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Eduard Feireisl

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ON THE EXISTENCE OF INFINITELY MANY PERIODIC SOLUTIONS FOR AN EQUATION OF A RECTANGULAR THIN PLATE

EDUARD FEIREISL, Praha

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In this paper, our aim is to demonstrate the existence of infinitely many time periodic solutions to the problem

{P}

(P1) $$u_{tt} + \Delta^2 u + f(x, y, t, u) = 0\,, \quad (x, y) \in Q\,, \quad t \in R\,,$$

(P2) $$u = \Delta u = 0\,, \quad (x, y) \in \partial Q\,, \quad t \in R\,,$$

(P3) $$u(x, y, t + T) = u(x, y, t)\,, \quad (x, y) \in Q\,, \quad t \in R$$

where $Q = (0, a) \times (0, b)$ is a rectangle and Δ denotes the Laplacian.

To be more precise, the weak solution of {P} can be found the norm of which in a certain space of periodic functions exceeds an arbitrarily chosen positive value. The crucial condition we assume in the sequel is that both a/b and $a^2/T\pi$ are rational numbers. It is an interesting task to find out how to treat the above problem if this is not the case. Moreover, the function f is supposed to satisfy some "reasonable" requirements specified in Section 2, (F1)–(F3).

In [3], an analogous problem is investigated in the case of a wave equation. The technique which is used in this work, however, does not seem to be of any help to us here. It is mainly its dependence on the d'Alembert operator that prevents us from applying it to our equation.

In order to cope with the given problem, we have employed the approximation method of Rayleigh-Ritz. In this way, we get a sequence of variational problems solvable with the aid of some topological methods (see Section 3). The approximate solutions we have obtained should converge to a weak solution of {P}. To accomplish it in a general situation, we have no alternative but to require the function f to be monotone with respect to u (the assumption (F1)). Nonetheless, this assumption can be avoided if f depends on the variable u only. It should be pointed out that the method we have just sketched works in this case as well. This is what made us abandon the dual action approach here (for this method see e.g. [4]).

1. PRELIMINARIES

We use the standard notation. In particular, the symbol $\mathbb{R}$ will denote the set of real numbers, $\mathbb{N}$ the set of positive integers, $\mathbb{Z}$ the set of all integers. Throughout the paper, the symbols c_i, $i \in \mathbb{N}$ are used to denote positive constants.

For definiteness we set $T = 2\pi$, $a = b = \pi$. The general case can be treated in a similar way. Appropriate spaces in which the solution of $\{P\}$ is to be looked for are the spaces L_p of periodic functions which are defined as the closure of all real functions smooth on $Q \times R$ and satisfying (P2), (P3) with respect to the norm

$$\|v\|_p = \left[\int_Q \int_0^{2\pi} |v|^p \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right]^{1/p}, \quad 1 \leqq p < +\infty .$$

A suitable basis to the space L_2 is formed by the eigenfunctions of the linear part of the equation (P1)–(P3), i.e. by the functions

$$e_{k,l,j}(x, y, t) = \begin{array}{ll} \sin(kx)\sin(ly)\sin(jt), & j > 0, \\ \sin(kx)\sin(ly)\cos(jt), & j \leqq 0 \end{array}$$

where $(k, l, j) \in I$, $I = \mathbb{N} \times \mathbb{N} \times \mathbb{Z}$.

The Fourier coefficients are determined by the formula

$$a_q(v) = \int_Q \int_0^{2\pi} v e_q \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \quad q \in I .$$

Note that our definition makes sense even for functions belonging to L_1.

Let us now pay attention to the system of the corresponding eigenvalues

$$\lambda_{k,l,j} = (k^2 + l^2)^2 - j^2, \quad \Lambda = \{\lambda_q\}_{q\in I} .$$

For later purposes, it seems to be convenient to introduce the sets

$$\{\Lambda \leqq z\} = \operatorname{span}\{e_q \mid \lambda_q \leqq z\}, \quad z \in \mathbb{R} .$$

We emphasize that the symbol span is used for all finite linear combinations (over $\mathbb{R}$).

