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OPTIMAL STOPPING AND IMPULSIVE CONTROL OF ONE-DIMENSIONAL DIFFUSION PROCESSES

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1. INTRODUCTION

In the optimal stopping time problem one looks for a stopping time θ such that stopping at time θ maximalizes the reward functional. There are two approaches to this problem. The first one, due to E. B. Dynkin, is based on the method of superharmonic majorants (cf. [7]). In [9] an analytical characterization of the reward function relying on this approach is given, it characterizes the reward function as a solution of the Stefan problem with free boundary. The Stefan problem seems not to be the right analytical tool, because the uniqueness of solution requires undesirable smoothness of the reward function. The second approach, the so-called method of penalization, was used by A. Bensoussan and J. L. Lions [2] together with variational inequalities, which provide a suitable analytical characterization. Variational inequalities are generalization of partial differential equations and they usually appear in problems of mathematical physics involving obstacles. Here we use basically the Dynkin's approach combined with the theory of variational inequalities. The analytical characterization is based on the observation that while the reward function is equal to the least supermedian majorant of the given function ψ, the solution of the variational inequality equals the least supersolution which majorizes ψ. It remains to find suitable conditions under which the supermedian functions and supersolutions coincide. This is done by lemma (5, 10).

The impulsive control enables us to shift the trajectory in a random time by a random vector. It comes out that this type of control is closely related to the optimal stopping time problem (cf. [1] and theorem (6, 3)). The analytical characterization is provided by quasivariational inequalities (cf. (7,3)) which are more general than variational inequalities and describe for instance the dam soak.

Except of Section 3, where general Markov processes are considered, we work with diffusion processes in $\langle\alpha, \beta\rangle \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$ determined by the second order ordinary differential operator

$$A\, u(x) = a(x)\, u''(x) + b(x)\, u'(x)$$

and boundary conditions

$$\varrho_\alpha u(\alpha) - \pi_\alpha e^{B(\alpha)} u'(\alpha) = 0$$
$$\varrho_\beta u(\beta) + \pi_\beta e^{B(\beta)} u'(\beta) = 0 ,$$

where a, b are continuous on (α, β), $a > 0$ there and $\varrho_\alpha, \pi_\alpha, \varrho_\beta, \pi_\beta \geqq 0$. We impose no restriction on boundary behaviour of a, b and that is why we need the weighted Sobolev spaces.

Section 2 contains preliminary material from analysis, in Section 3 the optimal stopping time problem is solved by means of probability theory. Weighted Sobolev spaces are investigated in Section 4, Section 5 contains analytical characterization of the probabilistic solution found in Section 3. Sections 6 and 7 contain probabilistic solution of the impulsive control problem and its analytical characterization, respectively. Lemma (5, 10) is proved in Section 8 and a brief introduction to the theory of weak solutions of ordinary differential equations can be found there.

I would like to express my thanks to Professor Petr Mandl for his kind guidance.

2. PRELIMINARIES

(2,1) Let H be a real Hilbert space with scalar product $[\cdot, \cdot]$ and norm $\|\cdot\|$, let $\mathfrak{f} \in H^*$ and let $\langle \cdot, \cdot \rangle$ be the duality between H and H^*. Let c be a continuous and coercive bilinear form on H, i.e. mapping $H \times H \to \mathbb{R}$, linear in each variable and for which

$$c(u, v) \leqq \text{const.} \|u\| \, . \, \|v\|$$
$$\text{const.} \|u\|^2 \leqq c(u, u)$$

hold for every $u, v \in H$. Let $K \subseteq H$ be a nonempty, closed convex set. The problem to find $u \in K$ such that

$$c(u, v - u) \geqq \langle \mathfrak{f}, v - u \rangle \quad \text{for any} \quad v \in K$$

is called an (abstract) *variational inequality.*

(2,2) *Under the assumptions* (2,1) *the variational inequality has one and only one solution.*

Proof. See [5]. ■

(2,3) We shall derive an important property of the solution of (2,1) if the space H is partially ordered (which is often the case with function spaces). Assume that a partial ordering $\leqq$ on H is given such that $(H, \leqq)$ is a vector lattice (cf. [3]) and that for any $v \in H$

$$c(v^+, v^-) = 0 .$$

(2,4) An element $v \in K$ is called a *supersolution* if

$$c(v, \varphi) \geqq \langle \mathfrak{f}, \varphi \rangle \quad \text{for any} \quad \varphi \in H , \quad \varphi \geqq 0 .$$

(2,5) *Let the set K satisfy the following conditions:*

(i) *If* $u, v \in K$ *then* $\min(u, v) \in K$.

(ii) *If* $u \in K$, $v \in H$ *and* $v \geqq 0$ *then* $u + v \in K$.

Then the solution u of (2,1) *equals the greatest lower bound of all supersolutions from K.*

Proof. Setting $v = u + \varphi$ in (2,1), where $\varphi \geqq 0$, we see that u is a supersolution. It remains to show that if $w \in K$ is a supersolution then $w \geqq u$. The element

$$v = u - (w - u)^- = \min(u, w)$$

belongs to K by (i) and hence

$$c(u, -(w - u)^-) \geqq \langle \mathfrak{f}, -(w - u)^- \rangle .$$

Setting $\varphi = (w - u)^-$ in the definition of a supersolution we get

$$c(w, (w - u)^-) \geqq \langle \mathfrak{f}, (w - u)^- \rangle .$$

Adding up

$$0 \leqq c(w - u, (w - u)^-) = -c((w - u)^-, (w - u)^-) \leqq 0$$

by (2,3). Hence $u \leqq w$. ∎

(2,6) *Let K_n be closed convex sets such that $K = \bigcap K_n \neq \emptyset$. Let u_n (u respectively) be the solution of* (2,1) *with respect to the set K_n (K respectively). Then $u_n \to u$ in H.*

Proof. Let $v \in K$. Then

$$\text{const.}\ \|u_n\|^2 \leqq c(u_n, u_n) \leqq c(u_n, v) + \langle \mathfrak{f}, u_n - v \rangle \leqq$$
$$\leqq \text{const.}\ \|u_n\| \cdot \|v\| + \|\mathfrak{f}\| \cdot \|u_n\| + \|\mathfrak{f}\| \cdot \|v\|$$

from where it follows that the sequence $\{u_n\}$ is bounded. We claim that $u_n \to u$ in H weakly. For if not then there would be a subsequence $\{u_{n_k}\}$ converging weakly to a $u^* \neq u$. For any $v \in K$ and k natural

$$c(u_{n_k}, v - u_{n_k}) \geqq \langle \mathfrak{f}, v - u_{n_k} \rangle ,$$

going to the limit

$$c(u^*, v - u^*) \geqq \langle \mathfrak{f}, v - u^* \rangle .$$

By (2,2) $u = u^*$, a contradiction. To finish the proof let us write

$$\text{const.}\ \|u - u_n\|^2 \leqq c(u - u_n, u - u_n) =$$
$$= c(u, u - u_n) - c(u_n, u - u_n) \leqq c(u, u - u_n) - \langle \mathfrak{f}, u - u_n \rangle \to 0$$

from the weak convergence of $\{u_n\}$. Hence $u_n \to u$ in H strongly. ∎

(2,7) If for any $u \in H$ a convex set K_u is given, then the problem to find a $u \in K_u$ such that

$$c(u, v - u) \geqq \langle \mathfrak{f}, v - u \rangle \quad \text{for any} \quad v \in K_u$$

is called an (abstract) *quasivariational inequality.*

(2,8) *Let X, Y be complete separable metric spaces, $A \subseteq X \times Y$ a Borel set such that for any $x \in X$*

$$\{y \in Y: (x, y) \in A\}$$

is a nonempty σ-compact set. Then there is a Borel measurable mapping φ: $X \to Y$ such that $(x, \varphi(x)) \in A$ for any $x \in X$.

Proof. This assertion is known as Uniformization theorem, see [8]. ■

3. PROBABILISTIC SOLUTION OF THE OPTIMAL STOPPING PROBLEM

(3,1) Let (Y_t, P^x, ζ) be a (time homogeneous) Fellerian, strong Markov process with respect to $\mathscr{F}_t = \sigma(Y_s: s \leqq t)$ with continuous trajectories and with values in a metric space X. Here ζ is interpreted as the termination time (cf. [4]). Let

f be a bounded Borel function on X,

ψ a continuous upper bounded function on X and

$\gamma > 0$.

