Ján Jakubík

Lexicographic product decompositions of a linearly ordered group

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 4, 553–563

Persistent URL: http://dml.cz/dmlcz/102116

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

LEXICOGRAPHIC PRODUCT DECOMPOSITIONS OF A LINEARLY ORDERED GROUP

JÁN JAKUBÍK, Košice

(Received June 11, 1985)

The system of all lexicographic product decompositions of a linearly ordered group G will be denoted by $L_0(G)$. On this set we can define a quasiorder in a natural way (by using the well-known theorem of A. I. Mal'cev concerning the existence of isomorphic refinements). In this paper we investigate the partially ordered set $L(G)$ corresponding to the quasiordered set $L_0(G)$ (in the sense of [1], Chap. II, § 1).

INTRODUCTION

For any two lexicographic product decompositions α and β of a linearly ordered group G A. I. Mal'cev [5] constructed a pair of new lexicographic product decompositions α' and β' of G such that

(i) α' is a refinement of α and β' is a refinement of β;

(ii) α' and β' are isomorphic.

This construction was generalized by Fuchs [2] (for lexicographic product decompositions of directed groups) and by the author [3], [4] (for a certain type of lexicographic decompositions of linearly ordered groupoids and for mixed product decompositions of directed groups).

Now let $L_0(G)$ be the set of all lexicographic product decompositions of linearly ordered group. For α and β in $L_0(G)$ we put $\alpha \leqq \beta$ if $\alpha' = \alpha$. The relation $\leqq$ is a quasiorder on the set $L_0(G)$. If α and β are elements of $L_0(G)$ such that $\alpha \leqq \beta$ and $\beta \leqq \alpha$, then they will be said to be equivalent. Let $L(G)$ be the set of all equivalence classes with the natural induced relation $\leqq$. Then $L(G)$ is a partially ordered set. For $\alpha \in L_0(G)$ let $c(\alpha)$ be the element of $L(G)$ containing α.

In [3] it was shown that $L(G)$ is a lattice and that (under the above notation) we have

$$c(\alpha') = c(\beta') = c(\alpha) \wedge c(\beta) .$$

The lattice $L(G)$ possesses a greatest element which will be denoted by I. In view of a result of Mal'cev [5] (concerning the existence of linearly ordered groups which have no lexicographic product decomposition with irreducible factors) the partially ordered set $L(G)$ need not have the least element. Hence $L(G)$ need not be a complete lattice.

In this paper we investigate certain convex l-subgroups of G corresponding to a given lexicographic decomposition α (called *lower sections* of α). Next, the following results concerning $L(G)$ are proved.

The lattice $L(G)$ is distributive. For each $\alpha \in L_0(G)$ with $c(\alpha) < I$ the interval $[c(\alpha), I]$ of $L(G)$ is a complete and completely distributive lattice. This lattice is a Boolean algebra if and only if it is finite.

1. PRELIMINARIES

The group operation in a linearly ordered group will be denoted by the symbol $+$ (the commutativity will not be assumed).

For the notion of lexicographic product of partially ordered groups cf., e.g., Fuchs [2], Chap. I, § 7. In accordance with [2], the lexicographic product of linearly ordered groups A_λ (where λ runs over a linearly ordered set Λ) will be denoted by $\Gamma_{\lambda \in \Lambda} A_\lambda$.

Throughout the whole paper we assume that G is a nonzero linearly ordered group.

Let φ be an isomorphism of G onto $\Gamma_{\lambda \in \Lambda} A_\lambda$. Then the triple $(G, \varphi, \Gamma_{\lambda \in \Lambda} A_\lambda)$ will be said to be a *lexicographic product decomposition of* G. When no misunderstanding can occur, then we also say that φ is a lexicographic product decomposition of G.

We put $\Lambda' = \{\lambda \in \Lambda : A_\lambda \neq \{0\}\}$. The set Λ' is linearly ordered by the induced linear order. Let $(G, \varphi, \Gamma_{\lambda \in \Lambda} A_\lambda)$ be a lexicographic product decomposition of G. For $g \in G$ let $\varphi(g) = \langle \ldots, g_\lambda, \ldots \rangle_{\lambda \in \Lambda}$. Put

$$\varphi'(g) = \langle \ldots, g_\lambda, \ldots \rangle_{\lambda \in \Lambda'} .$$

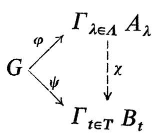
Then $(G, \varphi', \Gamma_{\lambda \in \Lambda'} A_\lambda)$ is also a lexicographic product decomposition of G.

