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NEWS AND NOTICES

SIXTY YEARS OF PROFESSOR MIROSLAV FIEDLER

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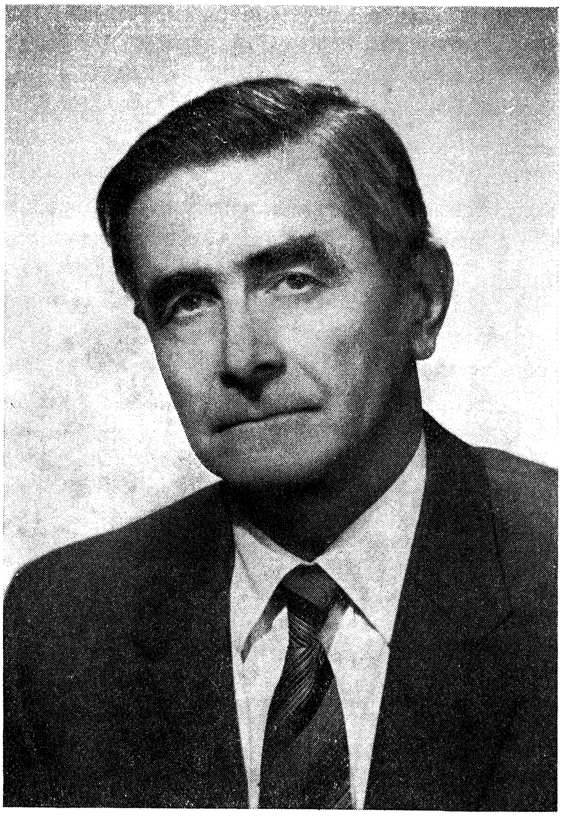
Professor RNDr. Miroslav Fiedler, DrSc., corresponding member of the Czechoslovak Academy of Sciences, an outstanding Czech mathematician, reaches sixty years of age this year. This anniversary offers an opportunity to reflect upon his research work and his activities in education and organization of science.

Miroslav Fiedler was born on 7 April, 1926 in Prague. Already when attending the secondary school he showed his talent by winning the mathematical problem-solving competition of the journal *Rozhledy matematicko-přírodovědecké*. After the leaving examination in 1945 he studied mathematics and physics at Charles University in Prague, graduating in 1950. In thesis [1] he generalized some results of his teacher Professor B. Bydžovský from classical algebraic geometry.

After receiving his degree of RNDr. (rerum naturalium doctor) M. Fiedler became research student at the Central Mathematical Institute, which was later incorporated in the Czechoslovak Academy of Sciences. With the Mathematical Institute of the Czechoslovak Academy of Sciences he has remained till now. His supervisor in the Institute was Academician Eduard Čech. M. Fiedler prepared his thesis on the geometry of simplexes [5], [6], [7], being one of the first who were granted the newly introduced degree of CSc. (candidate of sciences). As a research worker he continued to work in geometry, but soon extended his interests to the matrix theory, numerical methods, theory of graphs and applications of mathematics in economy. In 1963 he defended his thesis for the degree of DrSc. (doctor of sciences), in 1965 he was appointed full professor of mathematics at the Faculty of Mathematics and Physics of Charles University, and in 1981 he was elected corresponding member of the Czechoslovak Academy of Sciences. In the Institute he soon became head of a department and in 1984, after the Institute was divided into two sections, he was assigned head of one of them.

Already since the early sixties M. Fiedler is gradually gaining international reputation especially by his results in the theory of matrices. He has frequently obtained invitations to conferences and lectures. The first of his longer stays abroad was as visiting professor at California Institute of Technology in 1964, followed by several others. The general recognition of his scientific eminence found its expression also in his long-lasting membership in Editorial Boards of the journals Numerische Mathematik, Linear Algebra and its Applications, and Linear and Multilinear Algebra.

Although research has always been the main concern of Professor Fiedler, he has



never ceased teaching mathematics. His advanced lectures and seminars at Universities in Prague, Bratislava and Košice have been a considerable help in the professional education of numerous graduated and research students. M. Fiedler is Chief Editor of the Czechoslovak Mathematical Journal and member of Editorial Boards of several other journals, chairman of the National Mathematical Council, vice-chairman of the Scientific Board for Mathematics of the Academy and the member of the Presidium of the Union of Czechoslovak Mathematicians and Physicists, to mention only the most important of his offices.

