Czechoslovak Mathematical Journal

Robert H. Redfield

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Czechoslovak Mathematical Journal, Vol. 37 (1987), No. 4, 613–627

Persistent URL: http://dml.cz/dmlcz/102189

Czechoslovak Mathematical Journal, 37 (112) 1987, Praha

THE DUAL SPACE OF A TOTALLY ORDERED ABELIAN GROUP

R. H. REDFIELD*), Kelowna

(Received December 2, 1985)

1. INTRODUCTION

Let T be an abelian totally ordered group (*o-group*). The purpose of this paper is to suggest a way of studying the homomorphisms from T to the real numbers $\mathbb{R}$. This set of homomorphisms forms a group with respect to pointwise addition and, in general, a dual space of T will be a partially ordered subgroup of this group. The usual definition of the partially ordered subgroup, which takes as its positive cone the order-preserving homomorphisms (see [7], [11], [8]), has a major dawback: it always leads to an archimedean dual space [11].

We will propose below a different definition for the dual space, a definition which, for a non-archimedean base group, will yield a non-archimedean dual space. The homomorphisms which we will single out to be positive will be those which are locally order-preserving with respect to a fixed but arbitrary Banaschewski function. This Banaschwski function will have a dual Banaschewski function, and hence we will be able to form all higher dual spaces in the same way. At the very least, such a construction should allow a homomorphism between two base groups to lift in the usual way to a homomorphism between their dual spaces, and for our construction, this will indeed be the case. Furthermore, the evaluation map into the second dual space will be a one-to-one homomorphism, and all the odd-numbered higher dual spaces will be isomorphic as will all the even-numbered ones. These results will not surprisingly have as an immediate consequence the well-known embedding theorem of Hahn [5] and will also imply that the group of eventually constant sequences has two dual spaces (arising from two different Banaschewski functions) which are not isomorphic.

Now let T be an abelian o-group and let $\boldsymbol{P}$ (or if necessary $\boldsymbol{P}_T$) denote its set of convex subgroups. If S is a subgroup of T, let S^d denote its divisible closure in T: S^d is the subgroup of all $x \in T$ for which there exists a positive integer n such that $nx \in S$. Let $\boldsymbol{D}$ (or if necessary $\boldsymbol{D}_T$) denote the set of all subgroups S of T such that $S = S^d$. Clearly $\boldsymbol{P} \subseteq \boldsymbol{D}$.

*) Some of this work was done while the author was visiting Simon Fraser University where he was supported in part by NSERC grant A4044. The author wishes to thank the Department of Mathematics and Statistics at Simon Fraser and, in particular, N.R. Reilly for their hospitality.

The following generalizes a result of Banaschewski. He proved a similar result for divisible groups (see below and [2], p. 431).

Proposition 1.1. *For any abelian o-group T, there exists a function $\tau: \boldsymbol{P} \to \boldsymbol{D}$ such that*

(i) *if $P \subseteq Q$, $P, Q \in \boldsymbol{P}$, then $\tau(P) \supseteq \tau(Q)$;*

(ii) *for all $P \in \boldsymbol{P}$, $T = (P \oplus \tau(P))^d$.*

Proof. Let Φ be the set of all functions $\varphi: \boldsymbol{P} \to \boldsymbol{D}$ such that

(a) if $P \subseteq Q$, $P, Q \in \boldsymbol{P}$, then $\varphi(P) \supseteq \varphi(Q)$, and

(b) $P \cap \varphi(P) = \{0\}$ for all $P \in \boldsymbol{P}$.

The function which takes all $P \in \boldsymbol{P}$ to $\{0\} \in \boldsymbol{D}$ is clearly in Φ and hence $\Phi \neq \emptyset$. Define a binary relation $\leqq$ on Φ by letting $\varphi \leqq \gamma$ if and only if $\varphi(P) \subseteq \gamma(P)$ for all $P \in \boldsymbol{P}$. Clearly $(\Phi, \leqq)$ is a partially ordered set to which we may apply Zorn's Lemma, and hence we may pick an element μ in Φ which is maximal with respect to $\leqq$. Because $\mu \in \Phi$, it suffices to show that $T = (P + \mu(P))^d$ for all $P \in \boldsymbol{P}$. By way of contradiction suppose that $0 < x \in T \setminus (U + \mu(U))^d$ for some $U \in \boldsymbol{P}$, and define $\mu^*: \boldsymbol{P} \to \boldsymbol{D}$ by letting $\mu^*(P) = \mu(P)$ if $U \subset P$ and $\mu^*(P) = S^d$ if $U \supseteq P$, where S is the subgroup of T generated by x and $\mu(P)$. We claim that $\mu^* \in \Phi$. It is easy to see that μ^* satisfies (a). To see that μ^* satisfies (b), pick $P \in \boldsymbol{P}$ and $z \in P \cap \mu^*(P)$. If $U \subset P$, then $z = 0$ because $\mu \in \Phi$. Suppose on the other hand that $U \supseteq P$. Then we have $y \in \mu(U)$ and integers k and n such that $nz = kx + y$. Since $z \in U$, $kx = nz - y \in U + \mu(U)$, and by our choice of x, $k = 0$. Then $nz = y \in \mu(U)$; hence $z \in \mu(U) \subseteq \mu(P)$; hence $z = 0$. We conclude that μ^* satisfies (b) and hence that $\mu^* \in \Phi$. Clearly $\mu < \mu^*$, a contradiction of our choice of μ as maximal in $(\Phi, \leqq)$. Therefore $T = (P + \mu(P))^d$ for all $P \in \boldsymbol{P}$ and Proposition 1.1 follows.

We call an abelian o-group T equipped with a function τ satisfying conditions (i) and (ii) of Proposition 1.1 a *β-group*. If the function satisfies (i) and the stronger condition

$$\text{(ii)* } \quad T = P \oplus \tau(P) \quad \text{for all} \quad P \in \boldsymbol{P},$$

then we call T a *strong β-group*. Every divisible abelian o-group possesses a function $\tau: \boldsymbol{P} \to \boldsymbol{D}$ with respect to which it is a strong β-group [2].

2. DEFINITION OF THE DUAL SPACE

In this section we give the definition of the dual space. It is based on the set of archimedean subgroups of T which are generated by τ as follows. For a β-group T, let $\boldsymbol{A}$ (or if necessary $\boldsymbol{A}_T$ or $\boldsymbol{A}[T, \tau]$) be the set of all subgroups A of T such that there exist $P_A, Q_A \in \boldsymbol{P}$ such that P_A covers Q_A in the lattice $\boldsymbol{P}$ and $A = P_A \cap \tau(Q_A)$. Clearly $\boldsymbol{A} \subseteq \boldsymbol{D}$, and each $A \in \boldsymbol{A}$ is archimedean. We may also characterize $\boldsymbol{A}$ as follows. For any $S \subseteq T$, let $\langle S \rangle$ denote the convex subgroup of T generated by S (we abbreviate $\langle \{z\} \rangle$ by $\langle z \rangle$); for $0 \neq z \in T$, let $[z]$ denote the subgroup of T formed by all

$y \in T$ such that $y \ll |z|$, i.e., such that $ny < |z|$ for all integers n. Then $\boldsymbol{A} = \{\langle x \rangle \cap \cap \tau([x]) \mid 0 < x \in T\}$.

Proposition 2.1. *For $A \in \boldsymbol{A}$, $T = (A \oplus Q_A \oplus \tau(P_A))^d$, where P_A covers Q_A in the lattice $\boldsymbol{P}$ and $A = P_A \cap \tau(Q_A)$.*

Proof. We first show that $P_A = (Q_A \oplus A)^d$. We have ([3], p. 172)

$$P_A = P_A \cap (Q_A \oplus \tau(Q_A))^d \supseteq P_A \cap (Q_A \oplus \tau(Q_A)) = Q_A \oplus A\,.$$

Since $P_A \in \boldsymbol{P} \subseteq \boldsymbol{D}$, $P_A \supseteq (Q_A \oplus A)^d$. Conversely, if $z \in P_A$, then there exists a positive integer n such that $nz \in P_A \cap (Q_A \oplus \tau(Q_A)) = Q_A \oplus A$, i.e., $z \in (Q_A \oplus A)^d$. Hence $P_A = (Q_A \oplus A)^d$. Now let $x \in T$. Then there exists a positive integer n such that

$$nx \in P_A \oplus \tau(P_A) = (Q_A \oplus A)^d \oplus (P_A)\,,$$

i.e., $nx = w + z$, where $z \in \tau(P_A)$ and $mw \in Q_A \oplus A$ for some positive integer m. Then

$$mnx = mw + mz \in Q_A \oplus A \oplus \tau(P_A)\,,$$

i.e., $x \in (Q_A \oplus A \oplus \tau(P_A))^d$. This proves Proposition 2.1.