Let us have lexicographic product decompositions $(G, \varphi, \Gamma_{\lambda \in \Lambda} A_\lambda)$ and $(G, \psi, \Gamma_{t \in T} B_t)$.

If there exists an isomorphism μ of Λ' onto T' such that for each $\lambda \in \Lambda'$, there exists an isomorphism of A_λ onto $B_{\mu(\lambda)}$, then the lexicographic product decompositions φ and ψ are said to be *isomorphic*.

The lexicographic product decompositions φ and ψ will be considered as equal if there exists an isomorphism χ of $\Gamma_{\lambda \in \Lambda} A_\lambda$ onto $\Gamma_{t \in T} B_t$ such that the following conditions are fulfilled:

(i) the diagram



is commutative;

(ii) there exists an isomorphism i of Λ' onto T' (the meaning of T' is analogous to that of Λ', if Λ is replaced by T) and for each $\lambda \in \Lambda'$ there is an isomorphism χ_λ of A_λ onto $B_{i(\lambda)}$ such that, whenever $\langle \ldots, a_\lambda, \ldots \rangle_{\lambda \in \Lambda} \in \Gamma_{\lambda \in \Lambda} A_\lambda$ and $\chi(\langle \ldots, a_\lambda, \ldots \rangle_{\lambda \in \Lambda}) = \langle \ldots, b_t, \ldots \rangle_{t \in T}$, then for each $\lambda \in \Lambda'$ we have $b_{i(\lambda)} = \chi_\lambda(a_\lambda)$.

Under this notion of equality, the collection of all lexicographic product decompositions of G is a set; let us denote this set by $L_0(G)$.

Let us remark that if α and β are isomorphic lexicographic product decompositions of G, then α and β need not be equal.

Again, let φ and ψ be lexicographic product decompositions of G. For $\Lambda_1 \subseteq \Lambda$ we denote by $G(\Lambda_1)$ the set of all $g \in G$ such that $g_\lambda = 0$ for each $\lambda \in \Lambda \setminus \Lambda_1$ (where $\varphi(g) = \langle \ldots, g_\lambda, \ldots \rangle_{\lambda \in \Lambda}$). If $\Lambda_1 = \{\lambda_1\}$ is a one-element set, then we denote $G(\Lambda_1) = G(\lambda_1)$. A similar notation will be employed for $T_1 \subseteq T$.

The lexicographic product decomposition ψ will be said to be a *refinement of* φ if for each $\lambda_1 \in \Lambda'$ there exists $T_1 \subseteq T'$ such that the mapping defined by

$$g \to \langle \ldots, g_t, \ldots \rangle_{t \in T'}$$

for each $g \in G(\lambda_1)$ is a lexicographic product decomposition of $G(\lambda_1)$.

Now let

$$\alpha = (G; \varphi, \Gamma_{\lambda \in \Lambda} A_\lambda), \quad \beta = (G; \psi, \Gamma_{\mu \in M} B_\mu)$$

be any two lexicographic product decompositions of G.

For each $\mu \in M$ and $\lambda \in \Lambda$ let $C_{\mu\lambda}$ be as in [2], p. 27 (cf. also Section 2 below); further let $C_{\lambda\mu}$ be defined analogously. Then the following result is valid (cf. [5] (with another notation)):

1.1. Theorem. (Mal'cev [5].) *Let α and β be as above. Then there exist lexicographic product decompositions*

$$\alpha' = (G; \varphi_1; \Gamma_{\lambda \in \Lambda} \Gamma_{\mu \in M} C_{\lambda\mu}),$$
$$\beta' = (G; \psi_1; \Gamma_{\mu \in M} \Gamma_{\lambda \in \Lambda} C_{\mu\lambda})$$

such that

(i) *α' is a refinement of α and β' is a refinement of β;*

(ii) *α' and β' are isomorphic.*

If α and β are as above, then we denote

$$\alpha' = f(\alpha, \beta).$$

Hence we have $\beta' = f(\beta, \alpha)$.

It is easy to verify that if the relations $\alpha = \alpha_1$ and $\beta = \beta_1$ hold in $L_0(G)$ (cf. the above definition of equality in $L_0(G)$), then

$$f(\alpha, \beta) = f(\alpha_1, \beta_1)$$

is valid in $L_0(G)$.

For $\alpha, \beta \in L_0(G)$ we put $\alpha \leqq \beta$ if $f(\alpha, \beta) = \alpha$.