In spite of these energy- and time-consuming activities, Professor Fiedler has found enough time to be one of the leading personalities of the Mathematical Olympiad competitions in Czechoslovakia. Also his cooperation with the Ministry of Education in preparing and reviewing secondary school curricula and textbooks is generally appreciated. Here his interest is focused on talented pupils and classes with special mathematical training.

In the conclusion of the present biographical notes, let us mention just the most important distinctions and rewards granted to Professor Fiedler for his merits in research and education. In 1962 it was the silver medal of the Union of Czechoslovak Mathematicians and Physicists together with the first prize in the scientific competition; in 1968, the prize of the Czechoslovak Academy of Sciences for popularization of science (together with Academician Josef Novák and Professor Jan Vyšín); in 1978 the National Prize for a collection of papers on matrix theory (jointly with Professor Vlastimil Pták). In 1978 he obtained the medal of the Faculty of Mathematics and Physics of Charles University and in 1981, on the occasion of his 55th birthday, Bernard Bolzano silver medal of the Academy for his merits in mathematical sciences. Last but not least, he was elected honorary member of the Union of Czechoslovak Mathematicians and Physicists in 1984.

As we have already mentioned above, M. Fiedler first concentrated on geometry. His first papers [1], [3], [4] dealt with the algebraic geometry of curves and hypersurfaces in the n-dimensional space. Then he pursued a detailed study of n-dimensional simplexes [5]–[7], [11], [14]. Let us recall that a simplex is formed by $n + 1$ linearly independent points in the n-dimensional space, thus being a generalization of a triangle and a tetrahedron. This field turned out to be very fruitful and M. Fiedler returned to it later several times [24], [25], [38], [81], [97]. In addition to their metrical properties he also characterized their combinatorial ones, which was then a novelty. For example, let us associate an n-dimensional simplex with a graph with $n + 1$ vertices which correspond to the faces of the simplex and let us connect by an edge exactly those vertices whose corresponding faces form an acute angle. Fiedler proved that the resulting graph is connected and, conversely, for any connected graph with $n + 1$ vertices there exists a simplex with the above property. (Moreover, for each non-connected pair of vertices we can prescribe whether the corresponding angle should be right or obtuse.) Hence we can see that among all the angles formed by the faces of the simplex at least n are acute. Such simplexes which have exactly n acute angles and all the others right ones Fiedler called rectangular, characterizing them in a simple way: $n + 1$ vertices of a rectangular simplex can be completed to 2^n vertices of an n-dimensional rectangular box in such a way that the edges of the simplex opposite to the right angle are mutually perpendicular edges of the box. Conversely, if we take n mutually perpendicular edges of an arbitrary n-dimensional box such that they form a tree then they are opposite edges to right angles in some simplex. Moreover, the centre of the hypersphere described to this simplex has barycentric coordinates $1 - \frac{1}{2}s_i$, where s_i are the degrees of vertices in the above mentioned tree. Fiedler succeeded in describing the location of the centre of the hypersphere described to a simplex even in the general case by means of the configuration of the angles between faces. Especially deep were his new results on the relations of orthocentral simplexes (with all heights intersecting at one point) and equiaxial hyperquadrics and Hankel matrices (to which he has returned again lately).

The analytical techniques used by Fiedler to investigate simplexes required fine work with positive definite matrices. This led him to a more detailed study of the latter. Thus in [19] he estimated from below the trace of the matrix $(A - B)(B^{-1} - A^{-1})$ in terms of the norms of the matrices $A, B, A - B$. This estimate has an important consequence: a positive definite matrix is uniquely determined if some of its elements are given together with the elements of the inverse matrix at the other places. In [23] and [37] the relations between the diagonal elements a_{ii}, α_{ii} of two mutually inverse positive definite matrices were established. It was shown that they satisfy the relations

$$a_{ii} > 0\,, \quad \alpha_{ii} > 0\,, \quad a_{ii}\alpha_{ii} \geqq 1\,, \quad \sqrt{(a_{ii}\alpha_{ii})} - 1 \leqq \sum_{j \neq i} (\sqrt{(a_{jj}\alpha_{jj})} - 1)$$

and conversely, if $2n$ numbers a_{ii}, α_{ii} satisfy these inequalities, then they are diagonal elements of two mutually inverse positive definite matrices. There is also an interesting geometric interpretation to this theorem: namely, it yields necessary and sufficient conditions for the lengths of $2n$ vectors in order that they may form a biorthogonal basis of the n-dimensional space, further, for the angles between the mutually corresponding vectors of the biorthogonal basis and also for the lengths of heights of a spherical simplex. Fiedler [28] found similar necessary and sufficient conditions also for the diagonal elements of an M-matrix (the notion will be recalled below) and its inverse.