Proposition 2.1 says that any $x \in T$ has a multiple which may be written uniquely as a sum of elements from A, Q_A, and $\tau(P_A)$. We will be using this property, as well as related ones, continually and hence we adopt the following notation. If $x \in T$ and $S^d = T$ for a subgroup S of T, then there exists a positive integer n such that $nx \in S$ and we let $m\{x, S\}$ denote the minimal such n. For $V \in \boldsymbol{P}$, we abbreviate $m\{x, V \oplus \tau(V)\}$ by $m(x, V)$ and we let $\alpha_{x,V} \in V$ and $\beta_{x,V} \in \tau(V)$ be such that $m(x, V)\, x = \alpha_{x,V} + \beta_{x,V}$. For $A \in \boldsymbol{A}$, we abbreviate $m\{x, A \oplus Q_A \oplus \tau(P_A)\}$ by $m(x, A)$ and we let $x_A \in A$, $q_{x,A} \in Q_A$, and $p_{x,A} \in \tau(P_A)$ be such that $m(x, A)\, x = x_A + q_{x,A} + p_{x,A}$. For a strong β-group, we have $m(x, A) = 1$ for all $x \in T$ and $A \in \boldsymbol{A}$ and the proofs in the sequel simplify accordingly (cf. [9]).

Define a binary relation $\leqq$ on $\boldsymbol{A}$ by letting $A \leqq B$ if and only if $A \subseteq \langle B \rangle$. It is easy to see that $(\boldsymbol{A}, \leqq)$ is a totally ordered set. For a divisible group T, the following result is due to Banaschewski ([2], page 433).

Proposition 2.2. *For all $0 \neq x \in T$, the set $S(x) = \{A \in \boldsymbol{A} \mid x_A \neq 0\}$ is an inversely well-ordered subset of $(\boldsymbol{A}, \leqq)$.*

Proof. Let $0 \neq x \in T$ and construct $\boldsymbol{B}[x] \subseteq \boldsymbol{A}$ inductively as follows. Let $B[0] = \langle x \rangle \cap \tau([x])$. If $q_{x,B[0]} = 0$, let $\boldsymbol{B}[x] = \{B[0]\}$. Suppose that for an ordinal α, $B[\alpha]$ has been defined and $q_{x,B[\alpha]} \neq 0$. Then let $B[\alpha + 1] = \langle q_{x,B[\alpha]} \rangle \cap \tau([q_{x,B[\alpha]}])$. If $q_{x,B[\alpha+1]} = 0$, let $\boldsymbol{B}[x] = \{B[\beta] \mid \beta \leqq \alpha\}$. Suppose λ is a limit ordinal and $q_{x,B[\alpha]} \neq 0$ for all $\alpha < \lambda$. Let $V[\lambda] = \bigcap_{\alpha<\lambda} \langle B[\alpha] \rangle$. If $\alpha_{x,V[\lambda]} = 0$, let $\boldsymbol{B}[x] = \{B[\beta] \mid \beta < \lambda\}$. If $\alpha_{x,V[\lambda]} \neq 0$, let $B[\lambda] = \langle \alpha_{x,V[\lambda]} \rangle \cap \tau([\alpha_{x,V[\lambda]}])$. If $q_{x,B[\lambda]} = 0$, let $\boldsymbol{B}[x] = \{B[\beta] \mid \beta \leqq \lambda\}$.

Let $x \in T$ and $A < B$ in $\boldsymbol{A}$. For notational convenience, let $\boldsymbol{q} = q_{x,B}$. Then

$$m(x, A)\,[m(\boldsymbol{q}, A)\, x_B + \boldsymbol{q}_A + q_{\boldsymbol{q},A} + p_{\boldsymbol{q},A} + m(\boldsymbol{q}, A)\, p_{x,B}] = $$
$$= m(x, A)\, m(\boldsymbol{q}, A)\, m(x, B)\, x = m(x, B)\, m(\boldsymbol{q}, A)\,[x_A + q_{x,A} + p_{x,A}]\,.$$

We have $q_A, x_A \in A$, $q_{q,A}, q_{x,A} \in Q_A$, and $p_{q,A}, p_{x,A} \in \tau(P_A)$. Since $A < B$, $B \subseteq \tau(P_A)$; thus $x_B \in \tau(P_A)$ and $p_{x,B} \in \tau(P_A)$. Therefore, by the directness of the sum $Q \oplus A_A \oplus \tau(P_A)$, we must have

(1) $m(x, A)\,(q_{x,B})_A = m(x, B)\, m(q_{x,B}, A)\, x_A$.

Similarly, if $A \in \boldsymbol{A}$ and $A \subseteq P \in \boldsymbol{P}$, then

(2) $m(x, A)\,(\alpha_{x,P})_A = m(x, P)\, m(\alpha_{x,P}, A)\, x_A$.

It is clear from the construction of $\boldsymbol{B}[x]$ and (1) and (2) above that $\boldsymbol{B}[x] \subseteq S(x)$. Conversely, suppose that $A \in S(x)$, and let $P = \bigcap\{\langle B[\alpha]\rangle \mid A \subseteq \langle B[\alpha]\rangle\}$. If $P \neq \langle B[\alpha]\rangle$ for some α, then $P = V[\lambda]$ for some limit ordinal λ. By (2) above, $(\alpha_{x,V[\lambda]})_A \neq 0$ because $x_A \neq 0$; hence $\alpha_{x,V[\lambda]} \neq 0$ and $A \subseteq \langle \alpha_{x,V[\lambda]}\rangle$. Thus $B[\lambda]$ is defined and $A \leqq B[\lambda]$, i.e., $V[\lambda] = P \subseteq \langle B[\lambda]\rangle$, a contradiction. Therefore, $P = \langle B[\alpha]\rangle$ for some α. If $A \neq B[\alpha]$, we have $A < B[\alpha]$ because $A \subseteq P = \langle B[\alpha]\rangle$. Then by (1) above, $(q_{x,B[\alpha]})_A \neq 0$ because $x_A \neq 0$; hence $q_{x,B[\alpha]} \neq 0$ and $A \subseteq \langle q_{x,B[\alpha]}\rangle$. Thus $B[\alpha + 1]$ is defined and $A \leqq B[\alpha + 1]$, i.e., $\langle B[\alpha]\rangle = P \subseteq \langle B[\alpha + 1]\rangle$, a contradiction. Therefore, $A = B[\alpha] \in \boldsymbol{B}[x]$, and we conclude that $S(x) = \boldsymbol{B}[x]$. Since $\boldsymbol{B}[x]$ is inversely well-ordered by construction, $S(x)$ is inversely well-ordered. This proves Proposition 2.2.

Any group of homomorphisms $f\colon T \to \mathbb{R}$ will be a partially ordered group with respect to the following order [8]: $0 \lesssim f$ if and only if $0 \leqq f(x)$ whenever $0 \leqq x \in T$, and $g \lesssim f$ if and only if $0 \lesssim f - g$. The dual space is usually defined in just this way: it is the group generated by all the homomorphisms f with $0 \lesssim f$; as a directed group, it will always be archimedean [11]. To defined a more order-theoretically interesting dual space, we let $\boldsymbol{F}$ denote the finite subsets of $\boldsymbol{A}$ directed by inclusion, and for any function $f\colon T \to \mathbb{R}$, we define the *support* of f to be the set

$$\operatorname{Supp}(f) = \{A \in \boldsymbol{A} \mid f|_A \neq 0\}\,.$$

The *dual space* $T^\wedge$ of T then consists of all the functions $f\colon T \to \mathbb{R}$ satisfying the following conditions:

(I) f is a group-homomorphism;

(II) for all $A \in \boldsymbol{A}$, $0 \lesssim f\,|_A$ or $f\,|_A \lesssim 0$;

(III) $\operatorname{Supp}(f)$ is well-ordered;

(IV) for all $x \in T$, $f(x) = \lim\limits_{\Phi \in \boldsymbol{F}} \sum\limits_{A \in \Phi} m(x, A)^{-1} f(x_A)$,

where the limit is taken over the directed set $\boldsymbol{F}$ (see [6], pages 77–78, "Integration Theory, Junior Grade"). Define a binary relation $<$ on $T^\wedge$ as follows:

$0 < f$ if and only if $0 \neq f$ and $0 \prec f|_{\wedge \operatorname{Supp}(f)}$;

$g < f$ if and only if $0 < f - g$.

Here $\wedge \operatorname{Supp}(f)$ is the minimum element in the lattice $\operatorname{Supp}(f)$. Note that $T^\wedge$ depends upon τ as well as T; therefore, to avoid confusion we will sometimes use $(T, \tau)^\wedge$ instead of $T^\wedge$. Note also that if T is archimedean, then $<$ and $\prec$ coincide.

(In [9], we defined the dual space by choosing the functions f which satisfied (I), (II), (IV), and

(III)* $\operatorname{Supp}(f)$ is inversely well-ordered.

The order on the dual space was then defined by using the maximum of the support rather than the minimum. The proofs of the results in [9] parallel the proofs given here. The reason we choose the functions with well-ordered support here is that when we apply our construction to o-rings, we want convolution to be a well-defined operation on the second dual. For this to be true, we need (III) instead of (III)* – see [10].)