We assume that the function

$$x \mapsto E^x \int_0^{t \wedge \zeta} f(Y_s) \, e^{-\gamma s} \, ds$$

is continuous for any $t > 0$.

We introduce the reward functional

$$J_x(\theta) = E^x \left[\psi(Y_\theta) \, e^{-\gamma\theta} \mathbf{1}_{\theta < \zeta} + \int_0^{\theta \wedge \zeta} f(Y_t) \, e^{-\gamma t} \, dt \right]$$

and the reward function

$$u(x) = \sup J_x(\theta)$$

the sup being taken over all stopping times θ. Our aim is to characterize the function u and to find an optimal stopping time.

(3,2) We say that a Borel measurable function $v: X \to (-\infty, +\infty\rangle$ is *supermedian,* if v is lower bounded and

$$v(x) \geqq E^x \left[v(Y_\tau) \, e^{-\gamma\tau} \mathbf{1}_{\tau < \zeta} + \int_0^{\tau \wedge \zeta} f(Y_t) \, e^{-\gamma t} \, dt \right]$$

holds for all $x \in X$ and all stopping times τ.

(3,3) (i) *If v is supermedian, then*

$$v(x) \geqq E^x \int_0^{\zeta} f(Y_t) \, e^{-\gamma t} \, dt \geqq \text{const} > -\infty \, .$$

(ii) *If v is lower semicontinuous, lower bounded and*

$$v(x) \geqq E^x \left[v(Y_t) \, e^{-\gamma t} \, \mathbf{1}_{t < \zeta} + \int_0^{t \wedge \zeta} f(Y_s) \, e^{-\gamma s} \, ds \right]$$

holds for any $x \in X$ and any $t \in (0, \infty)$, then v is supermedian.

(iii) *If* $\{v_j\}_{j\in J}$ *is a countable system of supermedian functions, then the function* $v = \inf_J v_j$ *is supermedian.*

(iv) *If* $\{v_n\}$ *is an increasing sequence of supermedian functions, then the function* $v = \lim_n v_n$ *is supermedian.*

(v) *If* v *is supermedian and* τ *is the first exit time from an open set* V, *then the function*

$$g(x) = E^x\left[v(Y_\tau)\,e^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_t)\,e^{-\gamma t}\,dt\right]$$

is supermedian.

(vi) *If* v *is supermedian,* τ *the first exit time from an open set* V *and* $\mu \leqq \tau$ *an arbitrary stopping time, then*

$$E^x\left[v(Y_\tau)\,e^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_t)\,e^{-\gamma t}\,dt\right] \leqq$$

$$\leqq E^x\left[v(Y_\mu)\,e^{-\gamma\mu}\,\mathbf{1}_{\mu<\zeta} + \int_0^{\mu\wedge\zeta} f(Y_t)\,e^{-\gamma t}\,dt\right].$$

Proof. Assertions (i), (iii) and (iv) are clear. To prove (ii) note that

$$Z_t = v(Y_t)\,e^{-\gamma t}\,\mathbf{1}_{t<\zeta} + \int_0^{t\wedge\zeta} f(Y_s)\,e^{-\gamma s}\,ds$$

is a supermartingale with lower semicontinuous, lower bounded trajectories. Thus (ii) follows from the standard lemma on stopping of supermartingales. To prove (v) let us choose $x \in X$, $t > 0$ and let

$$\sigma = \inf\{s \geqq t\colon Y_s \notin V\}\,, \quad \sigma' = \inf\{s \geqq t - \tau\colon Y_s \notin V\}\,.$$

σ is clearly a stopping time, we claim so is σ'. Since

$$\sigma' = \sigma'\,\mathbf{1}_{\tau>t/2} + \sigma'\,\mathbf{1}_{\tau\leqq t/2} = \tau\,\mathbf{1}_{\tau>t/2} + \sigma'\,\mathbf{1}_{\tau\leqq t/2} \geqq \tfrac{1}{2}t$$

and $[t - \tau \leqq s] \in \mathscr{F}_s$ for $s \geqq \frac{1}{2}t$, it can be verified by standard methods (cf. [10]) that σ' is indeed a stopping time. We have

$$E^x\left[g(Y_t)\,e^{-\gamma t}\,\mathbf{1}_{t<\zeta} + \int_0^{t\wedge\zeta} f(Y_s)\,e^{-\gamma s}\,ds\right] =$$

$$= E^x\left[e^{-\gamma t}\,\mathbf{1}_{t<\zeta}\,E^{Y_t}\left[v(Y_\tau)\,e^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_s)\,e^{-\gamma s}\,ds\right]\right] + E^x\int_0^{t\wedge\zeta} f(Y_s)\,e^{-\gamma s}\,ds =$$

$$= E^x\left[v(Y_\sigma)\,e^{-\gamma(\sigma-\tau)}\,\mathbf{1}_{\sigma<\zeta} + \int_{\tau\wedge\zeta}^{\sigma\wedge\zeta} f(Y_s)\,e^{-\gamma(s-\tau)}\,ds\right] e^{-\gamma\tau} + E^x\int_0^{\tau\wedge\zeta} f(Y_s)\,e^{-\gamma s}\,ds =$$

$$= E^x\left[e^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta}\,\mathbf{1}_{\sigma>\tau}E^{Y_\tau}\left[v(Y_{\sigma'})\,e^{-\gamma\sigma'}\,\mathbf{1}_{\sigma'<\zeta} + \int_0^{\sigma'\wedge\zeta} f(Y_s)\,e^{-\gamma s}\,ds\right]\right] +$$

$$+ E^x[v(Y_\tau)\,\mathrm{e}^{-\gamma\tau}\,\mathbf{1}_{<\zeta}\,\mathbf{1}_{\sigma=\tau}] + E^x\int_0^{\tau\wedge\zeta} f(Y_s)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s \leqq$$

$$\leqq E^x\left[v(Y_\tau)\,\mathrm{e}^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_s)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s\right] = g(x)\,.$$

Now (v) follows from (ii).

To prove (vi) let us write

$$E^x\left[v(Y_\tau)\,\mathrm{e}^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] =$$

$$= E^x\left[\mathrm{e}^{-\gamma\mu}\,\mathbf{1}_{\mu<\zeta}E^x\left[v(Y_\tau)\,\mathrm{e}^{-\gamma(\tau-\mu)}\,\mathbf{1}_{\tau<\zeta} + \int_{\mu\wedge\zeta}^{\tau\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma(t-\mu)}\,\mathrm{d}t\,\middle|\,\mathscr{F}_\mu\right] + \right.$$

$$\left. + \int_0^{\mu\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] =$$

$$= E^x\left[\mathrm{e}^{-\gamma\mu}\,\mathbf{1}_{\mu<\zeta}E^{Y_\mu}\left[v(Y_\tau)\,\mathrm{e}^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] + \int_0^{\mu\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] \leqq$$

$$\leqq E^x\left[v(Y_\mu)\,\mathrm{e}^{-\gamma\mu}\,\mathbf{1}_{\mu<\zeta} + \int_0^{\mu\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right]. \quad \blacksquare$$

(3,4) *Let g be an arbitrary function on X, $A \subseteq X$. We denote by R_g^A the least supermedian function which majorizes g on the set A. Its existence, for g and A which are of interest, is proved in* (3,5). *Set*

$$E = \{x \in X\colon P^x[\zeta = 0] = 0\}\,,$$

$$F = \{x \in X\colon P^x[\zeta = 0] = 1\}\,.$$

Since (Y_t, P^x, ζ) is assumed Fellerian, the set E is open in X. By $0-1$ *law* (cf. [10]) *$E \cup F = X$. Moreover $u \geqq \psi$ on E and it is easily seen that it is sufficient for the function ψ to be defined on E, the values of ψ on F are irrelevant.*

(3,5) *If g is lower semicontinuous on E, then R_g^E exists and is lower semicontinuous on E.*

Proof. Put

$$v_0 = g\,,$$

$$v_n = \sup_{t\geqq 0} E^x\left[v_{n-1}(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathbf{1}_{t<\zeta} + \int_0^{t\wedge\zeta} f(Y_s)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s\right].$$

Clearly $v_0 \leqq v_1 \leqq \dots$, denote $v = \lim v_n$. The functions v_n are lower semicontinuous and hence v is lower semicontinuous. For $t > 0$

$$v_n(x) \geqq E^x\left[v_{n-1}(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathbf{1}_{t<\zeta} + \int_0^{t\wedge\zeta} f(Y_s)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s\right]$$

and by passing to the limit

$$v(x) \geqq E^x\left[v(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathbf{1}_{t<\zeta} + \int_0^{t\wedge\zeta} f(Y_s)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s\right].$$

By (3,3ii) v is supermedian.