1.2. Lemma. (Cf. [3].) *The relation $\leqq$ is a quasiorder on the set $L_0(G)$.*

If $\alpha \in L_0(G)$, then we denote $c(\alpha) = \{\alpha_1 \in L_0(G): \alpha \leqq \alpha_1 \text{ and } \alpha_1 \leqq \alpha\}$. Let $L(G)$ be the system $\{c(\alpha)\}_{\alpha \in L_0(G)}$. For $c(\alpha)$, $c(\beta) \in L(G)$ we put $c(\alpha) \leqq c(\beta)$ if $\alpha \leqq \beta$; then $L(G)$ is a partially ordered set under $\leqq$.

1.3. Proposition. (Cf. [4].) *$L(G)$ is a lattice under the relation $\leqq$. If $\alpha, \beta \in L_0(G)$, then in $L(G)$ we have*

$$c(f(\alpha, \beta)) = c(f(\beta, \alpha)) = c(\alpha) \wedge c(\beta).$$

Let α be as above. Assume that Λ' is a one-element set. Then $c(\alpha)$ is the greatest element in $L(G)$; we denote $c(\alpha) = I$.

The linearly ordered group G is said to be *lexicographically irreducible* if $L(G) = \{I\}$.

Let $\gamma = (G, \varphi_\gamma, \Gamma_{t \in T}\, C_t) \in L_0(G)$. It is obvious that the following conditions are equivalent:

(i) $c(\gamma)$ is the least element of $L(G)$.

(ii) If $t \in T$ and $C_t \neq \{0\}$, then C_t is lexicographically irreducible.

There exists a linearly ordered group H having no lexicographic product decomposition such that all nonzero factors of this decomposition are lexicographically irreducible (see Mal'cev [5]; cf. also Fuchs [2], p. 28).

Thus in view of the equivalence of the conditions (i) and (ii) we infer that the lattice $L(G)$ need not have the least element. In particular, $L(G)$ need not be a complete lattice.

2. LOWER SECTIONS

Let α and β be as in Section 1. Let Λ_1 be a subset of Λ such that, whenever $\lambda_1 \in \Lambda_1$, $\lambda \in \Lambda$ and $\lambda > \lambda_1$, then $\lambda \in \Lambda_1$. Then $G(\Lambda_1)$ is said to be a *lower section of G with respect to α*. Properties of lower sections will be investigated in the present Section.

Let λ be a fixed element in Λ. Put

$$I(\lambda) = \{\lambda_1 \in \Lambda: \lambda_1 < \lambda\},$$
$$I_1(\lambda) = \{\lambda_1 \in \Lambda: \lambda_1 \leqq \lambda\},$$
$$D(\lambda) = G(I(\lambda)), \quad D_1(\lambda) = G(I_1(\lambda)).$$

For $\mu \in M$ let $D(\mu)$ and $D_1(\mu)$ have analogous meanings (with respect to the lexicographic decomposition β).

Let $X \subseteq G$ and $\lambda \in \Lambda$. We denote by $X(A_\lambda)$ the natural projection of X into A_λ (under the lexicographic product decomposition α). Hence if X is a subgroup of G, then $X(A_\lambda)$ is a subgroup of A_λ.

Now the linearly ordered group $C_{\lambda\mu}$ (cf. Thm. 1.1; λ and μ are fixed elements of Λ or M, respectively) can be constructed as follows. We have

$$C_{\lambda\mu} = (D_1(\lambda) \cap G(\mu))(A_\lambda).$$

Analogously we have

$$C_{\mu\lambda} = (D_1(\mu) \cap G(\lambda)) (B_\mu) .$$

Let us suppose that $A_\lambda \neq \{0\}$ and $B_\mu \neq \{0\}$ for each $\lambda \in \Lambda$ and each $\mu \in M$.

From [3] (Sections 41 and 45) we infer:

2.1. Lemma. *The following conditions are equivalent:*

(i) $\alpha \leqq \beta$.

(ii) For each $\lambda \in \Lambda$ there is $\mu \in M$ such that

$$D(\mu) \subseteq D(\lambda) \subset D_1(\lambda) \subseteq D_1(\mu) .$$

2.1.1. Remark. It is easy to verify that if λ and μ are as in 2.1 (ii), then μ is uniquely determined by λ. If $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 < \lambda_2$, then for the corresponding elements μ_1 and μ_2 we have $\mu_1 \leqq \mu_2$ (under the assumption that $\alpha \leqq \beta$ is valid).

The following assertion is obvious.