Clearly each $A \in \boldsymbol{A}$ is archimedean and hence order-isomorphic to a subgroup of $(\mathbb{R}, +, \leqq)$. Thus ([4], page 46) the set of real-valued group-homomorphisms of A which are either order-preserving or order-reversing forms a totally ordered group with respect to $\lesssim$ and pointwise addition. Hence if both $f|_A$ and $g|_A$ are comparable to 0 with respect to $\lesssim$ then $(f + g)|_A$ is also comparable to 0 with respect to $\lesssim$. It is then easy to check that $(T^\wedge, +)$ is a divisible abelian group. In particular, if $f, g \in T^\wedge$, then $f - g \in T^\wedge$ and either $\operatorname{Supp}(f - g) = \emptyset$ or $\operatorname{Supp}(f - g) \neq \emptyset$. In the latter case, we have $(f - g)|_{\wedge \operatorname{Supp}(f-g)}$ comparable to 0 with respect to $\lesssim$ and hence $f - g$ comparable to 0 with respect to $\leqq$. In the former case, for all $x \in T$,

$$(f - g)(x) = \lim \sum (f - g)(x_A) = \lim \sum 0 = 0\,,$$

i.e., $f - g = 0$. It is then easy to verify

Theorem 2.3. $(T^\wedge, +, \leqq)$ *is a divisible abelian o-group.*

3. STRUCTURE OF THE DUAL SPACE

If (T, τ) is a β-group, then according to Theorem 2.3, $(T^\wedge, +, \leqq)$ is a divisible abelian o-group. We abbreviate $\boldsymbol{P}_{T^\wedge}$, the set of convex subgroups of $T^\wedge$, by $\boldsymbol{P}^\wedge$ and $\boldsymbol{D}_{T^\wedge}$, the set of divisible subgroups of $T^\wedge$, by $\boldsymbol{D}^\wedge$. We will establish a correspondence between $\boldsymbol{P}^\wedge$ and $\boldsymbol{P}$ which will enable us to make $T^\wedge$ a β-group in a natural way.

We first define some functions which are present in all dual spaces. For $0 < b \in T$, the group $\langle b \rangle \cap \tau([b]) = B \in \boldsymbol{A}$ is archimedean and hence ([4], page 46) there exists an order-preserving group-homomorphism $h: B \to \boldsymbol{R}$ such that $h(b_B) = m(b, B)$. Define $b^\wedge: T^\wedge \to \boldsymbol{R}$ by letting $b^\wedge(y) = m(y, B)^{-1}\, h(y_B)$. It is routine to show that for all $x, y \in G$,

$$m(x, B)\, m(y, B)\, (x + y)_B = m(x + y, B)\, [m(y, B)\, x_B + m(x, B)\, y_B]$$

and from this it follows that $b^\wedge$ is a group-homomorphism. Then clearly $b^\wedge \in T^\wedge$. We conclude that for all $0 < b \in T$, there exists $0 < b^\wedge \in T^\wedge$ such that $b^\wedge(b) = 1$ and, for all $A \in \boldsymbol{A}$ such that $\vee S(b) \neq A$, $b^\wedge|_A = 0$, where $\vee S(b)$ is the maximum element of the lattice $S(b)$.

For $P \in \boldsymbol{P}$ and $V \in \boldsymbol{P}^\wedge$, let

$$P^\wedge = \{f \in T^\wedge \mid f|_A = 0 \text{ for all } P \supseteq A \in \boldsymbol{A}\}\,, \quad \text{and}$$
$$V_\# = \{z \in T \mid f(z) = 0 \text{ for all } f \in V\}\,.$$

Proposition 3.1. *The function $P \to P^\wedge$ is an order-reversing bijection of $\boldsymbol{P}$ to $\boldsymbol{P}^\wedge$ whose order-reversing inverse is $V \to V_\#$.*

Proof. (a) $P^\wedge \in \boldsymbol{P}^\wedge$ and $V_\# \in \boldsymbol{P}$: It is easy to see that $P^\wedge \in \boldsymbol{P}^\wedge$, and that $V_\#$ is a subgroup of T. To see that $V_\#$ is convex, let $w \in V_\#$ and suppose that $0 < y < w$ in T. If $f(y) \neq 0$ for some $0 < f \in V$, then

$$\wedge \operatorname{Supp}(f) \leqq \vee S(y) \leqq \vee S(w) = \wedge \operatorname{Supp}(w^\wedge).$$

Thus $nf \geqq w^\wedge > 0$ for some positive integer n and hence $w^\wedge \in V$. But $w^\wedge(w) = 1 \neq 0$; thus $w \in T \setminus V_\#$, a contradiction. Therefore, $f(y) = 0$ for all $f \in A$, i.e. $y \in V_\#$, and hence $V_\#$ is convex. We conclude that $V_\# \in \boldsymbol{P}$.

(b) $P^\wedge{}_\# = P$: Let $p \in P^\wedge{}_\#$. If $p \in T \setminus P$, then $p^\wedge \in P^\wedge$, and since $p^\wedge(p) = 1 \neq 0$, $p \in T \setminus P^\wedge{}_\#$, a contradiction. We conclude that $P^\wedge{}_\# \subseteq P$. Conversely, let $p \in P$. If $f \in P^\wedge$, then $f(p_A) = 0$ for all $A \in S(p)$ and hence $f(p) = \lim \sum f(p_A) = 0$. Thus $p \in P^\wedge{}_\#$ and we conclude that $P^\wedge{}_\# \supseteq P$.

(c) $V_\#{}^\wedge = V$: Let $f \in V$. If $V_\# \supseteq A \in \boldsymbol{A}$, then $f(a) = 0$ for all $a \in A$, i.e. $f|_A = 0$. Hence $f \in V_\#{}^\wedge$ and we conclude that $V \subseteq V_\#{}^\wedge$. Conversely, let $f \in V_\#{}^\wedge$. Suppose that $f \in T^\wedge \setminus V$ and let $0 < x \in \wedge \operatorname{Supp}(f)$. If $g \in V$, then $\wedge \operatorname{Supp}(g) > \wedge \operatorname{Supp}(f)$ and hence $g(x) = 0$. Thus $x \in V_\#$. But for all $V_\# = V_\#{}^\wedge{}_\#$ by (a) and (b), and hence $f(x) = 0$, a contradiction. Thus $f \in V$ and we conclude that $V \supseteq V_\#{}^\wedge$.

By (a), (b) and (c), it suffices to show that both $P \to P^\wedge$ and $V \to V_\#$ reverse order. Firstly suppose that $P \subseteq Q$ in $\boldsymbol{P}$. If $f \in Q^\wedge$, then whenever $P \supseteq A \in \boldsymbol{A}$, $Q \supseteq A$, and hence $f|_A = 0$. Therefore $P^\wedge \supseteq Q^\wedge$. Secondly suppose that $V \subseteq W$ in $\boldsymbol{P}^\wedge$ and let $z \in W_\#$. If $f \in V$, then $f(z) = 0$ because also $f \in W$. Hence $z \in V_\#$ and therefore $V_\# \supseteq W_\#$. This proves Proposition 3.1.

To give $T^\wedge$ the structure of a β-group, we define for all $V \in \boldsymbol{P}^\wedge$,

$$\tau^\wedge(V) = \{f \in T^\wedge \mid f|_A = 0 \text{ for all } \tau(V_\#) \supseteq A \in \boldsymbol{A}\}.$$

Theorem 3.2. *$(T^\wedge, +, \leqq, \tau^\wedge)$ is a strong β-group.*

Proof. By Theorem 2.3, $T^\wedge$ is a divisible abelian o-group and clearly $\tau^\wedge: \boldsymbol{P}^\wedge \to \boldsymbol{D}^\wedge$. Thus it suffices to show that $\tau^\wedge$ satisfies conditions (i) and (ii)* of § 1. Suppose firstly that $V \subseteq W$ in $\boldsymbol{P}^\wedge$ and let $f \in \tau^\wedge(W)$. By Proposition 3.1, $V_\# \supseteq W_\#$ and hence $\tau(V_\#) \subseteq \tau(W_\#)$. Thus, if $\tau(V_\#) \supseteq A \in \boldsymbol{A}$, $\tau(W_\#) \supseteq A$ as well, and hence $f|_A = 0$. Thus, $f \in \tau^\wedge(V)$, and therefore $\tau^\wedge(V) \supseteq \tau^\wedge(W)$. It remains to show that $T^\wedge = V \oplus \tau^\wedge(V)$ for all $V \in \boldsymbol{P}^\wedge$.