Let z be a supermedian function satisfying $z \geqq g = v_0$ on E. Assuming $z \geqq v_{n-1}$ on E we get

$$z(x) \geqq E^x\left[z(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathbf{1}_{t<\zeta} + \int_0^{t\wedge\zeta} f(Y_s)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s\right] \geqq$$

$$\geqq E^x\left[v_{n-1}(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathbf{1}_{t<\zeta} + \int_0^{t\wedge\zeta} f(Y_s)\,\mathrm{e}^{-\gamma s}\,\mathrm{d}s\right],$$

hence $z \geqq v_n$ and by going to the limit $z \geqq v$. This proves $v = R_g^E$ on E. ■

(3,6) (i) *It holds*

$$u(x) = R_\psi^E(x)$$

and $\hat{\theta} = \inf\{t\colon Y_t \notin \{u > \psi\}\}$ *is an optimal stopping time, i.e.* $u(x) = J_x(\hat{\theta})$.

(ii) *If* $\tau \leqq \hat{\theta}$ *is a stopping time, then*

$$R_\psi^E(x) = E^x\left[R_\psi^E(Y_\tau)\,\mathrm{e}^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right].$$

Proof. If v is supermedian and $v \geqq \psi$ on E, then for any stopping time θ

$$v(x) \geqq E^x\left[v(Y_\theta)\,\mathrm{e}^{-\gamma\theta}\,\mathbf{1}_{\theta<\zeta} + \int_0^{\theta\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] \geqq$$

$$\geqq E^x\left[\psi(Y_\theta)\,\mathrm{e}^{-\gamma\theta}\,\mathbf{1}_{\theta<\zeta} + \int_0^{\theta\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right].$$

Hence $v \geqq u$ and thus $u \leqq R_\psi^E$. By (3,5) R_ψ^E is lower semicontinuous on E and hence each of the sets $\{R_\psi^E - \varepsilon > \psi\}$ is open in E. For $\varepsilon > 0$ we define

$$\theta_\varepsilon = \inf\{t\colon Y_t \notin \{R_\psi^E - \varepsilon > \psi\}\}$$

and

$$v_\varepsilon(x) = E^x\left[R_\psi^E(Y_{\theta_\varepsilon})\,\mathrm{e}^{-\gamma\theta_\varepsilon}\,\mathbf{1}_{\theta_\varepsilon<\zeta} + \int_0^{\theta_\varepsilon\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right].$$

We claim that

(1) $$\psi(x) \leqq v_\varepsilon(x) + \varepsilon \quad \text{for} \quad x \in E\,.$$

Suppose (1) fails and put

$$\beta = \sup_{x\in E}\,(\psi(x) - v_\varepsilon(x)) > \varepsilon\,.$$

In fact

$$\beta = \sup\,(\psi(x) - v_\varepsilon(x))\,,$$

the sup being taken over $E \cap \{R_\psi^E - \varepsilon > \psi\}$, since $\theta_\varepsilon = 0$ P^x-a.s. for $x \in E \cap$

$\cap \{R_\psi^E - \varepsilon \leqq \psi\}$. Thus we can choose $x_0 \in E \cap \{R_\psi^E - \varepsilon > \psi\}$ such that

$$\psi(x_0) - v_\varepsilon(x_0) > \beta - \varepsilon .$$

On the other hand $v_\varepsilon + \beta$ is supermedian by (3,3v) and majorizes ψ on E, hence

$$R_\psi^E(x_0) \leqq v_\varepsilon(x_0) + \beta < \psi(x_0) + \varepsilon < R_\psi^E(x_0) ,$$

a contradiction, which proves (1).

By (1) $v_\varepsilon + \varepsilon$ is a supermedian majorant of ψ on E and hence

$$u(x) \leqq R_\psi^E(x) \leqq v_\varepsilon(x) + \varepsilon = E^x\left[R_\psi^E(Y_{\theta_\varepsilon})\,\mathrm{e}^{-\gamma\theta_\varepsilon}\,\mathbf{1}_{\theta_\varepsilon<\zeta} + \right.$$

$$\left. + \int_0^{\theta_\varepsilon\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] + \varepsilon \leqq E^x\left[\psi(Y_{\theta_\varepsilon})\,\mathrm{e}^{-\gamma\theta_\varepsilon}\,\mathbf{1}_{\theta_\varepsilon<\zeta} + \right.$$

$$\left. + \int_0^{\theta_\varepsilon\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] + 2\varepsilon \leqq u(x) + 2\varepsilon .$$

From the continuity of trajectories follows $\theta_\varepsilon \nearrow \hat\theta$. Letting $\varepsilon \to 0+$ we get (i).

To prove (ii) let $\tau \leqq \hat\theta$ be a stopping time, we have

$$R_\psi^E(x) \geqq E^x\left[R_\psi^E(Y_\tau)\,\mathrm{e}^{-\gamma\tau}\,\mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] \geqq$$

$$\geqq E^x\left[R_\psi^E(Y_{\hat\theta})\,\mathrm{e}^{-\gamma\hat\theta}\,\mathbf{1}_{\hat\theta<\zeta} + \int_0^{\hat\theta\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] \geqq$$

$$\geqq E^x\left[\psi(Y_{\hat\theta})\,\mathrm{e}^{-\gamma\hat\theta}\,\mathbf{1}_{\hat\theta<\zeta} + \int_0^{\hat\theta\wedge\zeta} f(Y_t)\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t\right] = u(x) = R_\psi^E(x)$$

by (3,3vi). ■

4. WEIGHTED SOBOLEV SPACES

(4,1) Let $-\infty \leqq \alpha < 0 < \beta \leqq +\infty$ and let us consider a second order ordinary differential operator

$$L u(x) = a(x)\,u''(x) + b(x)\,u'(x) - \gamma\,u(x) ,$$

where a, b are continuous in (α, β) and a is positive there. Put

$$B(x) = \int_0^x \frac{b(s)}{a(s)}\,\mathrm{d}s .$$

We note that

$$L u(x) = a(x)\,\mathrm{e}^{-B(x)}[\mathrm{e}^{B(x)}\,u'(x)]' - \gamma\,u(x) .$$

(4,2) Let us introduce weighted Sobolev spaces corresponding to the differential operator L. If u, v are locally absolutely continuous functions in (α, β), then u', v'

exist almost everywhere and we may define

$$(u, v) = \int_{\alpha}^{\beta} u(x)\, v(x) \frac{e^{B(x)}}{a(x)}\, dx$$

$$[u, v] = \int_{\alpha}^{\beta} u'(x)\, v'(x)\, e^{B(x)}\, dx + \gamma \int_{\alpha}^{\beta} u(x)\, v(x) \frac{e^{B(x)}}{a(x)}\, dx$$

provided the integrals make sense. Put

$W = \{v\colon v$ is locally absolutely continuous in (α, β) and $[v, v] < \infty\}$.

W is clearly a linear space and $[\cdot, \cdot]$ is a scalar product on W. Let us denote $\|\cdot\|$ the norm induced by this scalar product. Let $\mathscr{D}$ denote the set of all infinitely differentiable functions with compact support contained in (α, β). We define W_0 as the closure of $\mathscr{D}$ in the topology of $(W, \|\cdot\|)$.

(4,3) *We remark that* $[\cdot, \cdot]$ *is the bilinear form corresponding to the differential operator L, that is*

$$[u, v] = -(Lu, v) \quad \text{for} \quad v \in W_0 \quad \text{and} \quad u \in W \quad \text{such that} \quad u'$$

is locally absolutely continuous.