Since the half of the fifties, as a consequence of the starting exploitation of computers, the interest of mathematicians in numerical methods began to grow. M. Fiedler was one of those who directed their interest to this field. His first contribution was to the numerical methods of solution of algebraic equations with one unknown by the classical Bernoulli-Whittaker and Gräffe methods [8], [9], [12], [30]. Both methods have a common weak point, namely the case when all roots of the equation have almost the same absolute value. And this is exactly the case dealt with in [9]. Here Fiedler used for the first time the perturbation method, which later simplified the techniques of proofs in a number of papers, making it possible to avoid limit processes. Particular attention was paid to the analysis of Gräffe method, which Fiedler improved substantially, increasing its efficiency and guaranteeing convergence even in the situations that had been obscure till then. He also modified it for the computation of the eigenvalue of a matrix with the maximum absolute value. At this point we already come to the numerical methods of linear algebra. Fiedler's interest first concentrated on the problems of convergence of iterative methods. It is here that he started his intensive cooperation with V. Pták, who is the co-author of many important Fiedler's papers from the matrix theory.*) In [10] the classical Gauss-Seidel iterative method of solution of the system of linear equations $(I - A)\, x = b$, i.e.

$$x_{n+1} = Ax_n + b\,,$$

is modified to the iteration process

$$(I - B)\, x_{n+1} = (A - B)\, x_n + b$$

and its rate of convergence with regard to the choice of the matrix B is discussed. Both authors recall with pleasure later discussions with R. S. Varga and his colaborators who, as appeared, were engaged in similar problems. The iterative method suggested in [18] for the evaluation of the spectrum of a symmetric matrix is based on the construction of unitary matrices $U_1, U_2, \ldots$ such that the sequence of matrices $U_k A U_k^*$ converges quadratically to a diagonal matrix. Iteration processes converging to an eigenvalue of an "almost decomposable" matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

i.e. a matrix where some of the partial matrices A_{12}, A_{21} has small norm [34], [36], [41], are based on the idea that under certain conditions the spectrum of the matrix A will be near to the union of spectra of the matrices A_{11}, A_{22}. The case particularly important for numerical computation is that with A_{11} of the size 1×1. Another problem from the numerical methods of linear algebra is the reduction of the size of the matrix to be inverted. The methods established in [31], [33] have important applications, especially in solving the Dirichlet problem by the method of nets and inverting ill-conditioned Leontiev matrices. Let us conclude the account of Fiedler's contributions to numerical methods by pointing out the fact that also his investigations the matrix theory were frequently inspired by problems of numerical character, and that their results in most cases are of importance for numerical practice.

Before proceeding to the analysis of the crucial part of Fiedler's research, let us mention some of his immediately applicable results. In [17] he studied the so called deformation equations, i.e. systems of linear algebraic equations which appear in solving frame constructions, and are also connected with the solution of electric circuits. His contribution to applications of mathematics in economy is of special importance. In [16], [22], [39] and [40] he substantially developed the methods of optimization of transport nets.

The theory of matrices is the field which is in the centre of Fiedler's professional interest. The papers most frequently quoted are [26], [43], [45], [47], [50], which actually represent a monograph devoted to the so called M-matrices., i.e. matrices with nonpositive nondiagonal elements and positive principal minors. This class

*) In the present survey we cannot, for practical reasons, distinguish Fiedler's own papers from those written jointly with other authors. We refer the reader, for precise information, to the list of publications at the end of the present text.

of matrices occurs in various connections – practical, numerical as well as theoretical (stability, electric circuits, convergence of iterative processes, majorization of other classes of matrices, localization of spectra, etc.). The properties and applications of M-matrices were studied from various view-points by A. Ostrowski, Ky Fan D. M. Kotel'anskij, R. S. Varga and others. The series of papers mentioned above represents a synthesis of the known facts, completes them by new results and applications and provides a unified and clear exposition of the theory. Here we find tens of necessary and sufficient conditions characterizing the M-matrices and matrices of related classes. An essential property of the M-matrices is that the real parts of all their eigenvalues are positive and that their inverses have nonnegative elements. Many of their properties are similar to those of positive definite matrices – Fiedler clarified this similarity and studied also their relations to matrices with dominating diagonal. The latter are again connected with the majorizing role the M-matrices play in certain classes of matrices. This is shown, for example, by the classical Kotel'anskij's theorem which is here improved and proved in a natural way: if $U = (u_{ij})$ is a complex matrix and $V = (v_{ij})$ an M-matrix, where $|u_{ii}| \geqq v_{ii}$, $|u_{ik}| \leqq \leqq |v_{ik}|$, then $|\det U| \geqq \det V$. Let us recall that this theory is applied in the problems of convergence of relaxation methods and in generalized processes of the Gauss-Seidel type.

