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A CHARACTERIZATION OF TOLERANCE-DISTRIBUTIVE TREE SEMILATTICES

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A tolerance on an algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ is a reflexive and symmetric binary relation T on A which has the Substitution Property with respect to $\mathscr{F}$, i.e., $(a_1, b_1) \in \in T, \ldots, (a_n, b_n) \in T$ implies $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in T$ for each n-ary operation $f \in \mathscr{F}$ and any elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of A. The set of all tolerances on $\mathfrak{A}$ forms an algebraic lattice $LT(\mathfrak{A})$ with respect to the set inclusion (see [4], [5]). Basic properties of this lattice were investigated in [4] and, especially for semilattices, in [5] and [6].

An algebra $\mathfrak{A}$ is called *congruence-distributive*, if the congruence lattice $Con(\mathfrak{A})$ is distributive. It is well-known that lattices and semilattices are congruence-distributive. Although tolerances are a generalization of congruences, the situation with them is quite different. We shall call an algebra $\mathfrak{A}$ *tolerance-distributive* (or *tolerance-modular*), if $LT(\mathfrak{A})$ is distributive (or modular, respectively). A class $\mathscr{K}$ of algebras is tolerance-distributive (or tolerance-modular), if each $\mathfrak{A} \in \mathscr{K}$ has this property.

It was proved in [2] and [3] that the variety $\mathscr{D}$ of all distributive lattices is the only non-trivial tolerance-distributive lattice variety. A variety of semilattices is tolerance-modular if and only if it is trivial, see [2]. The variety $\mathscr{D}$ of distributive lattices is the only non-trivial tolerance-modular variety [1]. H.-J. Bandelt [1] has investigated a weaker condition: a lattice L with the least element O is O-modular, if it does not contain a minimal non-modular sublattice containing the least element O. He has proved that every lattice L is tolerance-O-modular (i.e., $LT(L)$ is O-modular).

Our first results for tolerance lattices of semilattices were presented in [5]:

(i) The class of all semilattices is not tolerance-O-modular.

(ii) The class of all tree semilattices is tolerance-O-modular.

Recall that a semilattice S is a tree semilattice, if each interval of S (in the induced ordering) is a chain. The result (ii) has motivated our effort to characterize tolerance-modular or tolerance-distributive semilattices among tree semilattices.

Let S be a semilattice. Its operation will be denoted by the symbol $\vee$ and the induced ordering of S will be defined by: $x \leqq y$ if and only if $x \vee y = y$.

Let us introduce the relation $C(S)$ of comparability on S. We define $C(S) = \{(x, y) \mid x \vee y \in \{x, y\}\}$. Clearly, $C(S)$ is a reflexive and symmetric binary relation on S. It plays the key role in our investigation of tolerance-distributivity of S. First we ask whether $C(S)$ is a tolerance, i.e., whether $C(S) \in LT(S)$ and whether at least the intersection of $T \in LT(S)$ with $C(S)$ is in $LT(S)$.

Theorem 1. *Let S be a semilattice. The following conditions are equivalent:*
(1) *S is a tree semilattice.*
(2) *$T \cap C(S) \in LT(S)$ for each $T \in LT(S)$.*

Proof. (1) $\Rightarrow$ (2) Clearly, $T \cap C(S)$ is a reflexive and symmetric binary relation on S, since both T and $C(S)$ have these properties. It remains to prove the Substitution Property of $T \cap C(S)$. Let $(a, b) \in T \cap C(S)$, $(c, d) \in T \cap C(S)$. Then the Substitution Property of T implies $(a \vee c, b \vee d) \in T$. We only need to prove that also $(a \vee c, b \vee d) \in C(S)$. As $(a, b) \in C(S)$, $(c, d) \in C(S)$, we have four possibilities:

$$a \leqq b\,, \quad c \leqq d\,;$$
$$a \geqq b\,, \quad c \geqq d\,;$$
$$a \leqq b\,, \quad c \geqq d\,;$$
$$a \geqq b\,, \quad c \leqq d\,.$$

The first two of them imply trivially the comparability of $a \vee c$ and $b \vee d$, and the fourth is analogous to the third. Without loss of generality it suffices to study the third case. Then

$$a \leqq a \vee c \leqq b \vee c \quad \text{and} \quad a \leqq b \leqq b \vee d \leqq b \vee c\,,$$

thus both $a \vee c$, $b \vee d$ lie in the interval $[a, b \vee c]$. Since S is a tree semilattice, the interval is a chain and hence $a \vee c$ and $b \vee d$ are comparable, which proves (2).