Proof. For $v \in \mathscr{D}$ it is seen by integrating by parts, the general case follows from the density of $\mathscr{D}$. ∎

(4,4) For any interval $\langle x, y\rangle \subseteq (\alpha, \beta)$ the space $W \restriction \langle x, y\rangle$ formed by restrictions of functions from W to $\langle x, y\rangle$ is equal to the classical Sobolev space $W^{1,2}(x, y)$. Equality here means that the sets are equal and their norms are equivalent.

(4,5) (i) *W is a Hilbert space.*

(ii) *For* $v \in W$ *and* $x, y \in (\alpha, \beta)$

$$|v(x) - v(y)| \leqq \left|\int_{y}^{x} e^{-B(\xi)}\, d\xi\right|^{1/2} \cdot \|v\|$$

holds.

(iii) *If* $v \in W$ *and* $\lim_{x \to \alpha+} v(x) = \lim_{x \to \beta-} v(x) = 0$, *then* $v \in W_0$.

Proof. Assertion (i) follows from (4,4) and from the completeness of $L_2(e^B/a)$, $L_2(e^B)$. By Schwarz inequality

$$|v(x) - v(y)| = \left|\int_{y}^{x} v'(\xi)\, d\xi\right| \leqq \left|\int_{y}^{x} e^{-B(\xi)}\, d\xi\right|^{1/2} \left|\int_{y}^{x} |v'(\xi)|^2\, e^{B(\xi)}\, d\xi\right|^{1/2}$$

which gives (ii). Assertion (iii) will be proved under additional assumption that $-\infty < \alpha < \beta < +\infty$. The general case follows by transformation of the scale. Let $v \in W$ be such that $v(\alpha+) = v(\beta-) = 0$, extend v to $\langle -\infty, +\infty\rangle$ by letting $v(x) = 0$ for $x \notin (\alpha, \beta)$. We may assume without loss of generality that $(-2, 2) \subseteq$

$\subseteq (\alpha, \beta)$ and that $v'(0)$ exists and is finite. Now we define for $\varepsilon < 1$

$$v_\varepsilon(x) = \begin{cases} v(x - \varepsilon) & \text{for} \quad x \in \langle -\infty, -\varepsilon \rangle \\ v(x + \varepsilon) & \text{for} \quad x \in \langle \varepsilon, \infty \rangle \\ \dfrac{v(2\varepsilon) - v(-2\varepsilon)}{2\varepsilon}\, x + \dfrac{v(2\varepsilon) + v(-2\varepsilon)}{2} & \text{for} \quad x \in \langle -\varepsilon, \varepsilon \rangle . \end{cases}$$

The function v_ε belongs to W and its support is contained in (α, β). By (4,4) $v_\varepsilon \in W_0$. Using (ii) we get

$$\int_{-\varepsilon}^{\varepsilon} |v_\varepsilon'(x)|^2 \, e^{B(x)} \, dx + \int_{-\varepsilon}^{\varepsilon} |v_\varepsilon(x)|^2 \frac{e^{B(x)}}{a(x)} \, dx \leqq$$

$$\leqq \int_{-\varepsilon}^{\varepsilon} \left| \frac{v(2\varepsilon) - v(-2\varepsilon)}{2\varepsilon} \right|^2 e^{B(x)} \, dx + \text{const} \,.\, \varepsilon \leqq$$

$$\leqq \text{const} \int_{-\varepsilon}^{\varepsilon} e^{B(x)} \, dx + \text{const} \,.\, \varepsilon \leqq \text{const} \,.\, \varepsilon$$

and by continuity in L_2 of functions from L_2 follows

$$\int_{\alpha}^{-\varepsilon} |v'(x) - v_\varepsilon'(x)|^2 \, e^{B(x)} \, dx + \int_{\alpha}^{-\varepsilon} |v(x) - v_\varepsilon(x)|^2 \frac{e^{B(x)}}{a(x)} \, dx \leqq$$

$$\leqq \int_{\alpha}^{0} |v'(x) - v'(x - \varepsilon)|^2 \, e^{B(x)} \, dx + \int_{\alpha}^{0} |v(x) - v(x - \varepsilon)|^2 \frac{e^{B(x)}}{a(x)} \, dx \to 0 \,.$$

Now it is easily seen that $v_\varepsilon \to v$ in W_0. The proof is thus complete. ∎

(4,6) *Let* $\int_\alpha^\beta e^{-B(x)} \, dx < \infty$. *Then*

(i) $W_0 \subset\!\!\!\!\to \mathscr{C}_0\langle \alpha, \beta \rangle$, *i.e. there exists a constant K such that for any $v \in W_0$ holds $v \in \mathscr{C}_0\langle \alpha, \beta \rangle$ and*

$$\sup_{x \in \langle \alpha, \beta \rangle} |v(x)| \leqq K \|v\| \,.$$

(ii) $W_0 = \{v \in W : v(\alpha+) = v(\beta-) = 0\}$.

If moreover

$$\int_\alpha^\beta \frac{e^{B(x)}}{a(x)} \, dx < \infty$$

then

(iii) $W \subset\!\!\!\!\to \mathscr{C}\langle \alpha, \beta \rangle$.

Proof. Assertions (i) and (ii) follow easily from (4,5). From (4.5ii) we have

$$|v(x)| \leqq \left| \int_\alpha^\beta e^{-B(x)} \, dx \right|^{1/2} \|v\| + |v(y)| \leqq \text{const} \, \|v\| + |v(y)| \,.$$

Multiplying by $e^{B(y)}/a(y)$ and integrating with respect to y we get

$$|v(x)| \int_\alpha^\beta \frac{e^{B(y)}}{a(y)} \, dy \leqq \text{const} \, \|v\| \int_\alpha^\beta \frac{e^{B(y)}}{a(y)} \, dy + \int_\alpha^\beta |v'(y)| \frac{e^{B(y)}}{a(y)} \, dy \leqq$$

$$\leqq \text{const} \left[\|v\| + \left| \int_\alpha^\beta |v(y)|^2 \frac{e^{B(y)}}{a(y)} \, dy \right|^{1/2} \right] \leqq \text{const} \, \|v\|$$

and (iii) follows. ∎

(4,7) *For every $v \in W$ there exists a constant such that*

$$v(x) \leqq \text{const} \left(\left| \int_0^x e^{-B(y)} \, dy \right| + 1 \right)^{1/2}$$

holds for all $x \in (\alpha, \beta)$.

Proof. It follows from (4,5ii). ∎

5. ANALYTICAL CHARACTERIZATION OF THE PROBABILISTIC SOLUTION

The purpose of this section is to prove theorem (5,11).

(5,1) Consider a second order ordinary differential operator

$$A \, u(x) = a(x) \, u''(x) + b(x) \, u'(x) \tag{2}$$

in $\langle \alpha, \beta \rangle$, where a, b and B are as in (4,1), and boundary conditions

$$\varrho_\alpha \, u(\alpha) - \pi_\alpha e^{B(\alpha)} \, u'(\alpha) = 0 \tag{3}$$

$$\varrho_\beta \, u(\beta) + \pi_\beta e^{B(\beta)} \, u'(\beta) = 0 \, . \tag{4}$$

We say that a stochastic process is *determined by the operator* (2) *and boundary conditions* (3), (4) if (2) is its infinitesimal generator in the space $\{v \in \mathscr{C}\langle \alpha, \beta \rangle : v$ satisfies (3), (4)$\}$. Let us recall briefly several known facts (see e.g. [6]).