To see that $T^\wedge = V + \tau^\wedge(V)$, let $0 < g \in T^\wedge$. For $x \in T$, abbreviate $m(x, V_\#)$ by $\mu(x)$, $\alpha_{x,V}$ by α_x, $\beta_{x,V\#}$ by β_x, and define

$$g_1(x) = \mu(x)^{-1}\, g(\alpha_x), \quad \text{and} \quad g_2(x) = \mu(x)^{-1}\, g(\beta_x).$$

Then for $x, y \in T$.

$$g_1(x + y) = \mu(x + y)^{-1}\, g(\alpha_{x+y}) = [\mu(x)\,\mu(y)\,\mu(x + y)]^{-1}\, g(\mu(x)\,\mu(y)\,\alpha_{x+y}) =$$
$$= [\mu(x)\,\mu(y)\,\mu(x + y)]^{-1}\, g[\mu(x + y)\,\mu(y)\,\alpha_x + \mu(x + y)\,\mu(x)\,\alpha_y] = g_1(x) + g_1(y).$$

Furthermore, $g_1|_A = g|_A$ for all $V_\# \supseteq A \in \boldsymbol{A}$, and $g_1|_A = 0$ for all $A \in \boldsymbol{A}$ with $V_\# \subset \subset \langle A \rangle$. Thus, Supp (g_1) is well-ordered, and for all $A \in \boldsymbol{A}$, $g_1|_A$ is comparable to 0 with respect to $\lesssim$. Finally, let $x \in T$. For $A \in \boldsymbol{A}$, we have the following. If $V_\# \subset \langle A \rangle$, then

$$m(x, A)^{-1} g_1(x_A) = 0 = \mu(x)^{-1} m(\alpha_x, A)^{-1} g([\alpha_x]_A) .$$

If $A \subseteq V_\#$, then

$$\mu(x) m(\alpha_x, A) [x_A + q_{x,A} + p_{x,A}] = \mu(x) m(\alpha_x, A) m(x, A) x = \\ = m(x, A) m(\alpha_x, A) [\alpha_x + \beta_x] = m(x, A) [[\alpha_x]_A + y + z] ,$$

where $y \in Q_A$ and $z \in \tau(P_A)$. Therefore, by the directness of the sum $A \oplus Q_A \oplus \oplus \tau(P_A)$, we must have $\mu(x) m(\alpha_x, A) x_A = m(x, A) [\alpha_x]_A$, and hence in this case as well

$$m(x, A)^{-1} g_1(x_A) = \mu(x)^{-1} m(\alpha_x, A)^{-1} g([\alpha_x]_A) .$$

Therefore,

$$\lim \textstyle\sum m(x, A)^{-1} g_1(x_A) = \lim \sum \mu(x)^{-1} m(\alpha_x, A)^{-1} g([\alpha_x]_A) = \mu(x)^{-1} g(\alpha_x) = g_1(x) .$$

We conclude that $g_1 \in T^\wedge$; similarly $g_2 \in T^\wedge$. It is clear that $g_1 \in \tau^\wedge(V)$ and since clearly $g_2 \in V_\#^\wedge$, $g_2 \in V$ by Proposition 3.1. Since $g = g_1 + g_2$, $g \in V + \tau^\wedge(V)$; therefore $T^\wedge = V + \tau^\wedge(V)$.

To see that the sum is direct, let $f \in V \cap \tau^\wedge(V)$. Because $f \in \tau^\wedge(V)$, $f|_A = 0$ for all $\tau(V_\#) \supseteq A \in \boldsymbol{A}$. By Proposition 3.1, $V = V_\#^\wedge$, and thus, because $f \in V$, $f|_A = 0$ for all $V_\# \supseteq A \in \boldsymbol{A}$. We conclude that $f = 0$ and hence that $T^\wedge = V \oplus \tau^\wedge(V)$. This proves Theorem 3.2.

In cases where there is no ambiguity, we let $\boldsymbol{A}^\wedge$ abbreviate $\boldsymbol{A}_{T^\wedge}$, the archimedean subgroups of $T^\wedge$ distinguished by $\tau^\wedge$. For $A \in \boldsymbol{A}$ $(A = P_A \cap \tau(Q_A))$, we define $A^\wedge \in \boldsymbol{A}^\wedge$ by $A^\wedge = (Q_A)^\wedge \cap \tau^\wedge((P_A)^\wedge)$. (That $A^\wedge \in \boldsymbol{A}^\wedge$ follows from Proposition 3.1.) Note that if $0 < a \in A \in \boldsymbol{A}$, then $0 < a^\wedge \in A^\wedge$.

4. HOMOMORPHISMS OF β-GROUPS

In section we wish to investigate homomorphisms between β-groups. We will define such homomorphisms (called β-homomorphisms) and show that they lift in the usual way to homomorphisms between the corresponding dual spaces.

To be such a homomorphism, a function should preserve the group structure, the order structure, and the structure arising from the Banaschewski function. Let (T, τ) and (S, σ) be β-groups, and let $\Gamma: (T, \tau) \to (S, \sigma)$. Then Γ is a *β-homomorphism* if Γ satisfies the following conditions (cf. Example 6.10):

(i) Γ is a group-homomorphism;
(ii) Γ is *dense*: for all $Q \in \boldsymbol{P}_S$, $Q = \langle \Gamma(P) \rangle$ for some $P \in \boldsymbol{P}_T$;
(iii) Γ is a *Banaschewski homomorphism*: for all $P \in \boldsymbol{P}_T$, $\Gamma(\tau(P)) \subseteq \sigma(\langle \Gamma(P) \rangle)$.
(iv) Γ is *locally real*: for all $A \in \boldsymbol{A}_T$, $0 \prec \Gamma|_A$ or $\Gamma|_A \prec 0$;

Clearly, the composition of two β-homomorphisms is also a β-homomorphism.

Let $\Gamma\colon (T, \tau) \to (S, \sigma)$ be a β-homomorphism. For each $h \in S^\wedge$, we may define a function $\Gamma^\wedge(h)\colon T \to \mathbb{R}$ by letting $\Gamma^\wedge(h)(x) = h(\Gamma(x))$. This is the usual definition of the dual map $\Gamma^\wedge\colon S^\wedge \to T^\wedge$ ([1], [7], [11]). We will show (Theorem 4.2) that $\Gamma^\wedge$ is a well-defined β-homomorphism, after we first collect some elementary properties of β-homomorphisms.

Proposition 4.1. *Let $\Gamma\colon (T, \tau) \to (S, \sigma)$ be a β-homomorphism.*

(1) *If $x \ll |y|$ in T, then $\Gamma(x) \ll |\Gamma(y)|$ in S.*

(2) *Γ is one-to-one.*

(3) *For all $Q \in \boldsymbol{P}_S$, $\Gamma^{-1}(Q) \in \boldsymbol{P}_T$.*

(4) *The map $P \to \langle \Gamma(P) \rangle$ is an o-isomorphism of $\boldsymbol{P}_T$ onto $\boldsymbol{P}_S$.*

(5) *If $A \in \boldsymbol{A}_T$, there exists a unique $A^* \in \boldsymbol{A}_S$ with $\Gamma(A) \subseteq A^*$.*

(6) *For all $B \in \boldsymbol{A}_S$, $\Gamma^{-1}(B) \in \boldsymbol{A}_T$.*

(7) *The map $A \to A^*$ is an o-isomorphism of $\boldsymbol{A}_T$ onto $\boldsymbol{A}_S$.*

(8) *For all $A \in \boldsymbol{A}_T$ and $0 \neq x \in T$,*

$$\begin{aligned} m(\Gamma(x), A^*)\, \Gamma(x_A) &= m(x, A)\, \Gamma(x)_{A^*}, \\ m(\Gamma(x), A^*)\, \Gamma(q_{x,A}) &= m(x, A)\, q_{\Gamma(x),A^*}, \quad \textit{and} \\ m(\Gamma(x), A^*)\, \Gamma(p_{x,A}) &= m(x, A)\, p_{\Gamma(x),A^*}. \end{aligned}$$

Proof. (1) First suppose that $y \in A \in \boldsymbol{A}_T$, and note that $\Gamma(y) \neq 0$ because Γ is locally real and $y \neq 0$. If the conclusion is false, then $\Gamma(y) \in \langle \Gamma(x) \rangle$. Furthermore, $y \in A \subseteq \tau(\langle x \rangle)$ by hypothesis, and hence, since Γ is a Banaschewski homomorphism and $\langle \Gamma(\langle x \rangle) \rangle \supseteq \langle \Gamma(x) \rangle$,

$$\Gamma(y) \in \Gamma(\tau(\langle x \rangle)) \subseteq \sigma(\langle \Gamma(\langle x \rangle) \rangle) \subseteq \sigma(\langle \Gamma(x) \rangle).$$

Then $\Gamma(y) = 0$, a contradiction, and therefore, the conclusion holds for all $y \in A \in \boldsymbol{A}_T$. Now let y be any non-zero element of T, and let $A = \bigvee S(y)$. Then $y = y_A + q_{y,A}$, where $y_A \in A \in \boldsymbol{A}_T$ and $q_{y,A} \ll |y_A|$. Since $x \ll |y|$, $x \ll |y_A|$, and hence by the argument above, $\Gamma(x) \ll |\Gamma(y_A)|$. But also by the argument above, $\Gamma(q_{y,A}) \ll |\Gamma(y_A)|$, and hence, since $\Gamma(y) = \Gamma(y_A) + \Gamma(q_{y,A})$, we must have $\langle \Gamma(y_A) \rangle = \langle \Gamma(y) \rangle$. Therefore, $\Gamma(x) \ll |\Gamma(y)|$. (Cf. Example 6.11.)

(2) Let $0 < y \in T$. If $\langle y \rangle$ is archimedean, then $\langle y \rangle \in \boldsymbol{A}_T$, and since Γ is locally real, $\Gamma(y) \neq 0$. Otherwise, apply (1).