(2) $\Rightarrow$ (1). If S is not a tree semilattice, then it contains a subsemilattice $\{x, y, z, x \vee y\}$ with the diagram in Fig. 1. Then $(x, z) \in C(S)$, $(z, y) \in C(S)$. Let T be the least tolerance of $LT(S)$ containing the pairs (x, z) and (z, y). Then $(x, z) \in T \cap C(S)$, $(z, y) \in T \cap C(S)$ and $(x \vee z, z \vee y) = (x, y) \in T$. But x and y are not comparable, i.e. $(x, y) \notin C(S)$ thus $T \cap C(S)$ has not the Substitution Property and $T \cap C(S) \notin LT(S)$.

Corollary 1. *If S is a tree semilattice, then $C(S) \in LT(S)$.*

Proof. The assertion follows directly from (2) of Theorem 1 by putting $T = S \times S$.

First we shall study those tolerances T of $LT(S)$ for which $T \subseteq C(S)$.

Theorem 2. *Let S be a tree semilattice. Let $T_1 \in LT(S)$, $T_2 \in LT(S)$, $T_1 \subseteq C(S)$, $T_2 \subseteq C(S)$. Then $T_1 \vee T_2 = T_1 \cup T_2$.*

Proof. Let $T = T_1 \cup T_2$. Let $(a, b) \in T$, $(c, d) \in T$; then each of the pairs (a, b), (c, d) belongs to T_1 or to T_2. If $(a, b) \in T_1$, $(c, d) \in T_1$, then $(a \vee c, b \vee d) \in T_1 \subseteq T$. If $(a, b) \in T_2$, $(c, d) \in T_2$, then $(a \vee c, b \vee d) \in T_2 \subseteq T$. Suppose that $(a, b) \in T_1$,

$(c, d) \in T_2$. As $T_1 \in C(S)$, $T_2 \in C(S)$, the elements a, b are comparable and so are c, d. Let $a \leqq b$, $c \leqq d$. We have $a \leqq a \vee c \leqq b \vee d$, $a \leqq b \leqq b \vee d$. As S is a tree semilattice, the interval $[a, b \vee d]$ is a chain and thus $a \vee c$ and b are comparable. Analogously $c \leqq a \vee c \leqq b \vee d$, $c \leqq d \leqq b \vee d$, and $a \vee c$ and d are comparable as well. If $a \vee c \leqq b$, $a \vee c \leqq d$, then both b, d are in the interval $[a \vee c, b \vee d]$ and they are comparable. If $b \leqq d$, then $b \vee d = d$ and $(a \vee c, b \vee d) = (a \vee c, d) = (c \vee a, d \vee a) \in T_2 \subseteq T$, because $(c, d) \in T_2$, $(a, a) \in T_2$. If $b \geqq d$, then analogously $(a \vee c, b \vee d) \in T_1 \subseteq T$. If $a \vee c \geqq b$, $a \vee c \geqq d$, then $a \vee c \geqq b \vee d$; as $a \leqq b$, $c \leqq d$, we have also $a \vee c \leqq b \vee d$ and thus $a \vee c = b \vee d$. This implies $(a \vee c, b \vee d) \in \Delta \subseteq T$. If $b \leqq a \vee c \leqq d$, then $b \vee d = d$ and $(a \vee c, b \vee d) = (a \vee c, d) = (c \vee a, d \vee a) \in T_2 \subseteq T$. If $d \leqq a \vee c \leqq b$, then $b \vee d = b$ and $(a \vee c, b \vee d) = (a \vee c, b) = (a \vee c, b \vee c) \in T_1 \subseteq T$. Now let $a \leqq b$, $c \geqq d$. We have $a \leqq a \vee c \leqq b \vee c$, $a \leqq b \leqq b \vee d \leqq b \vee c$, thus $a \vee c$ and $b \vee d$ are comparable. If $a \vee c \leqq b \vee d$, then $b \leqq b \vee d$ and $a \leqq b$ imply $b \vee c = (b \vee a) \vee c = b \vee (a \vee c) \leqq b \vee (b \vee d) = b \vee d$. As $c \geqq d$ gives $b \vee c \geqq b \vee d$, we have $b \vee c = b \vee d$, thus $(a \vee c, b \vee d) = (a \vee c, b \vee c) \in T_1 \subseteq T$. If $a \vee c \geqq b \vee d$, then $b \leqq a \vee c$, $c \leqq a \vee c$ and $b \vee c \leqq a \vee c$; since also $a \vee c \leqq b \vee c$, we have $a \vee c = b \vee c$ and $(a \vee c, b \vee d) = (c \vee b, d \vee b) \in T_2 \subseteq T$. The cases $a \geqq b$, $c \geqq d$ and $a \geqq b$, $c \leqq d$ are analogous. We have proved that $T \in LT(S)$ and thus $T = T_1 \cup T_2 = T_1 \vee T_2$.