The theory of M-matrices developed in the above mentioned papers was later applied by M. Fiedler to the localization of the spectra of general matrices. This is the fundamental problem of the spectral matrix theory, consisting in determining the possibly smallest complex domain containing all eigenvalues of a given matrix. This problem is closely connected with the criteria of regularity of a matrix A, if we apply such a criterion to the matrix $\lambda I - A$. So, for example, the classical Hadamard's theorem on regularity of a matrix with a dominating diagonal yields the classical Geršgorin circles. The majorizing properties of M-matrices were essential especially for [21], [27]. Let us consider a linear operator A on a finite dimensional space decomposed into the direct sum of r subspaces $X_1 + X_2 + \ldots \ldots + X_r$. In each of these subspaces let us choose a vector norm g_i. Denote

$$p_{ij} = \sup_x \{g_i(P_i A P_j x);\ g_j(P_j(x)) \leqq 1\} \quad \text{for} \quad i \neq j\,,$$

$$p_{ii} = \sup_x \{g_i(P_i A P_i x);\ g_i(P_i(x)) \geqq 1\}\,,$$

where P_i denotes the projection to the subspace X_i. It can be proved that the matrix A is regular provided the matrix

$$\begin{pmatrix} p_{11}, & -p_{12}, & \ldots, & -p_{1r} \\ -p_{21}, & p_{22}, & \ldots, & -p_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, & \ldots, & p_{rr} \end{pmatrix}$$

is an M-matrix. This regularity criterion then yields the following localization of the spectrum: Given real numbers $c_1, c_2, \ldots, c_r$ such that

$$\begin{pmatrix} c_1, & -p_{12}, & \ldots, & -p_{1r} \\ -p_{21}, & c_2, & \ldots, & -p_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, & \ldots, & c_r \end{pmatrix}$$

is an M-matrix, let us denote by R_i the set of all complex numbers z satisfying

$$\inf_x \{g_i(P_i(A - zI)\,P_i x);\ g_i(P_i(x)) \geqq 1\} \leqq c_i\,.$$

Then the spectrum of the matrix A is contained in $R_1 \cup R_2 \cup \ldots \cup R_r$. By various choices of the decomposition of the space and the particular norms g_i we then obtain not only the classical theorems on localization of spectra, as for example the well known Geršgorin circles or Cassini ovals, but also much finer estimates. Also the paper [20] was of pioneering character. In the estimates of position of the spectrum known till its publication only the values of the elements of the matrix occurred. The paper [20] provided much more general estimates, involving only the norm of the offdiagonal part of the matrix, its results being valid for a wide class of norms including those most frequently used. If the results are formulated for block matrices the different role of diagonal and offdiagonal blocks stands out clearly.

For the localization spectra of symmetric and hermitian matrices on the real axis Fiedler developed an elementary method that yielded strong results. Namely, he noticed that if real or hermitian matrices A and B have eigenvalues $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$, respectively, u_1 and v_1 are the unit eigenvectors corresponding to α_1 and β_1, respectively, and the matrix

$$\begin{pmatrix} \alpha_1 & \varrho \\ \varrho & \beta_1 \end{pmatrix}$$

has eigenvalues γ_1, γ_2, then the matrix

$$\begin{pmatrix} A & \varrho u v^T \\ \varrho v u^T & B \end{pmatrix}$$

has the eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2$. Thus he obtained [67] a simple technique of estimating the eigenvalues of symmetric matrices and constructing special matrices with given spectra. In this way he also easily deduced Horn's conditions which are necessary and sufficient for the numbers $\lambda_1 \geqq \ldots \geqq \lambda_n$, $a_1 \geqq \ldots \ldots \geqq a_n$ to be eigenvalues and diagonal elements of a $n \times n$ symmetric matrix

$$\sum_{i=1}^{s} a_i \leqq \sum_{i=1}^{s} \lambda_i \ (s = 1, \ldots, n-1)\,, \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \lambda_i\,.$$