2.2. Lemma. *Let $\lambda \in \Lambda$ and $\mu \in M$. Suppose that $D(\lambda) \subset D(\mu) \subset D_1(\lambda)$ is valid. Then $D(\mu)$ cannot be expressed as a lower section with respect to α.*

2.3. Lemma. *Let $\alpha \leqq \beta$. Let H be a lower section of G with respect to β. Then H is a lower section of G with respect to α.*

Proof. For each λ_1 in Λ we denote by μ_1 the corresponding element in M (cf. 2.1). There exists $M_1 \subseteq M$ such that $H = G(M_1)$. Denote

$$\Lambda_1 = \{\lambda_1 \in \Lambda: \mu_1 \in M_1\} .$$

If $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda$ and $\lambda_2 > \lambda_1$, then in view of 2.1.1 we have $\lambda_2 \in \Lambda_1$.

Let $\lambda_1 \in \Lambda_1$. According to 2.1,

$$D_1(\lambda_1) \subseteq D_1(\mu_1)$$

and clearly $D_1(\mu_1) \subseteq G(M_1) = H$. Hence $D_1(\lambda_1) \subseteq H$ for each $\lambda_1 \in \Lambda_1$. Moreover, we have

$$G(\Lambda_1) = \bigcup_{\lambda_1 \in \Lambda_1} D_1(\lambda_1) ,$$

thus $G(\Lambda_1) \subseteq H$.

Let $h \in H$. Suppose that h does not belong to $G(\Lambda_1)$. Hence there is $\lambda_2 \in \Lambda$ such that $\lambda_2 \notin \Lambda_1$, $h(\lambda_2) \neq 0$ and $h(\lambda_3) = 0$ for each $\lambda_3 \in \Lambda$ with $\lambda_3 < \lambda_2$. Consider the element μ_2 in M corresponding to λ_2. In view of 2.1 the element μ_2 cannot belong to M_1 and hence h does not belong to H, which is a contradiction. Therefore $H = G(\Lambda_1)$. Thus H is a lower section in G with respect to α.

Let λ be a fixed element of Λ. The convex subgroup $D(\lambda)$ of G is comparable with all convex subgroups $D(\mu)$ of G, where μ runs over the set M. Denote

$$M_1 = \{\mu \in M: D(\mu) \subseteq D(\lambda)\} , \quad M_2 = \{\mu \in M: D(\mu) \supseteq D(\lambda)\} ,$$

$$H_1 = \bigcup_{\mu \in M_1} D(\mu) , \quad H_2 = \bigcap_{\mu \in M_2} D(\mu) .$$

(In the case $M_1 = \emptyset$ or $M_2 = \emptyset$ we set $H_1 = \{0\}$ or $H_2 = G$, respectively.) Clearly $H_1 \subseteq D(\lambda) \subseteq H_2$.

2.4. Lemma. *Suppose that $M_1 \neq \emptyset$ and that M_1 has no least element. Then $H_1 = H_2$.*

Proof. Let $0 < h \in H_2$. There exists $\mu \in M$ such that $h(\mu) > 0$ and $h(\mu') = 0$ for each $\mu' \in M$ with $\mu' < \mu$. We have $h \notin D(\mu)$. If $\mu \in M_2$, then $h \notin H_2$, which is a contradiction. Therefore μ belongs to M_1. Since M_1 has no least element, there is $\mu_1 \in M_1$ with $\mu_1 < \mu$. Then $h \in D(\mu_1)$, hence $h \in H_1$. Therefore $H_1 = H_2$.

2.5. Lemma. *Assume that each lower section of β is a lower section of α. Then $M_1 \neq \emptyset$.*

Proof. By way of contradiction, suppose that $M_1 = \emptyset$. We always have $\bigcap_{\mu \in M} D(\mu) = \{0\}$. In view of the assumption, $D(\lambda) \subset D(\mu)$ is valid for each $\mu \in M$, thus $D(\lambda) = \{0\}$. Since $D(\lambda) \subset D_1(\lambda)$ (because of $A_\lambda \neq \{0\}$), there exists $\mu \in M$ with $D(\lambda) \subset D(\mu) \subset D_1(\lambda)$. According to 2.2 we arrive at a contradiction.

2.6. Lemma. *Assume that each lower section of β is a lower section of α. Then M_1 has a least element.*

Proof. According to 2.5, $M_1 \neq \emptyset$. By way of contradiction, assume that M_1 has no least element. Then in view of 2.4, $H_1 = D(\lambda) = H_2$. Because $A_\lambda \neq \{0\}$, we infer that $D(\lambda) \subset D_1(\lambda)$.

Hence there exists $\mu_2 \in M_2$ such that

$$D(\lambda) \subseteq D(\mu_2) \subset D_1(\lambda)\,.$$

The relation $D(\lambda) = D(\mu_1)$ cannot be valid, because in such a case we would have $\mu_2 \in M_1$ and hence μ_2 would be the least element in M_1. Therefore in view of 2.2 we arrive at a contradiction.