(3) Since Γ is dense, there exists $P \in \boldsymbol{P}_T$ such that $\langle \Gamma(P) \rangle = Q$. Clearly $\Gamma^{-1}(Q) \supseteq P$. If $P \subseteq [x]$, then by (1), $\Gamma(P) \subseteq [\Gamma(x)]$, thus $Q \subseteq [\Gamma(x)]$, and hence $x \in T \setminus \Gamma^{-1}(Q)$. Therefore, $\Gamma^{-1}(Q) \subseteq P$.

(4) The map is one-to-one by (1) and onto by (3). Both the map and its inverse are clearly order-preserving.

(5) Let $P_A, Q_A \in \boldsymbol{P}_T$ be such that P_A covers Q_A in $\boldsymbol{P}_T$ and $A = P_A \cap \tau(Q_A)$. By (4), $\langle \Gamma(P_A) \rangle$ covers $\langle \Gamma(Q_A) \rangle$ in $\boldsymbol{P}_S$ and hence $A^* = \langle \Gamma(P_A) \rangle \cap \sigma(\langle \Gamma(Q_A) \rangle) \in \boldsymbol{A}_S$. Since Γ is a Banaschewski homomorphism, $\Gamma(A) \subseteq A^*$. The uniqueness of A^* is clear.

(6) Let $P_B, Q_B \in \boldsymbol{P}_S$ be such that P_B covers Q_B in $\boldsymbol{P}_S$ and $B = P_B \cap \sigma(Q_B)$. By (3) and (4), $\Gamma^{-1}(P_B)$ covers $\Gamma^{-1}(Q_B)$ in $\boldsymbol{P}_T$ and hence $A = \Gamma^{-1}(P_B) \cap \tau(\Gamma^{-1}(Q_B)) \in \boldsymbol{A}_T$. Since Γ is a Banaschewski homomorphism, $A \subseteq \Gamma^{-1}(B)$. Let $0 < a \in A$, $0 < x \in \Gamma^{-1}(B)$, and denote $m(x, A)\, x - x_A$ by x^*. Suppose that $x^* \neq 0$ so that by (2) both $\Gamma(a)$ and $\Gamma(x^*)$ are non-zero elements of B. If $p_{x,A} \neq 0$, then $|p_{x,A}| \gg a$, hence $|\Gamma(p_{x,A})| \gg \Gamma(a)$ by (1), and hence, since $x^* = q_{x,A} + p_{x,A}$, $|\Gamma(x^*)| \gg \Gamma(a)$. This is impossible, and therefore $p_{x,A} = 0$. Furthermore, $a \gg q_{x,A}$ and hence by (1), $|\Gamma(a)| \gg \Gamma(q_{x,A}) = \Gamma(x^*)$. This is a contradiction and we conclude that $x^* = 0$. Then $m(x, A)\, x = x_A \in A$ and hence $x \in A$. Therefore $A \supseteq \Gamma^{-1}(B)$.

(7) By (5), the map is well-defined. By (6), the map is onto. Clearly, for all $B \in \boldsymbol{A}_S$, $[\Gamma^{-1}(B)]^* = B$, and thus the map is one-to-one. It is clear from (4) that both the map and its inverse are order-preserving.

(8) (a) $\Gamma(x_A) \in A^*$: $\Gamma(A) \subseteq A^*$ by assumption. (b) $\Gamma(q_{x,A}) \in Q_{A^*}$: For any $0 < a \in A$, $q_{x,A} \ll |a|$, and hence $\Gamma(q_{x,A}) \ll |\Gamma(a)|$ by (1). (c) $\Gamma(p_{x,A}) \in \sigma(P_{A^*})$: Since Γ is a Banaschewski homomorphism, $\Gamma(p_{x,A}) \in \Gamma(\tau(P_A)) \subseteq \sigma(\langle \Gamma(P_A) \rangle)$, and as in (6), $P_A = \langle \Gamma(P_A) \rangle$. We also have

$$m(\Gamma(x), A^*)\, [\Gamma(x_A) + \Gamma(q_{x,A}) + \Gamma(p_{x,A})] =$$
$$= m(\Gamma(x), A^*)\, m(x, A)\, \Gamma(x) = m(x, A)\, [\Gamma(x)_{A^*} + q_{\Gamma(x),A^*} + p_{\Gamma(x),A^*}]\,.$$

The equations then follow from (a), (b), (c), and the directness of the sum $A^* \oplus Q_{A^*} \oplus \sigma(P_{A^*})$.

Theorem 4.2. *If $\Gamma: (T, \tau) \to (S, \sigma)$ is a β-homomorphism, then $\Gamma^\wedge$ is a well-defined β-homomorphism from $S^\wedge$ to $T^\wedge$.*

Proof. We will show that $\Gamma^\wedge$ is well-defined by showing that for any $h \in S^\wedge$, $\Gamma^\wedge(h): T \to \mathbb{R}$ satisfies the conditions of the definition of $T^\wedge$ in § 2. (I) Clearly $\Gamma^\wedge(h)$ is a group-homomorphism. (II) Let $A \in \boldsymbol{A}_T$. Since Γ is locally real, $\Gamma^\wedge(h)|_A$ is comparable to 0 with respect to $\lesssim$ by Proposition 4.1 (5). (III) Since Γ is locally real, the map $A \to A^*$ takes Supp $(\Gamma^\wedge(h))$ onto Supp (h); by Proposition 4.1 (8), it is an order-isomorphism. Hence, since Supp (h) is well-ordered, Supp $(\Gamma^\wedge(h))$ is well-ordered. (IV) Let $\boldsymbol{T}$ denote the finite subsets of $\boldsymbol{A}_T$ and $\boldsymbol{S}$ the finite subsets of $\boldsymbol{A}_S$. Then by Proposition 4.1 (7) and (8), for any $x \in T$,

$$\lim_{\Phi \in \boldsymbol{T}} \sum_{A \in \Phi} m(x, A)^{-1}\, \Gamma^\wedge(h)\, (x_A) =$$
$$= \lim_{\Phi \in \boldsymbol{T}} \sum_{A \in \Phi} [m(x, A)\, m(\Gamma(x), A^*)]^{-1}\, h(m(\Gamma(x), A^*)\, \Gamma(x_A)) =$$
$$= \lim_{\Phi \in \boldsymbol{T}} \sum_{A \in \Phi} [m(x, A)\, m(\Gamma(x), A^*)]^{-1}\, h(m(x, A)\, \Gamma(x)_{A^*}) =$$
$$= \lim_{\Phi \in \boldsymbol{S}} \sum_{B \in \Phi} m(\Gamma(x), B)^{-1}\, h(\Gamma(x)_B) = h(\Gamma(x)) = \Gamma^\wedge(h)\, (x)\,.$$

We conclude that $\Gamma^\wedge$ is a well-defined function from $S^\wedge$ to $T^\wedge$. It remains to show that Γ is a β-homomorphism.

(i) Clearly $\Gamma^\wedge$ is a group-homomorphism.

(ii) Let $W \in \boldsymbol{P}_{T^\wedge}$ and let $V = \langle \Gamma(W_\sharp) \rangle \in \boldsymbol{P}_S$. Note that $A \subseteq W_\sharp$ if and only if $A^* \subseteq V$. If $g \in V^\wedge$, then $g|_{A^*} = 0$ for all $A^* \subseteq V$, and hence $\Gamma^\wedge(g)|_A = 0$ for all $A \subseteq W_\sharp$, i.e., $\Gamma^\wedge(g) \in W_\sharp{}^\wedge$. By Proposition 3.1, $W_\sharp{}^\wedge = W$, and hence $\langle \Gamma^\wedge(V^\wedge) \rangle \subseteq$ $\subseteq W$. Conversely, let $w \in W$, let $M = \wedge \operatorname{Supp}(w)$, and let $0 \neq z \in M^*$. Note that since $W = W_\sharp{}^\wedge$ by Proposition 3.1, $W_\sharp \subset \langle M \rangle$ and hence $z^\wedge \in V^\wedge$. But $\operatorname{Supp}(\Gamma^\wedge(z^\wedge)) = \{M\}$, as in (III) above, and hence $w \in \langle \Gamma^\wedge(z^\wedge) \rangle \subseteq \langle \Gamma^\wedge(V^\wedge) \rangle$. Thus $\langle \Gamma^\wedge(V^\wedge) \rangle \supseteq W$, and therefore $\Gamma^\wedge$ is dense.

(iii) Let $V \in \boldsymbol{P}_{S^\wedge}$. Let $W = \langle \Gamma^\wedge(V) \rangle$ and note that $V_\sharp = \langle \Gamma(W_\sharp) \rangle$. We wish to show that $\Gamma^\wedge(\sigma^\wedge(V)) \subseteq \tau^\wedge(W)$. Let $0 \neq f \in \sigma^\wedge(V)$ and let $\tau(W_\sharp) \supseteq A \in \boldsymbol{A}_T$. Since Γ is a Banaschewski homomorphism, $A^* \subseteq \sigma(\langle \Gamma(W_\sharp) \rangle) = \sigma(V_\sharp)$ so that $f|_{A^*} = 0$ and hence $\Gamma^\wedge(f)|_A = 0$. Thus $\Gamma^\wedge(f) \in \tau^\wedge(W)$, and therefore $\Gamma^\wedge$ is a Banaschewski homomorphism.