Theorem 3. *Let S be a tree semilattice. Then $C(S)$ is a distributive element in the lattice $LT(S)$, i.e.*

$$C(S) \wedge (T_1 \vee T_2) = (C(S) \wedge T_1) \vee (C(S) \wedge T_2)$$

for any $T_1 \in LT(S)$, $T_2 \in LT(S)$.

Proof. We have $C(S) \wedge T_1 \subseteq C(S)$, $C(S) \wedge T_2 \subseteq C(S)$ and thus, according to Theorem 2, $(C(S) \wedge T_1) \vee (C(S) \wedge T_2) = (C(S) \cap T_1) \cup (C(S) \cap T_2) = C(S) \cap (T_1 \cup T_2)$. On the other hand, suppose that there exist elements x, y of S such that $(x, y) \in C(S) \wedge (T_1 \vee T_2)$, but $(x, y) \notin C(S) \cap (T_1 \cup T_2)$. Then (as the operation $\vee$ on S is commutative and associative) there exist elements x_1, x_2, y_1, y_2 of S such that $x = x_1 \vee x_2$, $y = y_1 \vee y_2$, $(x_1, y_1) \in T_1$, $(x_2, y_2) \in T_2$. As $(x, y) \in C(S)$, without loss of generality we may suppose that $x \geqq y$. Consider the elements $x_1 \vee y$, $x_2 \vee y$. As they are both greater than or equal to y, they are comparable. As $(x_1 \vee y) \vee (x_2 \vee y) = x_1 \vee x_2 \vee y = x \vee y = x$, we have either $x_1 \vee y = x$, or $x_2 \vee y = x$. Suppose that $x_1 \vee y = x$. Then $(x_1, y_1) \in T_1$, $(y, y) \in T_1$ imply $(x_1 \vee y, y_1 \vee y) = (x, y) \in T_1$. Analogously $x_2 \vee y = x$ implies $(x, y) \in T_2$. In both cases we have a contradiction with the assumption that $(x, y) \in C(S) \cap (T_1 \cup T_2)$. Hence $C(S) \wedge (T_1 \vee T_2) = C(S) \cap (T_1 \cup T_2) = (C(S) \wedge T_1) \vee (C(S) \wedge T_2)$, which was to proved.

Theorem 4. *Let S be a tree semilattice. The tolerances from $LT(S)$ which are* *contained in $C(S)$ form a sublattice $L_0(S)$ of $LT(S)$. The mapping φ: $T \mapsto T \cap C(S)$ is a homomorphism of $LT(S)$ onto $L_0(S)$.*

Proof. We have $C(S) \in LT(S)$ according to Corollary 1. The lattice $L_0(S)$ is the ideal of $LT(S)$ with the greatest element $C(S)$. If T_1, T_2 are in $LT(S)$, then obviously $\varphi(T_1) \wedge \varphi(T_2) = (T_1 \cap C(S)) \cap (T_2 \cap C(S)) = (T_1 \cap T_2) \cap C(S) = \varphi(T_1 \wedge T_2)$. Now consider $T_1 \vee T_2$. According to Theorems 2 and 3 we have $\varphi(T_1) \vee \varphi(T_2) = \varphi(T_1) \cup \varphi(T_2) = (T_1 \cap C(S)) \cup (T_2 \cap C(S)) = (T_1 \cup T_2) \cap C(S) = (T_1 \wedge T_2) \vee C(S) = \varphi(T_1 \vee T_2)$. Thus $\varphi(T_1 \vee T_2) = \varphi(T_1) \vee \varphi(T_2)$, and φ is a homomorphism. Finally, for each $T \in L_0(S)$ we have $\varphi(T) = T$, thus φ is a mapping of $LT(S)$ onto $L_0(S)$.

Corollary 2. *For a tree semilattice S the lattice $L_0(S)$ is a sublattice of the lattice of all subsets of $S \times S$ and hence it is distributive.*

Let S be a tree semilattice, let $T_0 \in L_0(S)$. By $L^*(T_0)$ we denote the set of all tolerances $T \in LT(S)$ such that $T \cap C(S) = T_0$.