2.7. Lemma. *Assume that each lower section of β is a lower section of α. Suppose that $\mu \in M$ and $D(\mu) = D(\lambda)$. Then $D_1(\lambda) \subseteq D_1(\mu)$.*

Proof. By way of contradiction, suppose that the relation $D_1(\lambda) \subseteq D_1(\mu)$ does not hold. Because $D_1(\lambda)$ and $D_1(\mu)$ are comparable, we have $D_1(\mu) \subset D_1(\lambda)$. But in this case $D_1(\mu)$ fails to be a lower section in α; since $D_1(\mu)$ is a lower section in β, we arrive at a contradiction.

2.8. Lemma. *Assume that each lower section of β is a lower section of α. Let μ_1 be the least element of M_1. Suppose that $D(\mu_1) \neq D(\lambda) \neq D_1(\mu_1)$. Then*

$$D(\mu_1) \subset D(\lambda) \subset D_1(\lambda) \subseteq D_1(\mu_1)\,.$$

Proof. We only have to verify that the relation $D_1(\lambda) \subseteq D_1(\mu_1)$ is valid. If $M_2 = \emptyset$, then $D_1(\mu_1) = G$. Let $M_2 \neq \emptyset$.

a) First, suppose that M_2 has no greatest element. Then $D_1(\mu_1) \subset D(\mu_2)$ for each

$\mu_2 \in M_2$ and thus

$$D_1(\mu_1) \subseteq \bigcap_{\mu_2 \in M_2} D(\mu_2) = H_2 .$$

Let $0 < h \in H_2$. There exists $\mu_3 \in M$ such that $h(\mu_3) > 0$ and $h(\mu) = 0$ for each $\mu \in M$ with $\mu < \mu_3$. If $\mu_3 < \mu_1$, then $\mu_3 \in M_2$, hence there is $\mu_2 \in M_2$ with $\mu_3 < \mu_2$ and so $h \notin D(\mu_2)$, thus h does not belong to H_2, which is a contradiction. Hence $\mu_3 \geqq \mu_1$, implying that $h \in D_1(\mu_1)$. Thus we have $H_2 = D_1(\mu_1)$. Because of $D(\lambda) \subseteq H_2$ we infer that $D(\lambda) \subseteq D_1(\mu_1)$. In view of the assumption, $D(\lambda) \neq D_1(\mu_1)$, hence $D(\lambda) \subset D_1(\mu_1)$. If $D_1(\mu_1) \subset D_1(\lambda)$, then $D_1(\mu_1)$ fails to be a lower section in α; because $D_1(\mu_1)$ is a lower section in β, we have a contradiction. Therefore $D_1(\lambda) \subseteq$ $\subseteq D_1(\mu_1)$.

b) Now suppose that M_2 has a greatest element μ_2. Then

$$D_1(\mu_1) = D(\mu_2) = H_2 .$$

Again, $D(\lambda) \subseteq H_2$. If $D_1(\lambda) \supset D_1(\mu_1)$, then $D_1(\mu_1)$ is not a lower section in α, which is a contradiction. Therefore $D_1(\lambda) \subseteq D_1(\mu_1)$. (In this part of the proof the assumption $D(\lambda) \neq D_1(\mu_1)$ is not needed.)

2.9. Lemma. *Assume that each lower section of β is a lower section of α. Let μ_1 be the least element of M_1. Suppose that $D(\mu_1) \neq D(\lambda)$. Then $D(\lambda) \neq D_1(\mu_1)$.*

Proof. The case $M_2 = \emptyset$ is trivial; let $M_2 \neq \emptyset$. First, suppose that M_2 has a greatest element μ_2. We apply part b) of the proof of 2.8 and we obtain $D(\lambda) \subset$ $\subset D_1(\lambda) \subseteq D_1(\mu_1)$; hence the relation $D(\lambda) = D_1(\mu_1)$ cannot hold.

Now suppose that M_2 has no greatest element. In the same way as in part a) of the proof of 2.8 we can verify that $H_2 = D_1(\mu_1)$. By way of contradiction, suppose that $D(\lambda) = D_1(\mu_1)$. Because $D(\lambda) \subset D_1(\lambda)$, there exists $\mu_2 \in M_2$ such that $D(\lambda) \subset$ $\subset D(\mu_2) \subset D_1(\lambda)$. Then $D(\mu_2)$ fails to be a lower section in α, which is a contradiction.