The analogous but much more difficult problem for nonnegative matrices is solved by an analogous method in [62]. Here necessary conditions are established, viz.

$$\lambda_1 \geqq a_1\,, \quad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i\,,$$

$$\sum_{i=1}^{s} \lambda_i + \lambda_k \geqq \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k\,, \quad (1 \leqq s < k \leqq n)$$

and it is shown that Horn's conditions supplemented by the condition $\lambda_n \leqq a_{k-1}$ $(k = 2, \ldots, n - 1)$ are sufficient in this case. This paper represents also a substantial contribution to the — not yet completely solved — problem of characterization of eigenvalues of nonnegative matrices.

Another original method used for studying spectra of nonnegative matrices is the one based on estimating the magnitude of perturbation caused in the spectrum of an irreducible matrix.

$$\begin{pmatrix} U & O \\ W & V \end{pmatrix},$$

consisting of the eigenvalues of the matrices U, V; by a change of the zero matrix O. Here the crucial role is played by the so called measure of irreducibility of the matrix A, i.e. the quantity

$$\min_{M \neq \emptyset} \sum_{\substack{i \in M \\ k \notin M}} a_{ik}\,,$$

which is the least of the sums of elements of a nondiagonal corner block of the matrix. Let us recall that the wellknown Perron-Frobenius theorem asserts that the spectral radius of an irreducible nonnegative matrix is its simple positive eigenvalue (the so called Perron eigenvalue). While [57] provides the strict approximation of the distance of Perron eigenvalue from the others in terms of the measure of irreducibility, in [55], [69] and [72] this measure serves to the localization of spectra of doubly stochastic matrices, i.e. nonnegative matrices with unit row and column sums. These estimates were further improved by combining the above method with the technique of the so called compound matrices [61], [88]. By a compound matrix of the k-th degree corresponding to a given matrix of a size $n \times n$ we mean a matrix of the size $\binom{n}{k} \times \binom{n}{k}$ consisting of all subdeterminants of the k-th order of the matrix A. The measure of irreducibility and the spectrum of the matrix A are related in a simple way to the measure of irreducibility and the spectrum of the compound matrix, and, applying the estimates of the spectrum to the compound matrix, we obtain finer estimates for the original matrix A.

Another related fundamental problem, to whose solution Fiedler contributed in [95], is the study of geometric properties of the numerical range of a matrix A, i.e. the set $W(A) = \{(Ax, x);\ (x, x) = 1\}$ in the Gaussian plane. It is known that $W(A)$ is a compact and convex set containing the spectrum of the matrix A. The dual object to the set of all supporting lines of the set $W(A)$ is an algebraic curve $C(A)$, whose convex hull is the set $W(A)$ and whose real foci are the eigenvalues of the matrix A. Fiedler found the point equation of the curve $C(A)$ and determined its curvature at the boundary points of the set $W(A)$. Further, he proved that the numerical range of a nonnegative matrix lies in the circle $|z| \leqq r$, where r is Perron eigenvalue of the matrix $\frac{1}{2}(A + A^{\mathrm{T}})$ and, moreover, r is a corner point of the set $W(A)$.

Recently, Fiedler has devoted himself to Hankel matrices and related classes of matrices. Hankel matrices have the same elements in diagonals perpendicular to the main diagonal and appear in the problems of interpolation of rational functions, in reciprocal difference quotients and also in some geometrical situations. It has turned out that this apparently ancient theory is far from being closed. By improving classical methods Fiedler succeeded in finding mutual correspondence between Hankel, Toeplitz (which have the same elements on diagonals parallel to the main diagonal), Bézout (which are, roughly speaking, inverses of Hankel matrices), Loewner (with elements $(c_i - d_j)/(y_i - z_j)$ with c, d, y, z given vectors) and Vandermonde (with elements x_i^j, x a given vector) matrices, in exhibiting the analogies between the properties of these classes of matrices and, on the basis of these results, in essentially extending the classical results [103]–[110], [114].