(iv) Let $D \in \boldsymbol{A}_{S^\wedge}$, and let $A \in \boldsymbol{A}_T$ be such that $\{A^*\} = \operatorname{Supp}(d)$ for all $0 \neq d \in D$. As above, $\operatorname{Supp}(\Gamma^\wedge(d)) = \{A\}$ for all $0 \neq d \in D$. Suppose that $0 \prec \Gamma|_A$. Then for all $0 < d \in D$, $0 \prec \Gamma^\wedge(d)|_A$, and hence $0 < \Gamma^\wedge(d)$. Similarly, if $\Gamma|_A \prec 0$, then $\Gamma^\wedge(d) < 0$ for all $0 < d \in D$. Since Γ is locally real, these are the only two possibilities. Therefore, $0 \prec \Gamma^\wedge|_D$ or $\Gamma^\wedge|_D \prec 0$, i.e., $\Gamma^\wedge$ is locally real. This proves Theorem 4.2.

Finally we note some special properties of a β-homomorphism Γ which lift to its dual map $\Gamma^\wedge$.

Theorem 4.3. *Let $\Gamma: (T, \tau) \to (S, \sigma)$ be a β-homomorphism.*

(1) *If Γ preserves order, then $\Gamma^\wedge$ also preserves order.*

(2) *If Γ is onto, then $\Gamma^\wedge$ is also onto.*

Proof. (1) We noted in the proof of Theorem 4.2 that for $h \in S^\wedge$, the map $A \to A^*$ is an order-isomorphism of $\operatorname{Supp}(\Gamma^\wedge(h))$ onto $\operatorname{Supp}(h)$. Therefore, since Γ preserves order and is locally real, $0 \prec \Gamma^\wedge(h)|_{\wedge \operatorname{Supp}(\Gamma^\wedge(h))}$ exactly when $0 \prec h|_{\wedge \operatorname{Supp}(h)}$.

(2) By Proposition 4.1 (2), Γ is a β-isomorphism: from this, it follows easily that $\Gamma^\wedge$ is onto.

In view of Proposition 4.1 (2), we call an order-preserving β-homomorphism an *o-β-monomorphism* (The "one-to-one π-homomorphisms" of [9] correspond to the o-β-monomorphisms here.) Not every β-homomorphism is an o-β-monomorphism (cf. Example 6.11).

5. THE SECOND DUAL

In this section we show that the evaluation map into the second dual is an o-β-monomorphism. As a consequence we are able to show that all odd-numbered dual spaces are o-β-isomorphic as are all even-numbered dual spaces.

For any β-group T, $T^\wedge$ is also a β-group by Theorem 3.2, an hence we may form the β-group $T^{\wedge\wedge}$. For $x \in T$, let $\Xi(x): T^\wedge \to \mathbb{R}$ be defined by letting $\Xi(x)(f) = f(x)$ for all $f \in T^\wedge$. We will show that $\Xi(x) \in T^{\wedge\wedge}$. Clearly, $\Xi(x)$ is a group-homomorphism and it is easy to see that for $A^\wedge \in \boldsymbol{A}^\wedge$, $\Xi(x)|_{A^\wedge}$ is comparable to 0 with respect to $\precsim$.

It is also clear that $\mathrm{Supp}\,(\Xi(x)) = \{A^\wedge \in \boldsymbol{A}^\wedge \mid A \in S(x)\}$. By Proposition 2.2, $S(x)$ is inversely well-ordered, and hence by Proposition 3.1, $\mathrm{Supp}\,(\Xi(x))$ is well-ordered. Furthermore, $T^\wedge$ is a strong β-group (Theorem 3.2) and hence $m(f, A^\wedge) = 1$ for all $f \in T^\wedge$ and $A^\wedge \in \boldsymbol{A}^\wedge$. Thus for $x \in T$, $f \in T^\wedge$, and $A \in \boldsymbol{A}$, we have

$$f_{A^\wedge}(x) = m(x, A)^{-1} f_{A^\wedge}(x_A + q_{x,A} + p_{x,A}) = m(x, A)^{-1} f_{A^\wedge}(x_A) =$$
$$= m(x, A)^{-1} \left[f_{A^\wedge} + q_{f,A^\wedge} + p_{x,A^\wedge}\right](x_A) = \mathrm{m}(x, A)^{-1} f(x_A),$$

and hence

$$\lim \sum \Xi(x)(f_{A^\wedge}) = \lim \sum f_{A^\wedge}(x) = \lim \sum m(x, A)^{-1} f(x_A) = f(x) = \Xi(x)(f).$$

Therefore, $\Xi(x) \in T^{\wedge\wedge}$, i.e. Ξ, the *evaluation map* [7], is a well-defined map from T to $T^{\wedge\wedge}$.

Theorem 5.1. *The evaluation map $\Xi: T \to T^{\wedge\wedge}$ is a well-defined o-β-monomorphism.*

Proof. We showed above that Ξ is well-defined, and it is clear that Ξ is a group-homomorphism. For $0 < x \in T$, $\wedge\,\mathrm{Supp}\,(\Xi(x)) = (\vee S(x))^\wedge$, and if $0 < f \in (\vee S(x))^\wedge$, then (by the computation above)

$$\Xi(x)(f) = f(x) = m(x, \vee S(x))^{-1} f(x_{\vee S(x)}) > 0,$$

i.e., $\Xi(x) > 0$. Therefore Ξ is order-preserving and one-to-one (cf. Proposition 4.1 (2)), and hence locally real. Since $\Xi(V_{\#\#}) \subseteq V$ for all $V \in \boldsymbol{P}^{\wedge\wedge}$, Ξ is dense; since $\Xi(\tau(P)) \subseteq \tau^{\wedge\wedge}(P^{\wedge\wedge})$ for all $P \in \boldsymbol{P}$, Ξ is a Banaschewski homomorphism. This proves Theorem 5.1.

That Ξ need not always be onto will be shown in § 6 (Example 6.9).

Theorem 5.2. *The function $\Xi^\wedge: T^{\wedge\wedge\wedge} \to T^\wedge$ is an o-β-isomorphism.*

Proof. By Theorems 5.1 and 4.3, $\Xi^\wedge$ is an o-β-monomorphism; in particular, $\Xi^\wedge$ is one-to-one by Proposition 4.1 (2). Let Υ be the evaluation map from $T^\wedge$ to $T^{\wedge\wedge\wedge}$. We will show that $\Xi^\wedge$ is onto by showing that $\Xi^\wedge \circ \Upsilon$ is the identity function on $T^\wedge$. If $x \in T$ and $f \in T^\wedge$, then

$$\Xi^\wedge(\Upsilon(f))(x) = \Upsilon(f)(\Xi(x)) = \Xi(x)(f) = f(x).$$

Thus $\Xi^\wedge \circ \Upsilon(f) = f$ and hence $\Xi^\wedge$ is onto. This proves Theorem 5.2.

For $n \geqq 1$, let $T^{\wedge(n)}$ denote the n^{th} dual space of T.

Corollary 5.3. *For all $n \geqq 1$, $T^{\wedge(2n-1)}$ is o-β-isomorphic to $T^\wedge$, and $T^{\wedge(2n)}$ is o-β-isomorphic to $T^{\wedge\wedge}$.*

6. EXAMPLES

For a totally ordered set Δ, the product $\prod_\Delta \mathbb{R}$ of copies of the real numbers $\mathbb{R}$ over Δ contains two lexicographically ordered o-groups. The o-group ${}_N\prod_\Delta \mathbb{R}$ is the group consisting of all functions in $\prod_\Delta \mathbb{R}$ with well-ordered support. The elements of ${}_N\prod_\Delta \mathbb{R}$ are ordered according to their values on the minimum elements in their supports. The o-group ${}_X\prod_\Delta \mathbb{R}$ is the group of all functions in $\prod_\Delta \mathbb{R}$ with inversely well-ordered support. The elements of ${}_X\prod_\Delta \mathbb{R}$ are ordered according to their values on the maximum elements in their supports. The corresponding sums are denoted by ${}_N\sum_\Delta \mathbb{R}$ and ${}_X\sum_\Delta \mathbb{R}$.