From 2.6–2.9 and 2.1 we obtain:

2.10. Lemma. *Assume that each lower section of β is a lower section of α. Then $\alpha \leqq \beta$.*

Lemmas 2.9 and 2.10 yield:

2.11. Theorem. *The following conditions are equivalent:*

(i) $\alpha \leqq \beta$.

(ii) *Each lower section of β is a lower section of α.*

If $c(\alpha) = c(\beta)$, then in view of 2.11, α and β have the same lower sections; these will be called also lower sections of $c(\alpha)$.

2.12. Corollary. *Let $\alpha, \beta \in L_0(G)$. The following conditions are equivalent:*

(i) $c(\alpha) \leqq c(\beta)$.

(ii) *Each lower section of $c(\beta)$ is a lower section of $c(\alpha)$.*

Let α and β be as above. Let R_1 be a congruence relation on the linearly ordered set Λ and let $p(R_1)$ be the partition of Λ corresponding to R_1. The set $p(R_1)$ is linearly ordered in the natural way (for distinct elements $\Lambda_1, \Lambda_2 \in p(R_1)$ we put $\Lambda_1 < \Lambda_2$ if $\lambda_1 < \lambda_2$ for each $\lambda_1 \in \Lambda_1$ and each $\lambda_2 \in \Lambda_2$).

For $\Lambda_1 \in p(R_1)$ we put $A_{\Lambda_1} = \Gamma_{\lambda \in \Lambda_1} A_\lambda$. Let φ_{R_1} be the mapping of G into $\Gamma_{\Lambda_1 p(R_1)} A_{\Lambda_1}$ such that, whenever $g \in G$ and

$$\varphi(g) = \langle \ldots, g_\lambda, \ldots \rangle_{\lambda \in \Lambda}\,,$$

then $\varphi_{R_1}(g) = \langle \ldots, g_{\Lambda_1}, \ldots \rangle_{\Lambda_1 \in p(R_1)}$, where

$$g_{\Lambda_1} = \langle \ldots, g_\lambda, \ldots \rangle_{\lambda \in \Lambda_1}\,.$$

Then we obtain a lexicographic product decomposition

$$\gamma = (G; \varphi_{R_1}, \Gamma_{\Lambda_1 \in p(R_1)} A_{\Lambda_1})\,.$$

Clearly γ is a refinement of α.

The following assertion is obvious.

3.1. Lemma. *Suppose that α and β are isomorphic lexicographic product decompositions of G; let μ be the corresponding isomorphism of Λ onto M. Let γ be as above. Then μ induces a partition $p(R_2)$ on M and (under analogous notation as above) we have a lexicographic product decomposition*

$$\delta = (G; \psi_{R_2}, \Gamma_{M_1 \in p(R_2)} B_{M_1})\,.$$

The lexicographic product decompositions γ and δ of G are isomorphic.

3.2. Lemma. *Let α and β be lexicographic product decompositions of G such that $c(\alpha) \leqq c(\beta)$. Then there exists a lexicographic product decomposition β_1 of G such that*

(i) *β_1 and β are isomorphic,*

(ii) *$\alpha \leqq \beta_1$.*

Proof. Denote $\alpha_{10} = f(\beta, \alpha)$. Next, let α_1 be the lexicographic product decomposition of G consisting of nonzero factors of α_{10}. In view of 1.1, α_1 is a refinement of β and α_1 is isomorphic to α (because of $f(\alpha, \beta) = \alpha$). Now it suffices to apply Lemma 3.1 for constructing β_1.

Let α and α_i $(i \in I)$ be lexicographic product decompositions of G such that $c(\alpha) \leqq c(\alpha_i)$ is valid for each $i \in I$. In view of 3.2, for each $i \in I$ there exists a lexicographic product decomposition α_{i0} of G such that α_{i0} is isomorphic to α_i and $\alpha \leqq \alpha_{i0}$. In particular, $c(\alpha_i) = c(\alpha_{i0})$.

Since α is a refinement of α_{i0}, there exists a congruence relation R_i on Λ such that we have

$$\alpha_{i0} = \gamma_i\,,$$

where

$$\gamma_i = (G; \varphi_{R_i}, \Gamma_{A_1 \in p(R_i)} A_{A_1})$$

(under the notation as above).

Put $R = \bigvee_{i \in I} R_i$ and

$$\gamma_0 = (G; \varphi_R, \Gamma_{A_1 \in p(R)} A_{A_1}) .$$

3.3. Lemma. (i) *For each $i \in I$ the relation $c(\gamma_{i0}) \leqq c(\gamma_0)$ is valid.* (ii) *If $\delta \in L_0(G)$ such that $c(\delta) \geqq c(\gamma_{i0})$ for each $i \in I$, then $c(\delta) \geqq c(\gamma_0)$.*

Proof. The assertion (i) is obvious. The assertion (ii) follows from 2.12.

