.2–3.4 together with (3.19) imply

**3.11. Theorem.** *Let $U: [a, b] \to M_n$, $U(a) = I$. Then the following conditions are equivalent:*

(3.20) *there exists $A: [a, b] \to M_n$ such that*

$$U(t) = (\mathrm{PP}) \int_a^t (I + A(s)\,\mathrm{d}s)\,;$$

(3.21) *$U(t)$ is regular for $t \in [a, b]$, the derivative $\dot{U}(t)$ exists a.e., $U$ satisfies* (3.5);

(3.22) *$U$ is an $\mathrm{ACG}_*$ function, $U(t)$ is regular for $t \in [a, b]$;*

(3.23) *there exists $A: [a, b] \to M_n$ such that*

$$U(t) = I + (\mathrm{PS}) \int_a^t A(s)\, U(s)\,\mathrm{d}s\,,$$

*and $U(t)$ is regular for $t \in [a, b]$.*

Notice that the matrix $A$ in (3.20) coincides with $A$ in (3.23) since (3.4) holds in both cases (for (3.20) this follows from Theorem 3.2).

**3.12. Theorem.** *Let $A: [a, b] \to M_n$, let $Q \in M_n$ be regular and let the integral* $(\mathrm{PP}) \int_a^t (I + A(s)\,\mathrm{d}s)$ *exist for $t \in (a, b)$,*

$$\lim_{t \to b-} (\mathrm{PP}) \int_a^t (I + A(s)\,\mathrm{d}s) = Q\,.$$

*Then the integral* $(\mathrm{PP}) \int_a^b (I + A(s)\,\mathrm{d}s)$ *exists and is equal to $Q$.*

Proof. Put $U(a) = I$, $U(t) = (\mathrm{PP}) \int_a^t (I + A(s)\,\mathrm{d}s)$ for $t \in (a, b)$, $U(b) = Q$. Then $U$ is an $\mathrm{ACG}_*$ function by (3.18) and the assertion follows from Theorem 3.11.

**4.1. Definition.** A solution of the equation

$$\dot{x} = A(t)\,x \tag{4.1}$$

with $A: [a, b] \to M_n$ is any $\mathrm{ACG}_*$ function $u: [a, b] \to R^n$ satisfying (4.1) for a.e. $t \in [a, b]$.

**4.2. Theorem.** *Let $A$ be PP-integrable on $[a, b]$, let $t_0 \in [a, b]$, $y \in R^n$. Then there is a unique solution $u$ of* (4.1) *satisfying the initial condition $u(t_0) = y$.*

Proof. Recall that $A: [a, b] \to M_n$ is called PP-integrable if the functions $V_1$, $V_2$ in (1.3), (1.4) are PP-integrable. Let $U$ be defined as in Lemma 2.3. It is directly verified that the function $U(t)\,U^{-1}(t_0)\,y$ is the desired solution (cf. Theorem 3.2). On the other hand, if we look for the solution of (4.1) in the form $u(t) = U(t)\,z(t)$, where $z$ is $\mathrm{ACG}_*$, then it is easily seen that $\dot{z}(t) = 0$ a. e., hence $z$ is a constant function by (3.17). The uniqueness of the desired solution immediately follows.

**4.3.** Remark. Let $A$ be PP-integrable on $[a, b]$. By (3.17) and (3.18) a function $u$ is a solution of (4.1) if and only if

$$u(t) = u(a) + (\mathrm{PS}) \int_a^t A(s)\,u(s)\,\mathrm{d}s$$

for $t \in [a, b]$.

**4.4.** Remark. Let $A$ be PP-integrable, let $U$ be the same as in Lemma 2.3. Then $U$ is a fundamental matrix of solutions of (4.1). The variation-of-constants formula is applicable to the equation

$$\dot{x} = A(t)\,x + g(t) \tag{4.2}$$

provided $g: [a, b] \to R^n$ is Lebesgue integrable on $[a, b]$ (i.e., $\int_a^b g(t)\,\mathrm{d}t \in R^n$) since $U(t) \int_a^t U^{-1}(s)\,g(s)\,\mathrm{d}s$ is an $\mathrm{ACG}_*$ function and satisfies (4.2) for a.e. $t \in [a, b]$.