We turn these o-groups into β-groups in the following ways. If P is a convex subgroup of ${}_N\prod_\Delta \mathbb{R}$, then there exists $\mathsf{N}(P) \subseteq \Delta$ such that $\delta \in \mathsf{N}(P)$ whenever $\delta \leqq \eta \in \mathsf{N}(P)$ and such that

$$P = \{f \in {}_N\prod_\Delta \mathbb{R} \mid f_\delta = 0 \text{ for all } \delta \in \mathsf{N}(P)\} .$$

Define

$$\nu(P) = \{f \in {}_N\prod_\Delta \mathbb{R} \mid f_\delta = 0 \text{ for all } \delta \in \Delta \setminus \mathsf{N}(P)\} .$$

Clearly $({}_N\prod_\Delta \mathbb{R}, \nu)$ is a strong β-group, and the corresponding definition for ${}_N\sum_\Delta \mathbb{R}$ makes $({}_N\sum_\Delta \mathbb{R}, \nu)$ also a strong β-group. If Q is a convex subgroup of ${}_X\prod_\Delta \mathbb{R}$, then there exists $\mathsf{X}(Q) \subseteq \Delta$ such that $\delta \in \mathsf{X}(Q)$ whenever $\delta \geqq \eta \in \mathsf{X}(Q)$ and such that

$$Q = \{f \in {}_X\prod_\Delta \mathbb{R} \mid f_\delta = 0 \text{ for all } \delta \in \mathsf{X}(Q)\} .$$

Define similarly to the previous case

$$\chi(Q) = \{f \in {}_X\prod_\Delta \mathbb{R} \mid f_\delta = 0 \text{ for all } \delta \in \Delta \setminus \mathsf{X}(Q)\} .$$

Clearly $({}_X\prod_\Delta \mathbb{R}, \chi)$ is a strong β-group, and the corresponding definition for ${}_X\sum_\Delta \mathbb{R}$ makes $({}_X\sum_\Delta \mathbb{R}, \chi)$ also a strong β-group. Included in the examples below are characterizations of the first and second dual spaces of the sums and products of $\mathbb{R}$ defined above.

Proposition 6.1. *For any β-group T, there exists an o-β-monomorphism $\Gamma\colon {}_N\sum_A \mathbb{R} \to T^\wedge$.*

Proof. For each $A \in A$, let $i_A\colon A \to \mathbb{R}$ be a one-to-one, order-preserving group-homomorphism (see [4], page 46). For $d \in {}_N\sum_A \mathbb{R}$ and $x \in T$, let $\Gamma(d)(x) = \sum_{A \in A} d_A i_A(x_A)$.

Example 6.2. $({}_X\prod_\Delta \mathbb{R})^\wedge$ is o-β-isomorphic to ${}_N\sum_\Delta \mathbb{R}$. As above, the function $\Gamma\colon {}_N\sum_\Delta \mathbb{R} \to ({}_X\prod_\Delta \mathbb{R})^\wedge$, defined by letting $\Gamma(d)(x) = \sum_{\delta \in \Delta} d_\delta x_\delta$ for $d \in {}_N\sum_\Delta \mathbb{R}$ and $x \in {}_X\prod_\Delta \mathbb{R}$, is an o-$\beta$-monomorphism. To see that Γ is onto, let $f \in ({}_X\prod_\Delta \mathbb{R})^\wedge$. For $\delta \in \Delta$, let $e^\delta \in {}_X\prod_\Delta \mathbb{R}$ be such that $(e^\delta)_\eta = 1$ if $\eta = \delta$ and $(e^\delta)_\eta = 0$ otherwise. For $A \in A[{}_X\prod_\Delta \mathbb{R}, \chi]$, let $\alpha \in \Delta$ be such that $e^\alpha \in A$. Suppose that $f(e^\delta) \neq 0$ for an infinite number of $\delta \in \Delta$, and let $z \in {}_X\prod_\Delta \mathbb{R}$ be such that $z_\delta = 1/(e^\delta)$ if $f(e^\delta) \neq 0$ and $z_\delta = 0$ otherwise. Then $f(z_A) = 1$ if $f(e^\alpha) \neq 0$ and $f(z_A) = 0$ otherwise: hence $\lim \sum f(z_A)$ does not exist. This contradicts condition (iv) of the definition of the dual space, and we conclude that $f(e^\delta) = 0$ for all but a finite number of $\delta \in \Delta$. Hence $f \in \Gamma({}_N\sum_\Delta \mathbb{R})$.

Proposition 6.3. *For any β-group T, there exists an o-β-monomorphism $\Gamma\colon T^\wedge \to {}_N\prod_A \mathbb{R}$.*

Proof. For each $A \in \mathbf{A}$, let $i_A: A \to \mathbb{R}$ be as in the proof of Proposition 6.1. If $f \in T^\wedge$, then $f|_A = r_A i_A$ for a unique $r_A \in \mathbb{R}$ ([4], page 46). If we let $\Gamma(f)_A = r_A$, then clearly $\Gamma(f) \in {}_N\prod_{\mathbf{A}}\mathbb{R}$ and Γ is a one-to-one, order-preserving group-homomorphism. It follows that Γ is locally real, and it is easy to see that Γ is a dense Banaschewski homomorphism.

Note that Proposition 6.3, together with Theorem 5.1, shows that there always exists a β-homomorphism from T to ${}_N\prod_{\mathbf{A}}\mathbb{R}$. This is essentially Hahn's Theorem [5]. From the work of [2], it is not surprising that Hahn's Theorem should follow in this way.

Example 6.4. $({}_X\sum_\Delta\mathbb{R})^\wedge$ is o-β-isomorphic to ${}_N\prod_\Delta\mathbb{R}$. For $f \in ({}_X\sum_\Delta\mathbb{R})^\wedge$ and e^δ as in Example 6.2, define Γ: $({}_X\sum_\Delta\mathbb{R})^\wedge \to {}_N\prod_\Delta\mathbb{R}$ by letting $\Gamma(f)_\delta = f(e^\delta)$. As in Proposition 6.3, Γ is an o-β-monomorphism. For $d \in {}_N\prod_\Delta\mathbb{R}$, let f: ${}_X\sum_\Delta\mathbb{R} \to \mathbb{R}$ be defined by letting $f(x) = \sum_{\delta\in\Delta} d_\delta x_\delta$ for all $x \in {}_X\sum_\Delta\mathbb{R}$. Clearly, $f \in ({}_X\sum_\Delta\mathbb{R})^\wedge$ and $\Gamma(f) = d$. Thus Γ is also onto.

Let ∇ denote the set Δ with the opposite order: $\gamma \leqq \delta$ in ∇ if and only if $\gamma \geqq \delta$ in Δ. It is straightforward to prove

Proposition 6.5. *$({}_X\prod_\nabla\mathbb{R}, \chi)$ is o-β-isomorphic to $({}_N\prod_\Delta\mathbb{R}, \nu)$, and $({}_X\sum_\nabla\mathbb{R}, \chi)$ is o-β-isomorphic to $({}_N\sum_\Delta\mathbb{R}, \nu)$.*

Example 6.6. $({}_N\prod_\Delta\mathbb{R})^\wedge$ is o-β-isomorphic to ${}_X\sum_\Delta\mathbb{R}$. By Proposition 6.5 and Theorem 4.3, $({}_N\prod_\Delta\mathbb{R}, \nu)^\wedge$ is o-β-isomorphic to $({}_X\prod_\nabla\mathbb{R}, \chi)^\wedge$; by Example 6.2, $({}_X\prod_\nabla\mathbb{R}, \chi)^\wedge$ is o-β-isomorphic to $({}_N\sum_\nabla\mathbb{R}, \nu)$; by Proposition 6.5, $({}_N\sum_\nabla\mathbb{R}, \nu)$ is o-β-isomorphic to $({}_X\sum_\Delta\mathbb{R}, \chi)$.

Example 6.7. $({}_N\sum_\Delta\mathbb{R})^\wedge$ is o-β-isomorphic to ${}_X\prod_\Delta\mathbb{R}$. Use Proposition 6.5, Theorem 4.3, and Example 6.4.

Proposition 6.8. *The evaluation maps for ${}_X\sum_\Delta\mathbb{R}$, ${}_N\sum_\Delta\mathbb{R}$, ${}_X\prod_\Delta\mathbb{R}$, and ${}_N\prod_\Delta\mathbb{R}$ are all onto and hence o-β-isomorphisms.*