3.4. Corollary. *$c(\gamma_0) = \bigvee_{i \in I} c(\gamma_i)$ in the lattice $L(G)$.*

Hence we have

3.5. Theorem. *Let $\alpha \in L_0(G)$. Then the interval $[c(\alpha), I]$ of the lattice $L(G)$ is a complete lattice.*

Under the notation as above let $R^0 = \bigwedge_{i \in I} R_i$. Put

$$\delta_0 = (G; \varphi_{R^0}, \Gamma_{A_1 \in p(R^0)} A_{A_1}) .$$

By applying 2.12 again we obtain:

3.6. Lemma. *For each $i \in I$ we have $c(\gamma_{i0}) \geqq c(\delta_0)$.* (ii) *If $\delta \in L_0(G)$ such that $c(\delta) \leqq c(\gamma_{i0})$ for each $i \in I$, then $c(\delta) \leqq c(\delta_0)$.*

Hence $c(\delta_0) = \bigwedge_{i \in I} c(\alpha_i)$.

In view of the construction of γ_0, from 2.12 and 3.4 we infer:

3.7. Proposition. *Let α_i $(i \in I)$ and γ be lexicographic product decompositions of G. Then the following conditions are equivalent:*

(i) $c(\gamma) = \bigvee_{i \in I} c(\alpha_i)$;

(ii) *for each lower section d in G we have:*

d is a lower section in $\gamma \Leftrightarrow d$ is a lower section in each α_i $(i \in I)$.

Similarly, in view of the construction of δ_0, 2.12 and 3.6 yield:

3.8. Proposition. *Let α_i $(i \in I)$ and δ be lexicographic product decompositions of G. Then the following conditions are equivalent:*

(i) $c(\delta) = \bigwedge_{i \in I} c(\alpha_i)$;

(ii) *for each lower section d in G we have*

d is a lower section in $\delta \Leftrightarrow$ there is $i \in I$ such that d is a lower section in α_i.

4. COMPLETE DISTRIBUTIVITY; COMPLEMENTS

Let I be a nonempty set and for each $i \in I$ let J_i be a nonempty set. Let Φ be the system of all functions $\varphi: I \to \bigcup_{i \in I} J_i$ such that $\varphi(i) \in J_i$ for each $i \in I$. Let α_{ij} be lexicographic product decompositions of G $(i \in I, j \in J_i)$.

4.1. Theorem. *For each $c(\alpha) \in L(G)$, the interval $[c(\alpha), I]$ of $L(G)$ is completely distributive.*

Proof. Suppose that $c(\alpha_{ij}) \in [c(\alpha), I]$ for all $i \in I$ and $j \in J_i$. In view of 3.5 and [1], Chap. V, § 5 we have to verify that the relation

$$\bigwedge_{i\in I} \bigvee_{j\in J_i} c(\alpha_{ij}) = \bigvee_{\varphi\in\Phi} \bigwedge_{i\in I} c(\alpha_{i,\varphi(i)})$$

is valid

Denote
$$u = \bigvee_{\varphi\in\Phi} \bigwedge_{i\in I} c(\alpha_{i,\varphi(i)}) ; \quad v = \bigwedge_{i\in I} \bigvee_{j\in J_i} c(\alpha_{ij}) .$$

Since $u \leqq v$, we have to verify that $v \leqq u$ is valid.

Let d be a lower section in u. From 3.7 we infer that for each $\varphi \in \Phi$, d is a lower section in $\bigwedge_{i\in I} c(\alpha_{i,\varphi(i)})$. Hence in view of 3.8, for each $\varphi \in \Phi$ there exists $i \in I$ (depending on φ) such that d is a lower section of $c(\alpha_{i,\varphi(i)})$.

By way of contradiction, assume that d fails to be a lower section in v. Hence in view of 3.8, for each $i \in I$, d fails to be a lower section in $\bigvee_{j\in J_i} c(\alpha_{ij})$. Therefore according to 3.7, for each $i \in I$ there exists $j = \varphi(i) \in J_i$ such that d fails to be a lower section in $c(\alpha_{i,\varphi(i)})$, which is a contradiction.

4.2. Corollary. *The lattice $L(G)$ is distributive. If $L(G)$ has a least element, then $L(G)$ is complete and completely distributive.*

In the remaining part of this Section (except in 4.4) we assume that there is $\alpha_0 \in L_0(G)$ such that $c(\alpha_0)$ is the least element of $L(G)$. Let

$$\alpha_0 = (G; \varphi_0; \Gamma_{t\in T} A_t^0) .$$

We also suppose that $A_t^0 \neq \{0\}$ for each $t \in T$.