(5,2) If

$$\int_\alpha^\beta \frac{e^{B(x)}}{a(x)} \, dx < +\infty \, , \quad \int_\alpha^\beta e^{-B(x)} \, dx < +\infty \, ,$$

then the boundaries α, β are called *regular*. If, in addition,

$$\varrho_\alpha \geqq 0 \, , \quad \pi_\alpha \geqq 0 \, , \quad \varrho_\alpha + \pi_\alpha > 0$$

$$\varrho_\beta \geqq 0 \, , \quad \pi_\beta \geqq 0 \, , \quad \varrho_\beta + \pi_\beta > 0$$

then operator (2) and boundary conditions (3), (4) determine a *Fellerian strong Markov process* $Y = (Y_t, P^x, \zeta)$ in $X = \langle \alpha, \beta \rangle$. If in particular

$$\pi_\alpha = \pi_\beta = 0 \, ,$$

then the trajectory vanishes after reaching the boundaries and $E = (\alpha, \beta)$. If

$$\varrho_\alpha = \varrho_\beta = 0\,,$$

then the trajectory is reflected by the boundaries. In this case $\zeta = +\infty$.

(5,3) If

$$-\infty < \int_\alpha^\beta \mathrm{e}^{-B(x)} \int_0^x \frac{\mathrm{e}^{B(y)}}{a(y)}\, \mathrm{d}y\, \mathrm{d}x < +\infty\,,$$

$$\int_\alpha^0 \frac{\mathrm{e}^{B(x)}}{a(x)} \int_x^0 \mathrm{e}^{-B(y)}\, \mathrm{d}y\, \mathrm{d}x = \int_0^\beta \frac{\mathrm{e}^{B(x)}}{a(x)} \int_0^x \mathrm{e}^{-B(y)}\, \mathrm{d}y\, \mathrm{d}x = +\infty\,,$$

then α, β are called *exit boundaries*. In this case necessarily

$$\pi_\alpha = \pi_\beta = 0\,, \quad \varrho_\alpha > 0\,, \quad \varrho_\beta > 0$$

and under this assumption operator (2) and boundary conditions (3), (4) determine a Fellerian strong Markov process $Y = (Y_t, P^x, \zeta)$ in $X = \langle \alpha, \beta \rangle$.

(5,4) If

$$\int_\alpha^0 \mathrm{e}^{-B(x)} \int_x^0 \frac{\mathrm{e}^{B(y)}}{a(y)}\, \mathrm{d}y\, \mathrm{d}x = \int_0^\beta \mathrm{e}^{-B(x)} \int_0^x \frac{\mathrm{e}^{B(y)}}{a(y)}\, \mathrm{d}y\, \mathrm{d}x = +\infty\,,$$

then the boundaries α, β are called *inaccessible*. In this case necessarily $\varrho_\alpha = \varrho_\beta = \pi_\alpha = \pi_\beta = 0$ (i.e. no boundary conditions). The operator (2) determines a Fellerian strong Markov process (Y_t, P^x, ζ) in $X = (\alpha, \beta)$, the boundaries are not reached.

(5,5) We have considered only those cases when the boundaries are of the same type. But it is easily seen how the results can be adapted to those cases when the boundaries are of different type.

(5,6) We shall define a Hilbert space V and a continuous bilinear form $c(u, v)$ on V now.

a) If the boundaries are either regular or exit and $\pi_\alpha = \pi_\beta = 0$ then put $V = W_0$, $c(u, v) = [u, v]$. By (4,6) $V \subseteq \mathscr{C}_0\langle \alpha, \beta \rangle$ and $V = \{v \in W: v(\alpha+) = v(\beta-) = 0\}$.

b) If the boundaries are regular and $\pi_\alpha > 0$, $\pi_\beta > 0$, then we may assume $\pi_\alpha = \pi_\beta = 1$. Put $V = W$, $c(u, v) = [u, v] + \varrho_\alpha\, u(\alpha)\, v(\alpha) + \varrho_\beta\, u(\beta)\, v(\beta)$. The bilinear form $c(u, v)$ is continuous by (4,6iii) and clearly coercive. By (4,6iii) $V \subseteq \mathscr{C}\langle \alpha, \beta \rangle$.

c) If the boundaries are inaccessible we set $V = W_0$ and $c(u, v) = [u, v]$. In this case $V \subseteq \mathscr{C}(\alpha, \beta)$ only.

To the reader unfamiliar with the notion of a weak solution of ordinary differential equation we recommend paragraphs (8,1) and (8,2) as a brief introduction.

(5,7) *Every $v \in V$ is continuous on X.*

(5,8) Assume (3,1) and moreover that

$$f \in L_2\left(\frac{\mathrm{e}^B}{a}\right).$$

Consider the following problem: To find $u \in K = \{v \in V: v \geqq \psi\}$ such that

$$c(u, v - u) \geqq (f, v - u) \quad \text{for any} \quad v \in K .$$

This is a variational inequality. If ψ has a majorant in V, then K is a nonempty closed convex subset of V and by (2,2) the problem has a unique solution. We are going to show that this solution coincides with R^E_ψ.

(5,9) Under additional regularity assumptions on u, the problem (5,8) has the following interpretation:

$$\begin{array}{ll} u \text{ is bounded}, & \varrho_\alpha u(\alpha) - \pi_\alpha e^{B(\alpha)} u'(\alpha) \geqq 0, \\ Lu \leqq -f \quad \text{a.e.}, & (\varrho_\alpha u(\alpha) - \pi_\alpha e^{B(\alpha)} u'(\alpha))(u(\alpha) - \psi(\alpha)) = 0, \\ u \geqq \psi, & \varrho_\beta u(\beta) + \pi_\beta e^{B(\beta)} u'(\beta) \geqq 0, \\ (Lu + f)(u - \psi) = 0 \quad \text{a.e.}, & (\varrho_\beta u(\beta) + \pi_\beta e^{B(\beta)} u'(\beta))(u(\beta) - \psi(\beta)) = 0. \end{array}$$

We will not make this precise, because it is not needed in the sequel. However, this formulation is mor instructive than (5,8).

(5,10) From now on by a *supersolution* we shall always mean supersolution with respect to the variational inequality (5,8) and by a supermedian function a supermedian function with respect to the Markov process described in this Section.

(i) *If v is a supersolution, then it is supermedian.*

(ii) *Every supermedian function is continuous on E.*

(iii) *If v is supermedian and has a majorant in V, then v is a supersolution.*

Proof. We postpone the proof until Section 8. ∎

(5,11) *If ψ has a majorant in V, then the reward function u of the problem* (3,1) *coincides with the solution of the problem* (5,8).

Proof. It follows from (2,5), (3,6i) and (5,10). ∎

6. PROBABILISTIC SOLUTION OF THE IMPULSIVE CONTROL PROBLEM

(6,1) Let $Y = (Y_t, P^x, \zeta)$ be the Markov process from (5,2), (5,3) or (5,4). Let

f be a non-positive bounded Borel function on X,

γ a positive constant,

k a negative constant,

$d: X \times X \to \langle -\infty, k \rangle$ a continuous function.

In case (5,6c) we assume moreover that $d(x, y) \to -\infty$ for $y \to \alpha+$ or $y \to \beta-$. The function d is interpreted as the reward of the jump from x to y. Let us assume that the process Y is defined on the space Ω of all left-continuous trajectories with right hand limits which have only finitely many discontinuities on every bounded interval and let us denote by $\mathscr{F}_t$ the σ-algebra of events up to time t on this space. If θ is a stopping time then ω_θ denotes the trajectory $t \mapsto \omega(t + \theta)$. *Impulsive control* is a sequence

$$C = \{\theta_1, F_1(x, \omega), \theta_2, F_2(x, \omega), \dots\},$$

where $\{\theta_i\}$ is a nondecreasing sequence of $\mathscr{F}_t$-stopping times and $F_i(x, \omega)$ are $\mathscr{F}_{\theta_i}$-measurable distribution functions of probability measures with support contained in X. A stochastic process $(Y_t, \tilde{P}^x, \zeta)$ is called a *controlled process* (by impulsive control C) if for any $t > 0$, any $x \in X$ and any bounded Borel function v on X

$$\tilde{E}^x[v(Y_{(\theta_i+t)\wedge\theta_{i+1}}) \mathbf{1}_{(\theta_i+t)\wedge\theta_{i+1}<\zeta} \mid \mathscr{F}_{\theta_i}] =$$

$$= \int E^y[v(Y_{t\wedge\bar{\theta}_{i+1}}) \mathbf{1}_{t\wedge\bar{\theta}_{i+1}<\zeta}] F_i(\mathrm{d}y, \omega) \mathbf{1}_{\theta_i<\zeta}$$

holds $\tilde{P}^x$ – a.s., where

$\bar{\theta}_{i+1}(\omega_{\theta_i}) = \theta_{i+1} - \theta_i$,

$\theta_0 = 0$,

F_0 = distribution function with unit jump at x.