Finally, we shall deal with the functionals (quasinorms)

$$|||v|||_\alpha = [\sum_{q\in I} |\lambda_q|^\alpha a_q^2(v)]^{1/2}, \quad \alpha \geqq 0 .$$

At the end of this section, we are going to state some technical results. First of all, we claim that

$$\sum_{\lambda_q \neq 0} |\lambda_q|^{-\alpha} < +\infty$$

whenever $\alpha > 1$. For the proof, see [2].

Using the above estimate and the Hölder inequality, we get after an easy computation

$$\|v\|_\infty \leqq c_1 |||v|||_\alpha \quad \text{for all} \quad v \in \{\Lambda \neq 0\}, \quad \alpha > 1 .$$

This yields (via the complex interpolation theory) that for fixed p, $p > 2$, a number

$\beta < 1$ can be found such that

(1) $$\|v\|_p \leqq c_2 |||v|||_\beta$$

holds for every function $v \in \{A \neq 0\}$, $\beta = \alpha(p - 2)/p$.

2. FORMULATION OF THE MAIN THEOREM

Definition 1. The function u is said to be a *weak solution of* $\{P\}$ if $u \in L_1, f(\cdot, u) \in L_1$ and

(2) $$\int_Q \int_0^{2\pi} u(\varphi_{tt} + \Delta^2 \varphi) + f(\cdot, u)\, \varphi \, dx \, dy \, dt = 0$$

for all functions φ sufficiently smooth and satisfying (P2), (P3).

It is possible to show that (2) is equivalent to

(3) $$\lambda_q \, a_q(u) + a_q(f(\cdot, u)) = 0 \quad \text{for all} \quad q \in I .$$

We proceed to the formulation of the main theorem we intend to prove.

Theorem 1. *Let us suppose that the function* $f \in C(\bar{Q} \times \mathbb{R}^2)$ *is* 2π*-periodic in* t *and satisfies*

(F1) $f(x, y, t, u)$ *is nondecreasing in* u *and there is* u_0 *such that* $f(\cdot, u)\, u \geqq 0$ *whenever* $|u| \geqq u_0$;

(F2) *setting* $F(x, y, t, u) = \int_0^u f(x, y, t, s) \, ds$, *the estimate*

$$\tfrac{1}{2} u f(x, y, t, u) - F(x, y, t, u) \geqq c_3(|f(x, y, t, u)|^{p'} + |u|^p) - c_4$$

holds for all x, y, t, u, *where* $p > 2$ *and* $1/p + 1/p' = 1$;

(F3) *at least one of the following conditions is fulfilled: either*

(a) f *does not depend on* t

or

(b) f *is an odd function in* u $(f(\cdot, -u) = -f(\cdot, u))$.

Then, for each $d > 0$, *there exists a weak solution* $u \in L_p$ *of the problem* $\{P\}$ *with* $\|u\|_p \geqq d$.

The remaining part of this paper will be devoted to the proof of Theorem 1.

3. THE APPROXIMATE PROBLEM

Let us start with reviewing some helpful results. Using (F1), (F2), it is a matter of routine to deduce the estimates

(4) $$|f(x, y, t, u)|^{p'} \leqq c_5 |u|^p + c_6 ,$$

(5) $$F(x, y, t, u) \leqq c_7 |u|^p + c_8 ,$$

(6) $$F(x, y, t, u) \geqq c_9 |u|^p - c_{10} \quad \text{for all} \quad x, y, t, u .$$

Now, we proceed to the statement of our approximate problem. The energy functional corresponding to $\{P\}$ is formally given as

$$J(v) = \tfrac{1}{2} \sum_{q \in I} \lambda_q \, a_q^2(v) + \int_Q \int_0^{2\pi} F(\cdot, v) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \, .$$

Consider a sequence $\{E_n\}_{n \in \mathbb{N}}$ of Hilbert spaces,

$$E_n = \operatorname{span} \{ e_{k,l,j} \mid k, l \leqq n, \ |j| \leqq n \} \, ,$$

endowed with, say, the L_2-norm. Obviously, the functional J becomes differentiable on the space E_n and consequently, the following definition makes sense:

Definition 2. By an approximate solution u_n of $\{P\}$ we mean any critical point of the functional J on E_n, i.e. the solution of the Euler equation

$$\sum_{q \in I} \lambda_q \, a_q(u_n) \, a_q(w) + \int_Q \int_0^{2\pi} f(\cdot, u_n) \, w = 0 \quad \text{for all} \quad w \in E_n \, . \tag{7}$$

We are going to derive some auxiliary assertions. Keeping the estimate (6) in mind, it is not difficult to see that J is bounded from below on the space $\{\Lambda \geqq z\} \cap E_n$ independently of $n \in \mathbb{N}$. Exactly speaking, for $z \in \mathbb{R}$ there is a constant $\Omega(z)$ such that

$$J(v) > \Omega(z) \quad \text{for all} \quad v \in \{\Lambda > z\} \, . \tag{8}$$

We intend to deduce some upper estimates of J.

Lemma 1. *Let $z \in \mathbb{R}$ be an arbitrarily chosen number. Then there is $\omega(z) < 0$ such that the estimate*

$$J(v) \leqq z \tag{9}$$

holds for all functions $v \in \{\Lambda \leqq \omega\}$, $|||v|||_\beta = 1$.

Proof. According to (5), we get

$$J(v) \leqq -\tfrac{1}{2} \sum_{\lambda_q \leqq \omega} |\lambda_q| \, a_q^2(v) + c_7 \|v\|_p^p + c_8 \, .$$

By the help of (1), we immediately have

$$J(v) \leqq -\tfrac{1}{2} |\omega|^{1-\beta} + c_2^p c_7 + c_8 \, ,$$

which implies (9) provided $|\omega|$ is sufficiently large. Q.E.D.

In order to establish the existence of critical points of J on E_n, a variant of a standard result of the calculus of variations is needed.

Lemma 2. *Let us consider the restriction of J on E_n. Assume that there are no critical values in the interval $[s, r]$, i.e.* $\operatorname{grad} J(v) \neq 0$ *whenever $J(v) \in [s, r]$.*

Then there exists a continuous mapping h: $E_n \to E_n$ such that

$$h(\{v \mid J(v) \leqq r\}) \subset \{v \mid J(v) \leqq s\} \, . \tag{10}$$

Moreover, either $h(v(\cdot, t + \tau)) = h(v)(\cdot, t + \tau)$ for all $\tau \in [0, 2\pi]$, $v \in E_n$ if we have the condition (F3) (a), *or h is an odd mapping provided* (F3) (b) *is true.*

Proof. Observe that it suffices to show our assertion in the case $s = z - \varepsilon$,

$r = z + \varepsilon$ where $z \in \mathbb{R}$ is a noncritical value and ε is a suitably chosen positive number. In view of (6), the sets $\{v \mid J(v) \leqq a\}$, $a \in \mathbb{R}$ are compact in E_n. Thus, taking into account this fact, the Taylor expansion of J yields

$$(11) \qquad J(v) = J(w) + \langle \text{grad } J(w), v - w \rangle + o(\|v - w\|_2)$$

for all $v, w \in \{v' \mid J(v') \leqq z + \varepsilon_1\}$, $\varepsilon_1 > 0$.

As a consequence of the preceding hypotheses, there are positive numbers δ_1, δ_2, ε_2 satisfying the relations

$$(12) \qquad \|\text{grad } J(v)\|_2 \geqq \delta_1$$

whenever $v \in \{w \mid J(w) \in [z - \varepsilon_2, z + \varepsilon_2]\}$,

$$(13) \qquad \|\text{grad } J(v)\|_2 \leqq \delta_2$$

for all $v \in \{w \mid J(w) \leqq z + \varepsilon_1\}$.

Consider now the mapping h_Θ defined by

$$h_\Theta(v) = v - \Theta \text{ grad } J(v), \quad v \in E_n .$$

Using (11) we obtain

$$J(h_\Theta(v)) = J(v) - \Theta \|\text{grad } J(v)\|_2^2 + o(\|\Theta \text{ grad } J(v)\|_2) .$$

Consequently, (12) and (13) together imply the existence of Θ_1 such that

$$(14) \qquad J(h_\Theta(v)) \leqq J(v) - \varepsilon_3 , \quad \varepsilon_3(\Theta) > 0$$

provided that $0 < \Theta < \Theta_1$, and $v \in \{w \mid J(w) \in [z - \varepsilon_1, z + \varepsilon_1]\}$.