Proof. Let T denote any of the four β-groups ${}_X\sum_\Delta\mathbb{R}$, ${}_N\sum_\Delta\mathbb{R}$, ${}_X\prod_\Delta\mathbb{R}$, or ${}_N\prod_\Delta\mathbb{R}$. For each $\delta \in \Delta$, let $\boldsymbol{\delta} \in T^\wedge$ be defined by letting $\boldsymbol{\delta}(x) = x_\delta$ for all $x \in T$. Then (1) for all $A \in \mathbf{A}$, there exists $\delta \in \Delta$ such that, for all $f \in A^\wedge$, $f = r\boldsymbol{\delta}$ for some $r \in \mathbb{R}$. Also, (2) for any $\boldsymbol{F} \in T^{\wedge\wedge}$, $\boldsymbol{F}(r\boldsymbol{\delta}) = r\,\boldsymbol{F}(\boldsymbol{\delta})$ for all $r \in \mathbb{R}$ and $\delta \in \Delta$. To see that (2) holds, note that if $\boldsymbol{F}(\boldsymbol{\delta}) = 0$, then the equality obviously holds. If $\boldsymbol{F}(\boldsymbol{\delta}) \neq 0$, then define F_1, F_2: $\mathbb{R} \to \mathbb{R}$ by letting $F_1(r) = \boldsymbol{F}(r\boldsymbol{\delta})$ and $F_2(r) = r\,\boldsymbol{F}(\boldsymbol{\delta})$. These are both non-zero order-preserving or order-reversing group-homomorphisms and hence by [4], page 46, $F_1 = dF_2$ for some $0 \neq d \in \mathbb{R}$. Since $F_1(1) = F_2(1)$, $d = 1$, and hence $F_1 = F_2$, i.e. (2) holds. Now let $\boldsymbol{F} \in T^{\wedge\wedge}$ and define $z \in \prod_\Delta\mathbb{R}$ by letting $z_\delta = \boldsymbol{F}(\boldsymbol{\delta})$. If $z \in T$, then by (1) and (2) above, $\boldsymbol{F}(f) = \Xi(z)(f)$ for all $f \in A^\wedge \in \mathbf{A}^\wedge$. Since both $\boldsymbol{F}$ and $\Xi(z)$ are uniquely determined by their behaviour on the elements of $\mathbf{A}^\wedge$, we must have $\Xi(z) = \boldsymbol{F}$. Thus, it suffices to show that $z \in T$. If $T = {}_X\prod_\Delta\mathbb{R}$, then because $\boldsymbol{F}$ has well-ordered support in $\mathbf{A}^\wedge$, z has inversely well-ordered support in $\mathbf{A}$ (cf. Proposition

3.1), and hence $z \in T$. Similarly, if $T = {}_N\prod_\Delta \mathbb{R}$, then $z \in T$. If $T = {}_X\sum_\Delta \mathbb{R}$, then $T^{\wedge\wedge}$ is o-β-isomorphic to $({}_N\prod_\Delta \mathbb{R})^\wedge$ by Example 6.4 and Theorem 4.3. By Example 6.6, $T^{\wedge\wedge}$ is then o-β-isomorphic to ${}_X\sum_\Delta \mathbb{R}$. Thus F has finite support, and hence $z \in T$. If $T = {}_N\sum_\Delta \mathbb{R}$, a similar argument using Theorem 4.3 and Examples 6.2 and 6.7 shows that $z \in T$. Proposition 6.8 then follows from Theorem 5.1.

In spite of Proposition 6.8, the evaluation map is not always onto: By Theorem 5.1, an evaluation map $\Xi\colon T \to T^{\wedge\wedge}$ is always one-to-one, and by Theorem 2.3, $T^{\wedge\wedge}$ is divisible. Therefore, if T is not divisible, then Ξ cannot be onto. The next example shows that the evaluation map need not be onto even if T is a divisible β-group.

Example 6.9. Let $\mathbb{Q}$ denote the rational numbers and define $({}_X\prod_\Delta \mathbb{Q}, +, \leqq, \chi)$ analogously to $({}_X\prod_\Delta \mathbb{R}, +, \leqq, \chi)$. It is clear that $({}_X\prod_\Delta \mathbb{Q}, +, \leqq, \chi)$ is a divisible β-group and that $({}_X\prod_\Delta \mathbb{Q})^\wedge$ is o-β-isomorphic to $({}_X\prod_\Delta \mathbb{R})^\wedge$. Thus, by Theorem 4.3, $({}_X\prod_\Delta \mathbb{Q})^{\wedge\wedge}$ is o-β-isomorphic to $({}_X\prod_\Delta \mathbb{R})^{\wedge\wedge}$ and hence, by Proposition 6.8, to ${}_X\prod_\Delta \mathbb{R}$. Therefore, if the evaluation map $\Xi\colon {}_X\prod_\Delta \mathbb{Q} \to ({}_X\prod_\Delta \mathbb{Q})^{\wedge\wedge}$ were onto, it would induce, by Proposition 4.1 (5), an order-isomorphism from $\mathbb{Q}$ to $\mathbb{R}$. Since such a function cannot exist, we conclude that Ξ cannot be onto.

The following example illustrates the dual relationship between local reality and density: The dual of a non-dense map need not be locally real and the dual of a non-locally real map need not be dense.

Example 6.10. Let $\Delta = \{1, 2\}$ with the usual order. Define $\Gamma\colon \mathbb{R} \to {}_X\prod_\Delta \mathbb{R}$ by $\Gamma(r) = (0, r)$. Clearly Γ is a locally real Banaschewski homomorphism and a group-homomorphism but is not dense. It is also clear (cf. Example 6.2) that $\Gamma^\wedge\colon {}_N\sum_\Delta \mathbb{R} \to \mathbb{R}$ is defined by $\Gamma^\wedge(r, s) = s$. Thus $\Gamma^\wedge$ is a dense Banaschewski homomorphism and a group-homomorphism but is not locally real. Furthermore, $\Gamma^{\wedge\wedge}\colon \mathbb{R} \to {}_X\prod_\Delta \mathbb{R}$ is defined by $\Gamma^{\wedge\wedge}(r) = (0, r)$, and hence, as noted above, $\Gamma^{\wedge\wedge}$ is locally real but not dense.

We claimed in § 4 that not every β-homomorphism is an o-β-monomorphism. The following example shows that a β-homomorphism need be neither order-preserving nor order-reversing.

Example 6.11. As in Example 6.10, let $\Delta = \{1, 2\}$ with the usual order. Then $\Gamma\colon {}_X\prod_\Delta \mathbb{R} \to {}_X\prod_\Delta \mathbb{R}$ defined by $\Gamma(r, s) = (-r, s)$ is a β-homomorphism which is neither order-preserving nor order-reversing.

Our final example shows that different Banaschewski functions on the same o-group T may give rise not only to dual spaces which are different subgroups of the group of homomorphisms from T into $\mathbb{R}$ but also to dual spaces which are not even β-isomorphic.

Example 6.12. Let $\mathbb{Z}$ denote the integers, and let T denote the o-subgroup of eventually constant sequences in ${}_X\prod_{\mathbb{Z}} \mathbb{R}$: T consists of all those $x \in {}_X\prod_{\mathbb{Z}} \mathbb{R}$ such that for some $N \in \mathbb{Z}$, $x_n = x_m$ whenever $m, n \leqq N$. For $i \in \mathbb{Z}$, let c^i denote the long constant determined by i: $(c^i)_n = 0$ if $n > i$ and $(c^i)_n = 1$ if $n \leqq i$. For any $x \in T$, let $x^i = x_i - x_{i+1}$. Note that $x^i = 0$ for all but a finite number of i and $x = \sum_{i \in Z} x^i c^i$. Let τ_1 denote the usual Banaschewski function on G (derived from the function χ defined above for the entire product): For any convex subgroup P of G, $x \in \tau_1(P)$ if and only if $x_n = 0$ whenever $p_n \neq 0$ for some $p \in P$. Let τ_2 denote the following different Banaschewski function on T: For any convex subgroup P of T, $y \in \tau_2(P)$ if and only if there exists $N \in \mathbb{Z}$ such that (i) $n < N$ whenever $p_n \neq 0$ for some $p \in P$ and (ii) $y_n = y_m$ whenever $m, n \leqq N$. Each $A \in A[T, \tau_2]$ is then of the form $\{rc^i \mid r \in \mathbb{R}\}$ for some i. If $f: T \to \mathbb{R}$ is defined by letting $f(x) = \sum_{i \geqq 1} x^i$, then $f \in (T, \tau_2)^\wedge \setminus (T, \tau_1)^\wedge$, and hence $(T, \tau_1)^\wedge \neq (T, \tau_2)^\wedge$.

To see that $(T, \tau_1)^\wedge$ is not even β-isomorphic to $(T, \tau_2)^\wedge$, suppose that $\Theta: (T, \tau_2)^\wedge \to (T, \tau_1)^\wedge$ is a β-isomorphism and define $\Psi: (T, \tau_2) \to ({}_X\sum_Z \mathbb{R}, \chi)$ by letting $\Psi(x)_n = x^n$. Clearly, Ψ is a β-isomorphism and hence by Theorem 4.3,

$$\Psi^{\wedge\wedge} \circ \Theta^\wedge: (T, \tau_1)^{\wedge\wedge} \to (T, \tau_2)^{\wedge\wedge} \to ({}_X\textstyle\sum_Z \mathbb{R}, \chi)^{\wedge\wedge}$$

is also a β-isomorphism. By Proposition 6.8, $({}_X\sum_Z \mathbb{R}, \chi)^{\wedge\wedge}$ is β-isomorphic to $({}_X\sum_Z \mathbb{R}, \chi)$ and hence by Theorem 5.1, there exists a one-to-one β-homomorphism $\Upsilon: (T, \tau_1) \to ({}_X\sum_Z \mathbb{R}, \chi)$. By Proposition 4.1 (5), $\Upsilon({}_X\sum_Z \mathbb{R}) = {}_X\sum_Z \mathbb{R}$, a contradiction. We conclude that the β-isomorphism Θ cannot exist, and thus that $(T, \tau_1)^\wedge$ and $(T, \tau_2)^\wedge$ cannot be β-isomorphic.

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Author's address: Kelowna, British Columbia, Canada V1Y 7N5. *Current address:* Hamilton College, Clinton, NY 13323, U.S.A.