Let us note that (cf. [4])

$$\zeta(\omega_{\theta_i}) = \zeta(\omega) - \theta_i .$$

The impulsive control C is called *admissible* if

$$\tilde{E}^x \mathrm{e}^{-\gamma\theta_i} \mathbf{1}_{\theta_i<\zeta} \to 0 .$$

To an admissible control we assign the reward functional

$$J_x(C) = \tilde{E}^x \left[\sum_{i=1}^{\infty} \int \mathrm{d}(Y_{\theta_i}, y) F_i(\mathrm{d}y, \omega) \mathrm{e}^{-\gamma\theta_i} \mathbf{1}_{\theta_i<\zeta} + \int_0^{\zeta} f(Y_t) \mathrm{e}^{-\gamma t} \mathrm{d}t \right].$$

$J_x(C)$ is well defined with the possible value $-\infty$. Our aim is to characterize the reward function

$$u(x) = \sup J_x(C),$$

the sup being taken over all admissible controls, and to find an optimal impulsive control.

(6,2) *Let us denote, for* $x \in X$,

$$M\, v(x) = \sup \int [v(y) + d(x, y)] F(\mathrm{d}y),$$

where the sup *is taken over all probability distribution functions F concentrated on X. The operator M satisfies*:

(i) *M is order preserving.*

(ii) *If* $v_n \nearrow v$, *then* $Mv_n \to Mv$ *pointwise.*

(iii) *If v is continuous and bounded on X, then there exists a Borel measurable mapping* $x \mapsto \hat{F}_x$ *such that*

$$M\, v(x) = \int [v(y) + d(x, y)] \hat{F}_x(\mathrm{d}y)$$

and $\hat{F}_x$ is a distribution function of a probability measure with support contained in X.

(iv) *Mv is continuous and upper bounded on E whenever v is continuous and upper bounded on E.*

Proof. (iii) follows from (2,8) and properties of d. The other assertions are clear. ∎

(6,3) *There exists a least supermedian function u satisfying $u \geqq Mu$ on E. This function is continuous and bounded on E and it is the only continuous and bounded function on E which satisfies $u = R^E_{Mu}$. Moreover*

$$u(x) = \sup_C J_x(C)$$

and there exists an optimal impulsive control, i.e. an admissible impulsive control $\hat{C}$ such that $u(x) = J_x(\hat{C})$.

Proof. Let us define

$$u_0 = E^x \int_0^\zeta f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t$$

and

$$u_n = R^E_{Mu_{n-1}} \quad (n = 1, 2, \dots)\,.$$

From (3,3i) follows $u_1 \geqq u_0$. We claim that

$$u_0 \leqq u_1 \leqq \dots \leqq 0\,.$$

Indeed, all u_n are clearly nonpositive and assuming $u_{n-1} \leqq u_n$ we have from (6,2i)

$$Mu_{n-1} \leqq Mu_n \leqq u_{n+1}\,.$$

Since u_n is the least supermedian majorant of Mu_{n-1}, we have $u_n \leqq u_{n+1}$ as claimed. We denote $u = \lim u_n$, u is supermedian by (3,3iv) and $u \geqq Mu$ by (6,2ii). We claim it is the desired function. Let z be a supermedian function satisfying $z \geqq Mz$ on E. We may assume that $z \leqq 0$ for we can take function $\min(z, 0)$ otherwise. By (3,3i) $z \geqq u_0$, let us assume $z \geqq u_{n-1}$ on E. By theorem (3,6i) which can be used according to (5,10ii) and (6,2iv)

$$u_n(x) = \sup_\theta E^x \left[Mu_{n-1}(Y_\theta)\, \mathrm{e}^{-\gamma\theta}\, \mathbf{1}_{\theta<\zeta} + \int_0^{\theta\wedge\zeta} f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t \right] \quad (x \in E)$$

and hence by $Mu_{n-1} \leqq Mz$

$$u_n(x) \leqq \sup_\theta E^x \left[Mz(Y_\theta)\, \mathrm{e}^{-\gamma\theta}\, \mathbf{1}_{\theta<\zeta} + \int_0^{\theta\wedge\zeta} f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t \right] \quad (x \in E)\,.$$

Again by (3,6i) the right hand side equals the least supermedian majorant of Mz. Since z is a supermedian majorant of Mz it follows $u_n \leqq z$ and by a limit passage $u \leqq z$ on E. This shows that u is the least supermedian function satisfying $u \geqq Mu$ on E. Letting $n \to \infty$ in the definition of u_n one obtains $u = R^E_{Mu}$. The continuity of u follows from (5,10ii), the boundedness from $u_0 \leqq u \leqq 0$.

Let now u be a continuous bounded function on E satisfying $u = R^E_{Mu}$ and let $C = \{\theta_i, F_i(x, \omega)\}$ be an admissible impulsive control. We have by supermedianity of u and the fact $u \geqq Mu$

$$u(y)\, \mathrm{e}^{-\gamma\theta_i}\, \mathbf{1}_{\theta_i<\zeta} \geqq E^y\left[Mu(Y_{\bar\theta_{i+1}})\, \mathrm{e}^{-\gamma\bar\theta_{i+1}}\, \mathbf{1}_{\bar\theta_{i+1}<\zeta} + \int_0^{\bar\theta_{i+1}\wedge\zeta} f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t\right] \mathrm{e}^{-\gamma\theta_i}\, \mathbf{1}_{\theta_i<\zeta}\,.$$

Integrating, using the definition of a controlled process and the definition of M we get

$$\tilde{E}^x \int u(y)\, \mathrm{e}^{-\gamma\theta_i}\, \mathbf{1}_{\theta_i<\zeta}\, F_i(\mathrm{d}y, \omega) \geqq \tilde{E}^x\left[Mu(Y_{\theta_{i+1}})\, \mathrm{e}^{-\gamma\theta_{i+1}}\, \mathbf{1}_{\theta_{i+1}<\zeta} + \int_{\theta_i\wedge\zeta}^{\theta_{i+1}\wedge\zeta} f(Y_t)\mathrm{e}^{-\gamma t}\, \mathrm{d}t\right] \geqq$$

$$\geqq \tilde{E}^x\left[\int (u(y) + d(Y_{\theta_{i+1}}, y))\, F_{i+1}(\mathrm{d}y, \omega)\, \mathrm{e}^{-\gamma\theta_{i+1}}\, \mathbf{1}_{\theta_{i+1}<\zeta} + \int_{\theta_i\wedge\zeta}^{\theta_{i+1}\wedge\zeta} f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t\right]$$

for $i = 0, 1, \ldots$. Adding up for $i = 0, 1, \ldots, n - 1$

$$u(x) \geqq \sum_{i=1}^{n} \tilde{E}^x\left[\int d(Y_{\theta_i}, y)\, F_i(\mathrm{d}y, \omega)\, \mathrm{e}^{-\gamma\theta_i}\, \mathbf{1}_{\theta_i<\zeta} + \int_0^{\theta_n\wedge\zeta} f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t\right] +$$

$$+ \tilde{E}^x\left[\int u(y)\, F_n(\mathrm{d}y, \omega)\, \mathrm{e}^{-\gamma\theta_n}\, \mathbf{1}_{\theta_n<\zeta}\right],$$

letting $n \to \infty$ and using the admissibility of C we obtain

$$u(x) \geqq J_x(C)\,.$$

For the converse inequality let us define an optimal impulsive control. Put

$$\hat\theta_0 = 0\,,$$
$$\hat\theta_{i+1} = \inf\{t \geqq \hat\theta_i : u(Y_t) = Mu(Y_t)\}\,,$$
$$\hat F_{i+1}(y, \omega) = \hat F_{Y_{\hat\theta_{i+1}}}(y) \quad (\text{see } (6{,}2\mathrm{iii}))\,,$$
$$\hat C = \{\hat\theta_i, \hat F_i(x, \omega)\}\,.$$