In view of (13),

$$J(h_\Theta(v)) \leqq z - \varepsilon_4 , \quad 0 < \varepsilon_4 < \varepsilon_1$$

if $0 < \Theta < \Theta_2$ and $v \in \{w \mid J(w) \leqq z - \varepsilon_1\}$.

From these facts we deduce the existence of $\Theta > 0$, $\varepsilon > 0$ such that

$$h_\Theta(\{v \mid J(v) \leqq z + \varepsilon\}) \subset \{v \mid J(v) \leqq z - \varepsilon\} .$$

Clearly, the mapping h_Θ possesses all properties mentioned in Lemma 2. Q.E.D.

Now, we are in a position to prove the existence of suitable approximate solutions to $\{P\}$. The following lemma contains a desirable result.

Lemma 3. *For an arbitrarily chosen $z \in \mathbb{R}$ there exist a number $\gamma(z)$ and a sequence $\{u_n\}_{n \in \mathbb{N}}$ of approximate solutions to $\{P\}$ satisfying*

$$(15) \qquad J(u_n) \in [\gamma(z), z] \quad \text{for all} \quad n \in \mathbb{N}$$

(exactly speaking, for all $n \geqq n_0$, n_0 being a certain positive integer).

Proof. Without loss of generality, assume $z < 0$. Let us denote

$$S_n = E_n \cap \{v \mid |||v|||_\beta = 1\} \cap \{\Lambda \leqq \omega(z)\} ,$$

where $\omega(z)$ is the number the existence of which is ensured by Lemma 1. Moreover, consider n sufficiently large in order to have $S_n \neq \emptyset$.

Choose z_1 in such a way that $\{\Lambda \leqq z_1\} \subsetneqq \{\Lambda \leqq \omega(z)\}$ and set $\gamma(z) = \Omega(z_1)$, Ω appearing in (8).

We claim that there is at least one critical value of J in the interval $[\gamma(z), z]$.

Assume the contrary. Thus we have the mapping h which has been constructed in Lemma 2. Taking an orthogonal projection P onto the space $E_n \cap \{\Lambda \leqq z_1\}$ in E_n, we are able to define a new mapping η, $\eta: S_n \to S_n \cap \{\Lambda \leqq z_1\}$,

$$\eta(v) = \frac{P\,h(v)}{|||P\,h(v)|||_\beta}\,.$$

In view of (8), (9), (10), this step is fully justified.

From the topological point of view, η maps the sphere S_n into its proper subsphere. If (F3) (b) holds, we get a contradiction with the well known Borsuk-Ulam theorem since η is an odd mapping.

If the condition (F3) (a) is fulfilled, we consider an orthogonal action of the group S^1 on E_n defined by the formula

$$T_\tau\, v(\cdot, t) = v(\cdot, t + \tau)\,, \quad \tau \in [0, 2\pi]/\{0, 2\pi\} \sim S^1\,.$$

The mapping η is S^1-equivariant in accordance with [1]. We close up analogously as in the situation above by employing the S^1-version of the theorem of Borsuk-Ulam presented e.g. in [1]. In comparison with the general case investigated in [1], things are much simpler here because there are no fixed points of this action on S_n (see [1] for details). Q.E.D.

4. PASSING TO THE LIMIT

Our eventual goal is to carry out the limit process in the sequence $\{u_n\}_{n\in\mathbb{N}}$ which appears in Lemma 3.