Theorem 5. *Let S be a tree semilattice, let $T_0 \in L_0(S)$. The set $L^*(T_0)$ is a sublattice of $LT(S)$; its least element is T_0, its greatest element is $\tilde{T}_0 = \{(x, y) \mid (x, x \vee y) \in T_0 \mathbin{\&} (y, x \vee y) \in T_0\}$.*

Proof. Let $T_1 \in L^*(T_0)$, $T_2 \in L^*(T_0)$. Consider the homomorphism φ from Theorem 3. We have $\varphi(T_1 \vee T_2) = \varphi(T_1) \vee \varphi(T_2) = T_0 \vee T_0 = T_0$, $\varphi(T_1 \wedge T_2) = \varphi(T_1) \wedge \varphi(T_2) = T_0 \wedge T_0 = T_0$ and thus $T_1 \vee T_2 \in L^*(T_0)$, $T_1 \wedge T_2 \in L^*(T_0)$ and $L^*(T_0)$ is a sublattice of $LT(S)$. Obviously $T_0 \in L^*(T_0)$ and $T_0 \subseteq T$ for each $T \in L^*(T_0)$, hence T_0 is the least element of $L^*(T_0)$. Now consider $\tilde{T}_0$. Let $(a, b) \in \tilde{T}_0 \cap C(S)$. Then a, b are comparable; without loss of generality let $a \leqq b$. As $(a, b) \in \tilde{T}_0$, we have $(a, b) = (a, a \vee b) \in T_0$; thus $\tilde{T}_0 \cap C(S) = T_0$ and $\tilde{T}_0 \in L^*(T_0)$. Let $T \in L^*(T_0)$ and let $(c, d) \in T$. As T has the Substitution Property, we have $(c, c \vee d) \in T$, $(d, c \vee d) \in T$. As $c \leqq c \vee d$, $d \leqq c \vee d$, we have $(c, c \vee d) \in C(S)$, $(d, c \vee d) \in C(S)$, hence $(c, c \vee d) \in T \cap C(S)$, and so $(d, c \vee d)$. Hence $(c, d) \in \tilde{T}_0$. As T and (c, d) were chosen arbitrarily, we have $T \subseteq \tilde{T}_0$ for each $T \in L^*(T_0)$ and $\tilde{T}_0$ is the greatest element of $L^*(T_0)$.

The tolerance $T \in LT(S)$ for which $T \subseteq C(S)$ is easily recognized.

Theorem 6. *Let S be a tree semilattice, let T be a tolerance on S. Then the following two assertions are equivalent:*

(i) $T \in L_0(S)$.

(ii) *$T \subseteq C(S)$ and, if a, b, x are elements of S such that $a \leqq x \leqq b$ and $(a, b) \in T$, then $(x, b) \in T$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds. Obviously $T \in C(S)$. As $(a, b) \in T$, $(x, x) \in T$ and $T \in L_0(S) \subseteq LT(S)$, we have $(a \vee x, b \vee x) = (x, b) \in T$.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. Let $(a, b) \in T$, $(c, d) \in T$. If b and d are incomparable, then $a \vee c = b \vee d$, because S is a tree semilattice, and thus $(a \vee c,$

$b \vee d) \in \Delta \subseteq T$. If $b \leqq d$, then $b \vee d = d$ and $c \leqq a \vee c \leqq b \vee d = d$. But then $(c, d) \in T$ implies $(a \vee c, b \vee d) = (a \vee c, d) \in T$ according to (ii). The case $d \leqq b$ is analogous. We have proved that $T \in LT(S)$. As $T \subseteq C(S)$, we have $T \in L_0(S)$.

Now we are ready to prove our main theorem characterizing tolerance-distributive and tolerance-modular tree semilattices.