Theorem (3,6i) yields for $i = 0, 1, \ldots$

$$u(y)\, \mathrm{e}^{-\gamma\hat\theta_i}\, \mathbf{1}_{\hat\theta_i<\zeta} = E^y\left[Mu(Y_\theta)\, \mathrm{e}^{-\gamma\theta}\, \mathbf{1}_{\theta<\zeta} + \int_0^{\theta\wedge\zeta} f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t\right] \mathrm{e}^{-\gamma\hat\theta_i}\, \mathbf{1}_{\hat\theta_i<\zeta}\,,$$

where $\theta = \inf\{t \geqq 0 : u(Y_t) = Mu(Y_t)\}$. Integrating and using the definition of a controlled process we get for $i = 0, 1, \ldots$

$$\tilde{E}^x \int u(y)\, \mathrm{e}^{-\gamma\hat\theta_i}\, \hat F_i(\mathrm{d}y, \omega)\, \mathbf{1}_{\hat\theta_i<\zeta} = \tilde{E}^x\left[Mu(Y_{\hat\theta_{i+1}})\, \mathrm{e}^{-\gamma\hat\theta_{i+1}}\, \mathbf{1}_{\hat\theta_{i+1}<\zeta} + \int_{\hat\theta_i\wedge\zeta}^{\hat\theta_{i+1}\wedge\zeta} f(Y_t)\mathrm{e}^{-\gamma t}\, \mathrm{d}t\right].$$

Adding from 0 to $n - 1$ and using the definition of M one obtains

$$u(x) = \tilde{E}^x\left[\sum_{i=1}^{n} \int d(Y_{\hat\theta_i}, y)\, \hat F_i(\mathrm{d}y, \omega)\, \mathrm{e}^{-\gamma\hat\theta_i}\, \mathbf{1}_{\hat\theta_i<\zeta} + \int_0^{\hat\theta_n\wedge\zeta} f(Y_t)\, \mathrm{e}^{-\gamma t}\, \mathrm{d}t + \right.$$

$$\left. + \int u(y)\, \hat F_n(\mathrm{d}y, \omega)\, \mathrm{e}^{-\gamma\hat\theta_n}\, \mathbf{1}_{\hat\theta_n<\zeta}\right].$$

Using the boundedness of u and f and the fact that $d(x, y) \leqq k < 0$ it is easily seen that

$$\hat{E}^x e^{-\gamma \hat{\vartheta}_i} \mathbf{1}_{\hat{\vartheta}_i < \zeta} \to 0 ,$$

hence $\hat{C}$ is an admissible control. Letting $n \to \infty$ we get

$$u(x) = J_x(\hat{C}) .$$

This completes the proof. ■

7. ANALYTICAL CHARACTERIZATION

(7,1) Let the notation and assumptions be as in (5,6), (5,8) and (6,1) and let us assume that Mv is upper bounded for any $v \in V$. This requirement is clearly satisfied when V consists of bounded functions, which always happens in cases (5,6a), (5,6b) but only sometimes in case (5,6c). If V contains unbounded functions then the boundedness of Mv can be achieved by assuming

$$d(x, y) \left(\left| \int_0^y e^{-B(s)} \, ds \right| + 1 \right)^{-1/2} \to -\infty$$

for $y \to \alpha +$ and $y \to \beta -$ uniformly in $x \in (\alpha, \beta)$. This is a strengthening of $d(x, y) \to \to -\infty$ and its sufficiency follows from (4,7).

(7,2) Consider the problem:

To find $u \in V$ such that $u \geqq Mu$ and $c(u, v - u) \geqq (f, v - u)$ for any $v \in V$, $v \geqq Mu$.

This is a quasivariational inequality in the sense of (2,7).

(7,3) *Assume* (7,1). *Then the problem* (7,2) *has a unique solution and this solution coincides with the reward function of the optimal impulsive control problem* (6,1).

Proof. The function $u = R^E_{Mu}$ from (6,3) satisfies $u \leqq 0$ and hence has a majorant in V. Thus u is a solution of (7,2) by (2,5) and (5,10). Conversely, if u is a solution of (7,2), then $u = R^E_{Mu}$ by (2,5) and (5,10). The function Mu is upper bounded by assumption (7,1), hence it follows that u is bounded. Since u is continuous on E by (5,7) we may use (6,3) to establish $u(x) = \inf J_x(C)$. This shows also the uniqueness of solution of (7,2). ■

8. APPENDIX

(8,1) *Let $G \subseteq X$ be open in X, $v \in V$, let $f \in L_2(e^B/a)$ be bounded and let c be as in* (5,6). *Suppose that*

$$c(v, \varphi) = (f, \varphi) \quad \textit{for any} \quad \varphi \in V, \quad \operatorname{supp} \varphi \subseteq G .$$

Then

(i) *v' is locally absolutely continuous in G and $a(x)\, v''(x) + b(x)\, v'(x) - \gamma\, v(x) = -f(x)$ a.e. in G,*

(ii) *if $\alpha \in G$ then v satisfies* (3) *of* (5,1), *if $\beta \in G$ then v satisfies* (4) *of* (5,1).

Proof. Let us choose $\langle x, y\rangle \subseteq G \cap (\alpha, \beta)$ arbitrarily. Since $a(\xi) > \text{const} > 0$ in $\langle x, y\rangle$ there exists a function w with w' absolutely continuous in $\langle x, y\rangle$ such that

$$a(\xi)\, w''(\xi) + b(\xi)\, w'(\xi) - \gamma\, w(\xi) = -f(\xi) \quad \text{a.e. in} \quad \langle x, y\rangle\,,$$
$$v(\xi) = w(\xi) \quad \text{for} \quad \xi \in X \setminus (x, y)\,.$$

Multiplying the a.e. equation by $\varphi(\xi)\,(e^{B(\xi)}/a(\xi))$, where $\varphi \in V$, $\operatorname{supp}\varphi \subseteq \langle x, y\rangle$, integrating and using (4,3) we get

$$(f, \varphi) = (-Lw, \varphi) = [w, \varphi] = c(w, \varphi)\,.$$

Hence

$$c(v - w, \varphi) = 0 \quad \text{for any} \quad \varphi \in V, \quad \operatorname{supp}\varphi \subseteq \langle x, y\rangle\,.$$

Taking $\varphi = v - w$ we see that

$$c(v - w,\ v - w) = 0\,.$$

Hence $v = w$ and (i) is proved. The assertion (ii) is clear in cases (5,6a) and (5,6c). Let (5,6b) hold and $\alpha \in G$, choose $\varphi \in V$ with $\operatorname{supp}\varphi \subseteq \langle\alpha, 0)$ and $\varphi(\alpha) = 1$. Then by (i)

$$0 = c(v, \varphi) - (f, \varphi) = -\int_\alpha^\beta [e^{B(\xi)}\, v'(\xi)]'\, \varphi(\xi)\, d\xi +$$
$$+ \gamma \int_\alpha^\beta v(\xi)\, \varphi(\xi) \frac{e^{B(\xi)}}{a(\xi)}\, d\xi - \int_\alpha^\beta f(\xi)\, \varphi(\xi) \frac{e^{B(\xi)}}{a(\xi)}\, d\xi -$$
$$- e^{B(\alpha)}\, v'(\alpha)\, \varphi(\alpha) + \varrho_\alpha\, v(\alpha)\, \varphi(\alpha) = \varrho_\alpha\, v(\alpha) - \pi_\alpha\, e^{B(\alpha)}\, v'(\alpha)\,.$$

Hence v satisfies (3), the assertion concerning β is analogical. ∎

(8,2) *The problem*:
To find $u \in V$ such that

$$c(u, v) = (f, v) \quad \textit{for any} \quad v \in V$$

is equivalent to the following one:
To find u bounded with first derivative locally absolutely continuous such that

$$Lu = -f \quad \textit{a.e. and } u \textit{ satisfies } (3), (4)\,.$$

Proof. This is a special case of the following technical lemma. ∎

(8,3) *Let $G \subseteq X$ be open in X, let $v \in V$ be bounded and let $f \in L_2(e^B/a)$ be bounded. Consider the following problems*:

$$(5) \qquad \begin{cases} w \text{ bounded with } w' \text{ locally absolutely continuous in } G \\ a(x)\, w''(x) + b(x)\, w'(x) - \gamma\, w(x) = -f(x) \quad \text{a.e. in } G \\ \varrho_\alpha\, w(\alpha) - \pi_\alpha\, e^{B(\alpha)} w'(\alpha) = 0 \quad \textit{if} \quad \alpha \in G \\ \varrho_\beta\, w(\beta) + \pi_\beta\, e^{B(\beta)} w'(\beta) = 0 \quad \textit{if} \quad \beta \in G \\ w(x) = v(x) \quad \text{for} \quad x \in X \setminus G \end{cases}$$

and

$$(6) \qquad \begin{cases} w \in K = \{z \in V: z(x) = v(x) \quad \textit{for} \quad x \in X \setminus G\} \\ c(w, z - w) \geqq (f, z - w) \qquad \textit{for any} \quad z \in K\,. \end{cases}$$

They both have a unique solution and these solutions coincide.