Let $\alpha \in L_0(G)$ and let us deal with the question under what conditions $c(\alpha)$ possesses a complement in $L(G)$. The distributivity of $L(G)$ implies that if the complement of $c(\alpha)$ exists, then it is uniquely determined.

Without loss of generality we can assume that α_0 is a refinement of α (cf. Lemma 3.2). Hence there is a partition R_α of T such that for each $\lambda \in \Lambda$ there is a clas T_λ of this partition having the property that A_λ is isomorphic to $\Gamma_{t\in T_\lambda} A_t^0$. (The situation is analogous to that described in Section 3.) Also, R_α is linearly ordered in the natural way (again, cf. Sec. 3).

4.3. Lemma. *Assume that the set T is finite. Then $c(\alpha)$ possesses a complement in $L(G)$.*

Proof. We denote by $t(R_\alpha)$ the class in R_α containing the element t of T. For $t_1, t_2 \in T$ we put $t_1 R t_2$ if some of the following conditions is valid:
(i) $t_1(R_\alpha)$ covers or is covered by $t_2(R_\alpha)$ in $p(R_\alpha)$ and either $t_1(R_\alpha) \neq t_1$ or $t_2(R_\alpha) \neq t_2$;
(ii) $t_1 = t_2$.
Then R is a congruence relation on T; $R \wedge R_\alpha$ and $R \vee R_\alpha$ are the least and the greatest congruence on T, respectively. There exists $\beta \in L_0(G)$ such that $R_\beta = R$; we have (cf. 3.7 and 3.8)

$$c(\beta) \wedge c(\alpha) = c(\alpha_0) , \quad c(\beta) \vee c(\alpha) = I .$$

The following assertion is obvious.

4.4. Lemma. *The following conditions are equivalent*:
(i) *The set $L(G)$ is finite.*
(ii) *The set $L(G)$ has a least element α_0 such that (under the notation as above) the set T is finite.*

4.5. Corollary. *Let $L(G)$ be finite,* card $L(G) \neq 1$. *Then $L(G)$ is a Boolean algebra.*

4.6. Lemma. *Let T be infinite. Then there exists $\alpha \in L_0(G)$ such that $c(\alpha)$ has no complement in $L(G)$.*

Proof. Because T is infinite there exists $t_0 \in T$ such that some of the following conditions is fulfilled:
(i) t_0 fails to be a least element of T and no element of T is covered by t_0.
(ii) t_0 fails to be a greatest element of T and no elemenet of T covers t_0.

Assume that (i) holds. (In the case when (ii) is valid we proceed analogously.) There exists a subset T_0 of T such that T_0 is well-ordered (under the induced linear order), $t_0 \notin T_0$ and sup $T_0 = t_0$ holds in T.

For t and t' in T we put tRt' if either $t = t'$ or there exist elements t_1 and t_2 in T_0 such that t_1 is covered by t_2 in T_0 and

$$t_1 \leqq t < t_2\,, \quad t_1 \leqq t' < t_2$$

is valid. Then R is a congruence relation on T. Let $\alpha \in L_0(G)$ such that $R_\alpha = R$. If $\beta \in L_0(G)$ such that $c(\beta)$ is a complement of $c(\alpha)$ in $L(G)$, then R_β is a complement of R in the lattice of all congruence relations of the linearly ordered set T. But it is easy to verify that R has no complement. Hence $c(\alpha)$ has no complement in $L(G)$.

From 4.4, 4.5 and 4.6 we obtain:

4.7. Theorem. *Let* card $L(G) > 1$. *Then the following conditions are equivalent*:
(i) *$L(G)$ is finite.*
(ii) *$L(G)$ is a Boolean algebra.*

References

[1] *G. Birknoff:* Lattice theory, Providence 1967.
[2] *L. Fuchs:* Partially ordered algebraic systems, Oxford 1963.
[3] *J. Jakubík:* Lexicographic products of partially ordered groupoids. Czech. Math. J. *14*, 1964, 281—305. (In Russian.)
[4] *J. Jakubík:* The mixed product decompositions of partially ordered groups. Czech. Math. J. *20*, 1970, 184—206.
[5] *A. I. Mal'cev:* On ordered groups. Izv. Akad. Nauk SSSR, seria matem. *13*, 1949, 473—482. (In Russian.)

Author's address: 040 01 Košice, Ždanovova 6, Czechoslovakia (Matematický ústav SAV, dislokované pracovisko).