**4.5. Lemma.** *Let $k \geqq 0$ be an integer. Let $B: [a, b] \to M_n$ be continuous, let $c \in [a, b]$, $B(c) = 0$. Let $\dot{B}(t)$ exist for every $t \neq c$ and let $\dot{B}$ be continuous on $[a, c) \cup (c, b]$. Let $\int_a^b \|\dot{B}B^{k+1}\|\,\mathrm{d}t < \infty$. Put $A(t) = \dot{B}(t)\,[I - B(t) + B^2(t) - \dots$ $\dots + (-1)^k B^k(t)]$. Then $A$ is PP-integrable on $[a, b]$.*

Proof. In a neighbourhood of $c$ let us apply the substitution $x = (I + B(t))\,v$ to the equation (4.1). We obtain

$$\dot{v} = (I + B(t))^{-1}\,\dot{B}(t)\,B^{k+1}(t)\,v\,. \tag{4.3}$$

By the assumptions the matrix of coefficients of the equation (4.3) is Lebesgue integrable and the equation (4.3) has an absolutely continuous fundamental matrix of solutions $V$. It follows that $U(t) = (I + B(t))\,V(t)$ is regular (in a neighbourhood of $c$) and $U$ is an $\mathrm{ACG}_*$ function. Hence $A$ is PP-integrable by Theorem 3.11.

**4.6.** Remark. We shall make use of Lemma 4.5 to construct functions $A_1$, $A_2$:

$[-1, 1] \to M_2$ such that

(4.4) $$A_1 \text{ is PS-integrable but not PP-integrable},$$

(4.5) $$A_2 \text{ is PP-integrable but not PS-integrable}.$$

**4.7.** Examples. Let $\alpha > 0$, $2\alpha + 1 < \beta < 3\alpha + 1$,

(4.6) $$S = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1, & 0 \\ 0, & 1 \end{pmatrix},$$

$B(t) = t^\alpha (S \sin t^{-\beta} + C \cos t^{-\beta})$ for $t \in [-1, 1]$, $t \neq 0$, $B(0) = 0$.
We have

(4.7) $$S^2 = C^2 = I, \quad SC = -CS = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}.$$

(i) Take $k = 0$, $A_1(t) = \dot{B}(t)$ for $t \neq 0$.

Applying the substitution $v = (I + B(t))\, x$ to

(4.8) $$\dot{x} = A_1(t)\, x$$

we obtain

(4.9) $$\dot{v} = D(t)\, v$$

with $D(t) = (I + B(t))^{-1} \dot{B}(t) B(t)$. Making use of (4.6) and (4.7) we obtain

(4.10) $$\dot{B}(t) B(t) = t^{2\alpha-\beta} SC + O(|t|^{2\alpha-1}),$$
$$D(t) = t^{2\alpha-\beta} SC + O(|t|^{\varepsilon-1}), \quad \varepsilon > 0.$$

Put

(4.11) $$W(t) = \exp\left((2\alpha + 1 - \beta)^{-1} t^{2\alpha+1-\beta} SC\right)$$

for $t \in (0, 1]$ and look for the fundamental matrix $V$ of (4.9) on $(0, 1]$ in the form

(4.12) $$V(t) = W(t)\, Z(t).$$

We have

$$\dot{Z} = W^{-1}(t)\,(D(t) - t^{2\alpha-\beta} SC)\, W(t)\, Z.$$

The form of $SC$ implies that both $W(t)$ and $W^{-1}(t)$ are bounded and by (4.10) $Z(0) = \lim_{t\to 0+} Z(t)$ exists and is regular. Since $2\alpha + 1 - \beta < 0$, it follows by (4.11) and (4.12) that neither of the limits $\lim_{t\to 0+} V(t)$, $\lim_{t\to 0+} (I + B(t))^{-1} V(t)$ exists and $A_1$ is not PP-integrable, since $(I + B(t))^{-1} V(t)$ is the fundamental matrix of solutions of (4.8) on (0,1).

(ii) Take $k = 1$, $A_2(t) = \dot{B}(t)\,(I - B(t))$.

(4.10) implies that $A_2$ is not PS-integrable, while $A_2$ is PP-integrable by Lemma 4.5, since $\dot{B}(t) B^2(t) = O(|t|^{3\alpha-\beta})$ by (4.10) and $3\alpha - \beta > -1$.

**4.8.** Example. Let $\gamma > 1$, $\alpha > 2(\gamma - 1)$, $\sigma_{lm} \in R$ for $l, m = 1, 2, \ldots, n$ and put

$$A_{lm}(t) = t^{-\gamma} \exp(i\sigma_{lm} t^{-\alpha})$$

for $t \in [-1, 1]$, $t \neq 0$, $A_{lm}(0) = 0$; $A(t) = (A_{lm}(t))$. Integrating by parts we obtain that there exists a matrix function $B$ such that $\dot{B}(t) = A(t)$ for $t \neq 0$, and $B(t) = O(t^{-\gamma+\alpha+1})$ in a neighbourhood of $t = 0$. Since $A(t) = O(t^{-\gamma})$ in a neighbourhood of $t = 0$, the function $A$ is PP-integrable by Lemma 4.5.

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