Proof. The uniqueness of (5) is proved in [6], the uniqueness of (6) follows from (2,2). If w is the solution of (6), then, by (8,1), it satisfies (5) possibly except of the boundedness. Thus it remains to show w is bounded. It is sufficient to prove the boundedness of w in case (5,6c) only, since in the other cases every function from V is bounded. It is also sufficient to show the boundedness of w on $(\alpha, 0)$, so assume without loss of generality that $(\alpha, 0) \subseteq G$. Now we have to require familiarity with [6, Chapter II]. It is shown there that the general solution of (5) is of the form

$$w(x) = c_+ u_+(x) + c_- u_-(x) + w_0(x)\,,$$

where c_+, c_- are constants, u_- and w_0 are bounded on $(\alpha, 0)$ and u_+ satisfies

$$u_+(x) \geqq \gamma \int_x^0 \int_y^0 u_+(s) \frac{\mathrm{e}^{B(s)}}{a(s)} \,\mathrm{d}s\, \mathrm{e}^{-B(y)} \,\mathrm{d}y$$

(see [6, Chapter II, formula (17)]). Using the definition of an inaccessible boundary it easily follows that u_+ does not satisfy (4,7). Hence $c_+ = 0$ and w is bounded by results of [6]. ∎

(8,4) *If v is a supersolution, then it is supermedian.*

Proof. If v is a bounded supersolution, then v is continuous by (5,7). We denote $w = R_v^E$ and $G = \{w > v\}$. The set G is open in E by (3,5). From (3,6ii) follows that for any $x \in X$ and any $U \subseteq G$ open in G

$$w(x) = E^x\left[w(Y_\tau)\, \mathrm{e}^{-\gamma\tau}\, \mathbf{1}_{\tau<\zeta} + \int_0^{\tau \wedge \zeta} f(Y_t)\, \mathrm{e}^{-\gamma t} \,\mathrm{d}t\right],$$

where τ is the first exit time from U. This is known to imply (cf. [4, Chapter 15]) that w satisfies (5). By (6)

$$c(w, v - w) \geqq (f, v - w)$$

and using the fact that v is a supersolution we get

$$\|w - v\|^2 \leqq c(w - v, w - v) \leqq 0\,.$$

Hence $w = v$ and v is supermedian. If v is not bounded we take functions $w_n: = R^E_{v \wedge n} \nearrow R^E_v = w$. ∎

(8,5) *If v is supermedian, then it is continuous on E.*

Proof. See [4, Chapter 15]. ∎

(8,6) If v is supermedian and has a majorant in V, then it is a supersolution.

Proof. By (2,2) there exists one and only one function $w \in V$ such that

$$w \geqq v \quad \text{and} \quad c(w, z - w) \geqq (f, z - w) \quad \text{for any} \quad z \in V, \quad z \geqq v .$$

The function w is continuous by (5,7), let us denote $G = \{w > v\}$. Since v has a majorant in V it is continuous on the whole X, hence G is open in X. If $\varphi \in V$ and $\operatorname{supp} \varphi \subseteq G$, then for ε small enough $w + \varepsilon\varphi > v$, hence

$$c(w, \varphi) = (f, \varphi) .$$

This identity holds for all functions from the closure of $\{\varphi \in V\colon \operatorname{supp} \varphi \subseteq G\}$, hence (6) by (4,5iii). From (5) follows by well-known methods

$$w(x) = E^x\left[w(Y_\tau)\, e^{-\gamma\tau}\, \mathbf{1}_{\tau<\zeta} + \int_0^{\tau\wedge\zeta} f(Y_t)\, e^{-\gamma t}\, dt\right],$$

where τ is the first exit time from G. We have for $x \in G$

$$0 > v(x) - w(x) \geqq E^x[(v - w)(Y_\tau)\, e^{-\gamma\tau}\, \mathbf{1}_{\tau<\zeta}] = 0 ,$$

a contradiction, which shows $v = w$ and thus v is a supersolution by (2,5). ■

This completes the proof of (5,10).

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A, L	...	second order ordinary differential operators ... (4,1), (5,1)
a, b	...	coefficients of A, L ... (4,1)
B	...	(4,1)
C	...	impulsive control
c	...	continuous coercive bilinear form ... (2,1), (5,6)
$\mathscr{C}(\alpha, \beta)$	...	the space of continuous functions on (α, β)
$\mathscr{C}\langle\alpha, \beta\rangle$	...	the space of continuous function on (α, β) with one-sided limits at α, β
$\mathscr{C}_0\langle\alpha, \beta\rangle$	...	the space of continuous functions v on (α, β) with $v(\alpha+) = v(\beta-) = 0$
d	...	(6,1)
$\mathscr{D}$	...	the set of all infinitely differentiable functions with support contained in (α, β)
E	...	(3,4)
E^x	...	expectation with respect to P^x
$F_i(x, \omega)$	...	$\mathscr{F}_{\theta_i}$-measurable distribution function
$\mathfrak{f}$	...	continuous linear form on H ... (2,1)
f	...	bounded Borel function on X
$\mathscr{F}_t$	...	nondecreasing system of σ-algebras
$H, [\cdot, \cdot], \lVert\cdot\rVert, \langle\cdot, \cdot\rangle$	...	Hilbert space with scalar product, norm and duality pairing ... (2,1)
H^*	...	the space of continuous linear forms on H
J_x	...	reward functional ... (3,1), (6,1)
K, K_n	...	nonempty closed convex subsets of a Hilbert space
k	...	negative constant
$L_2(w)$	...	L_2 space with weight w, i.e. $\{v: \int_\alpha^\beta \lvert v(x)\rvert^2 w(x)\,dx < \infty\}$ with norm $\sqrt{(\int_\alpha^\beta \lvert v(x)\rvert^2 w(x)\,dx)}$
M	...	(6,2)
$\mathbb{R}$	...	the set of real numbers
R_g^E	...	(3,4)
supp v	...	support of the function v, i.e. the closure of $\{x: v(x) \neq 0\}$
$(u, v), [u, v]$	...	(4,2)
V, W, W_0	...	weighted Sobolev spaces (4,2), (5,6)
$W^{1,2}$	...	classical Sobolev space of square integrable absolutely continuous functions with first derivative square integrable
X	...	metric space in Section 3, otherwise either $X = \langle\alpha, \beta\rangle$ or $X = (\alpha, \beta)$
$Y = (Y_t, P^x, \zeta)$	...	Fellerian, strong Markov process with continuous trajectories ... ... (3,1), (5,2), (5,3), (5,4)
$(Y_t, \tilde{P}^x, \zeta)$	...	controlled process ... (6,1)

$\langle\alpha, \beta\rangle$	... interval in $\mathbb{R} \cup \{-\infty, +\infty\}$
γ	... positive constant
ψ	... continuous bounded function on X
$\theta, \tau, \sigma, \sigma'$	... stopping times
$\varrho_\alpha, \pi_\alpha, \varrho_\beta, \pi_\beta$	... coefficients in boundary conditions ... (5,1)
ω_θ	... the trajectory $t \mapsto \omega(t + \theta)$ ... (6,1)
$\mathbf{1}_A$	... the characteristic function of the set A
$\hookrightarrow$	... imbedding of function spaces ... (4,6)
$\blacksquare$	... end of proof

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