To accomplish it, we set $w = u_n$ in (7). Using (15), we easily deduce

$$\int_Q \int_0^{2\pi} F(\cdot, u_n) - \tfrac{1}{2} f(\cdot, u_n)\, u_n \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \in [\gamma(z), z]\,.$$

Taking into account (F2), we get in turn

$$\text{(16)} \qquad \begin{aligned} &\{u_n\}_{n\in\mathbb{N}} \text{ is bounded in } L_p\,, \\ &\{f(\cdot, u_n)\}_{n\in\mathbb{N}} \text{ is bounded in } L_{p'}\,. \end{aligned}$$

Moreover, the condition (F2) now implies

$$\text{(17)} \qquad c_{11} \int_Q \int_0^{2\pi} f(\cdot, u_n)\, u_n \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \geqq -(z + c_{12})\,.$$

Let us insert $w = \sum_{|\lambda_q| \geqq q_0} \operatorname{sgn}(\lambda_q)\, a_q(u_n)\, e_q$ in (7), $q_0 > 0$. Keeping the estimate (1) in mind, we obtain as a consequence of (16)

$$\sum_{|\lambda_q| \geqq q_0} |\lambda_q| \, a_q^2(u_n) \leqq c_{13} |||w|||_\beta \leqq c_{13} q_0^{(\beta-1)/2} \Big[\sum_{|\lambda_q| \geqq q_0} |\lambda_q| \, a_q^2(u_n) \Big]^{1/2} .$$

Since $\beta < 1$, we have obtained the following lemma.

Lemma 4. *For arbitrary $\varepsilon > 0$ there exists q_0 such that*

$$\sum_{|\lambda_q| \geqq q_0} |\lambda_q| \, a_q^2(u_n) < \varepsilon \ \textit{for all} \quad n \in \mathbb{N} .$$

Passing to a subsequence (denoted $\{u_n\}_{n\in\mathbb{N}}$ again), we get according to (16)

$$\begin{aligned} u_n &\to u \quad \text{weakly in} \quad L_p , \\ f(\cdot, u_n) &\to g \quad \text{weakly in} \quad L_{p'} . \end{aligned} \tag{18}$$

Moreover, using Lemma 4 we can deduce

$$\lim_{n\to\infty} \sum_{q\in I} \lambda_q \, a_q^2(u_n) = \sum_{q\in I} \lambda_q \, a_q^2(u) . \tag{19}$$

For fixed $w \in E_n$, we can pass to the limit in (7):

$$\sum_{q\in I} \lambda_q \, a_q(u) \, a_q(w) + \int_Q \int_0^{2\pi} g w \, dx \, dy \, dt = 0 . \tag{20}$$

The only thing we have to prove is the equality $g = f(\cdot, u)$. To overcome this inherent difficulty of nonlinear problems, arguments of monotonicity are used. Setting $w = u_n$ in (7) and letting $n \to \infty$, we conclude

$$\lim_{n\to\infty} \int_Q \int_0^{2\pi} f(\cdot, u_n) \, u_n \, dx \, dy \, dt = - \sum_{q\in I} \lambda_q \, a_q^2(u) . \tag{21}$$

Let us now insert $w = u_n$ in (20). We have

$$\sum_{q\in I} \lambda_q \, a_q^2(u) = - \int_Q \int_0^{2\pi} g u \, dx \, dy \, dt . \tag{22}$$

Combining (21), (22), we obtain a desirable relation

$$\lim_{n\to\infty} \int_Q \int_0^{2\pi} f(\cdot, u_n) \, u_n \, dx \, dy \, dt = \int_Q \int_0^{2\pi} g u \, dx \, dy \, dt . \tag{23}$$

The standard argument of Minty immediately gives $g = f(\cdot, u)$ since the function f can be understood as a continuous monotone operator from L_p into $L_{p'}$ via (F1), (4).

We have proved that u is a weak solution of $\{P\}$. In order to ensure $\|u\|_p \geqq d$, it suffices to choose the number z appearing in (12) small enough ($z < 0$).

Theorem 1 has been proved.

Remark. If it is possible to restrict our considerations to the space of symmetric functions (see [5]), the estimate (19) enables us to carry out the limit process via compactness only. The assumption (F1) can be dropped.

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Author's address: 115 67 Praha 1, Žitná 25, Czechoslovakia (Matematický ústav ČSAV), *present address:* 120 00 Praha 2, Karlovo nám. 13, Czechoslovakia (FSI ČVUT).