Another special class of matrices in which M. Fiedler is interested are the tridiagonal matrices. They are useful in numerical mathematics: many methods of computing eigenvalues and eigenvectors of matrices reduce the problem from a general matrix to a tridiagonal one. The original Fiedler's method [89], [92] is based on reducing the order of a tridiagonal matrix: if we know one simple eigenvalue and the corresponding eigenvector, we are able to construct a tridiagonal matrix of order $n - 1$, whose eigenvalues coincide with the remaining eigenvalues of the original matrix, and from whose eigenvectors the eigenvectors of the original matrix can be computed. Another advantage of the method consists in the fact that in the case of a tridiagonal M-matrix and a positive eigenvalue it yields again a tridiagonal M-matrix. The tridiagonal matrices are of interest also from the purely theoretical view-point [49], [90]. For instance, irreducible symmetric tridiagonal matrices and matrices resulting from them by simultaneous permutations of rows and columns are exactly those symmetric matrices whose rank can be, by changing their diagonal elements, reduced at most by one.

From a more general view-point we can regard classes of matrices as points in a space. As a rule, important classes are then represented as polyhedra or cones, the geometrical and topological properties of these sets being connected with the algebraic and operator properties of the matrices of the class in question. M. Fiedler worked also in this field, his main interest being directed to the class of all linear operators which map a given polyhedral cone into another [63], [75], [82], [83]. He showed that such a class generates also a polyhedral cone, and investigated its properties. In particular, he described matrix properties of operators which correspond to the extreme rays. Generally, the diagonals of a polyhedral cone are related to the linear dependence of its extreme rays. Cones having exactly two diagonals are generated by $n + 1$ extreme rays, between which there is a unique linear dependence; they are called the minimal cones. When studying cones of all operators which map a minima cone into a minimal cone Fiedler found that their extremal operators can have an arbitrary rank h except $h = 2$. Properties of classes of operators positive with respect to a given cone, which are a generalization of positive definite matrices and M-matrices, were studied in [58], [63], [64], [100].

It was already in his early paper [13] that Fiedler described relations between the signs of elements of a symmetric matrix and those of the coordinates of the eigenvectors corresponding to its largest and smallest eigenvalues, and also the relations between the signs of elements of a positive definite matrix and of its inverse. He resumed several times the research in the location of positive, negative and zero elements in matrices satisfying certain conditions, in [73], [94], [99], [113]. The disclosure of the relations between the signs of coordinates of an eigenvector of an acyclic matrix and the position of the corresponding eigenvalue among the other eigenvalues, in particular its multiplicity, had interesting consequences in the graph theory. Fiedler's solution of the problem of characterization of the sign pattern of matrices inverse to the nonnegative ones got considerable response. He showed that the set of all possible sign patterns of matrices inverse to the fully indecomposable nonnegative matrices coincides with the set of all possible sign structures of fully indecomposable matrices with vanishing row- and column sums. If each sign structure $S = (z_{ij})$ of the size $n \times n$ is assigned the directed graph $G(S)$ with vertices $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ in which an edge (a_i, b_j) occurs iff $z_{ij} > 0$ and an edge (b_i, a_j) occurs iff $z_{ij} < 0$, then the above mentioned set is formed exactly by such structures S which are assigned a strongly connected graph $G(S)$.

Here we should mention Fiedler's work in the boundary field of the matrix theory and the graph theory. Indeed, some properties of matrices depend only on the location of zero and nonzero elements and thus can be studied by purely combinatorial methods. On the other hand, the study of combinatorial properties of graphs can sometimes be reduced to the study of algebraic properties of their incidence matrices. We have already mentioned that Fiedler used these methods dealing with simplexes. Another opportunity of applying them appeared in the numerical methods of linear algebra, in particular, the optimization of the choice of pivots in elimination methods [35], [76], [77], [84]. The language of the graph theory essentially simplifies the approach to the elimination process since it is independent of the order in which the pivots are selected, as well as of the particular values of the nonzero elements. This led Fiedler to creating the graph-theoretical analogue of Schur's complement of a matrix. This approach is suitable especially for numerical problems with large sparse matrices. Indeed, it makes it possible to choose the strategy of pivoting so that it causes no unnecessary increase of the number of nonzero elements and that the so called contour of the matrix, i.e. its part in which the nonzero elements occur, is as small as possible. In terms of graphs Fiedler also formulated the algorithm in [31], which reduces the inversion of a large matrix to that of its blocks of smaller size.