Theorem 7. *Let S be a tree semilattice. Then the following three assertions are equivalent:*

(i) *S is a chain or S contains a maximal chain S_0 and an element $z \in S_0$ such that each element of $S - S_0$ is covered by z.*

(ii) *$LT(S)$ is distributive.*

(iii) *$LT(S)$ is modular.*

Proof. (i) $\Rightarrow$ (ii). If S is a chain, then $LT(S) = L_0(S)$ and (ii) holds according to Corollary 2. Suppose that there exists a maximal chain S_0 in S and an element $z \in S_0$ such that each element of $S - S_0$ is covered by z. Let $T_1 \in LT(S)$, $T_2 \in LT(S)$ and suppose that $T_1 \vee T_2 \neq T_1 \cup T_2$. Let $(a, b) \in T_1 \vee T_2 - T_1 \cup T_2$. If a and b are comparable, then $(a, b) \in \varphi(T_1 \vee T_2) = \varphi(T_1) \cup \varphi(T_2) \subseteq T_1 \cup T_2$ according to Theorem 3, which is a contradiction. Thus suppose that a, b are incomparable. All elements of S which are greater than or equal to z are comparable with all elements of S; therefore $a < z$, $b < z$ and at least one of the elements a, b belongs to $S - S_0$. Without loss of generality let $a \in S - S_0$; then a is a minimal element of S and all elements less than b (if any) form a chain. There exist pairs $(c, d) \in T_1$, $(e, f) \in T_2$ such that $c \vee e = a$, $d \vee f = b$. As a is a minimal element of S, we have $c = e = a$. As all elements less than b form a chain, at least one of the elements d, f is equal to b. Hence at least one of the pairs (c, d), (e, f) is equal to (a, b), and $(a, b) \in T_1 \cup T_2$, which is a contradiction. As T_1, T_2, (a, b) were chosen arbitrarily, we have proved that $T_1 \vee T_2 = T_1 \cup T_2$ for any two tolerances T_1, T_2 from $LT(S)$. Since also $T_1 \wedge T_2 = T_1 \cap T_2$, the lattice $LT(S)$ is a sublattice of the lattice of all subsets of $S \times S$, and it is distributive.

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (ii). Suppose that (i) does not hold. Let S_0 be a maximal chain in S. Suppose that there exist elements x, x', y, y' such that $x \in S_0$, $y \in S_0$, $x \neq y$, $x' \in S - S_0$, $y' \in S - S_0$, x is the least element of S_0 greater than x' and y is the least element of S_0 greater than y'. As S_0 is a maximal chain in S, the element y is not a minimal element of S and there exists an element $y'' \in S_0$ such that $y'' < y$. Let $T_1 = C(S) \cup \{(x', y'), (y', x')\}$, $T_2 = C(S) \cup \{(x', y''), (y'', x')\}$, $T_3 = C(S) \cup \cup \{(x', y''), (y'', x'), (x', y), (y, x')\}$, $T_4 = C(S) \cup \{(x', y), (y, x'), (x', y'), (y', x'), (x', y''), (y'', x')\}$. Each of these tolerances is in $LT(S)$ (the proof is left to the reader) and together with $C(S)$ they form a sublattice of $LT(S)$ whose diagram is in Fig. 2. Hence $LT(S)$ is not modular. We have proved that if $LT(S)$ is modular, then at most one element of the maximal chain S_0 of S may have the property that it is the least element of S_0 greater than an element of $S - S_0$. If this is fulfilled and (i) does not hold, then there are elements x, x', y, y' of S such that $x < x' < z$, $y < y' < z$, and each of the elements x, x' is incomparable with each of the elements y, y'. Let $T_1 = C(S) \cup \{(x, y'), (y', x)\}$, $T_2 = C(S) \cup \{(x', y), (y, x')\}$, $T_3 = C(S) \cup \{(x', y), (y, x'), (x', y'), (y', x')\}$, $T_4 = C(S) \cup \{(x, y'), (y', x), (x', y), (y, x'), (x', y'), (y', x')\}$. These tolerances are in $LT(S)$ and together with $C(S)$ they form a sublattice of $LT(S)$ whose diagram is in Fig. 2. Hence $LT(S)$ is not modular. The only possibility for S to be modular is to fulfill (i).

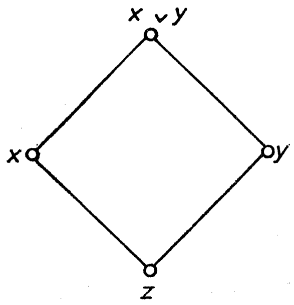

Fig. 1

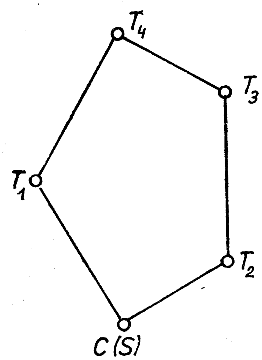

Fig. 2

Remark. In the terminology of [5] we may say that a tree semilattice S is tolerance-modular and tolerance-distributive if and only if it is either a chain, or a star semilattice, or the union of a chain and a star semilattice.

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