A nice example of combinatorial aspects of linear algebra is the classical Kirchhoff's theorem which determines the number of skeletons of a graph by means of the principal minors of the $(n-1)$-st order of its Laplace matrix. In [15] Fiedler gave an original proof of this result, generalizing it to directed labelled graphs and minors of the other orders. Further deep relations between the algebraic properties of the Laplace matrix $L(G)$ and the combinatorial properties of the nondirected graph G are established in [59], [68], [70], [74]. The matrix just mentioned has the off diagonal elements a_{ik} equal to -1 or 0 according to whether the graph G has the edge (i, k) or not, and the diagonal elements a_{ii} equal to the number of edges incidental with the i-th vertex. Evidently, the matrix $L(G)$ is symmetric, positive semidefinite and singular. Hence its eigenvalues are nonnegative, the least being equal to zero. Fiedler noticed the interesting behaviour of the least but one eigenvalue $\lambda(G)$ of the matrix $L(G)$, which increases with the increasing connectivity of the graph G, and thus can be considered a measure of its connectivity. Even the elements of the eigenvector corresponding to the eigenvalue $\lambda(G)$ have certain combinatorial properties.

The survey of Fiedler's work shows his effort to grasp the common general principles of linear algebra, combinatorics and geometry. In [80], these branches are joined by the theory of electric circuits. The following four objects are studied:

(A) the set of all finite nondirected graphs on n vertices with edges labelled by positive numbers;

(B) the set of all real symmetric $n \times n$-matrices of rank $n-1$ with nonpositive off-diagonal elements and zero row sums;

(C) the set of all connected electric circuits with n nodes consisting of resistors;

(D) the set of all classes of mutually congruent $(n-1)$-dimensional simplexes with no obtuse angles.

It is shown that these four models together with their numerical characteristics are mutually isomorphic.

In oor survey we have concentrated mainly on more comprehensive sets of thematically related Fiedler's works. From numerous lesser papers which do not belong to any of the reviewed domains, let us recall at least several easily formulated results, distinguished also by their aesthetical qualities:

If r is the degree of the minimal polynomial of a square matrix A then there exists a principal submatrix of A of order r and of rank at least $r-1$ [93].

The smallest eigenvalue of the matrix $(a_{ij}b_{ij})$, where $A = (a_{ij})$, $B = (b_{ij})$ are positive definite Hermitian matrices, is not less than the smallest eigenvalue of the matrix AB^{T} [102].

If two Hermitian matrices A, B have eigenvalues $\alpha_1 \geqq \ldots \alpha_n$ and $\beta_1 \geqq \ldots \geqq \beta_n$, respectively, then

$$\min_P \prod_{i=1}^n (\alpha_i + \beta_{P_i}) \leqq \det(A + B) \leqq \max_P \prod_{i=1}^n (\alpha_i + \beta_{P_i}),$$

where the maximum and minimum are taken over all permutations P of the indices $1, \ldots, n$ [54].

If A is an M-matrix of order n then $\operatorname{tr}(A^{\mathrm{T}}A^{-1}) \leqq n$, where the equality occurs iff there is a positive diagonal matrix D such that DAD^{-1} is symmetric [111].

Fiedler's works excel not only by the strength of his results, which as a rule are definitive and cannot be further improved, but also by their brilliant style and clear organization of the material. The characteristic feature of his methods is an ingenious combination of relatively elementary principles and the adequacy of the tools adopted. His results have become part of monographs, being frequently quoted by authors throughout the world, and have considerably enriched our mathematical knowledge.

LIST OF PUBLICATIONS OF PROFESSOR MIROSLAV FIEDLER

Papers

[1] Hyperosculating points of algebraic plane curves and their generalization in S_r (Czech). Thesis, Praha 1950.

[2] Solution of one problem of Professor E. Čech (Czech). Čas pěst.. mat. *77* (1952), 65—75.

[3] On certain matrices and the equation for parameters of singular points of a rational curve (Czech). Čas. pěst. mat. *77* (1952), 243—265, 321—346.

[4] Rational curves with the maximal number of real nodal points (Czech). Čas. pěst. mat. *79* (1954), 157—161 (with *L. Granát*).

[5] Geometry of simplexes in E_n, I (Czech). Čas. pěst. mat. *79* (1954), 270—297.

[6] Geometry of simplexes in E_n, II (Czech), Čas. pěst. mat. *80* (1955), 462—476.

[7] Geometry of simplexes in E_n, III (Czech), Čas. pěst. mat. *81* (1956), 182—223.

[8] Über das Gräffesche Verfahren. Czechoslovak Math. J. *5* (*80*) (1955), 506—516.

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