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Pushpa K. Jain; Khalil Ahmad; S. M. Maskey

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DOMINATION AND EQUIVALENCE OF SEQUENCES OF SUBSPACES IN DUAL SPACES

P. K. JAIN, Kuwait, K. AHMAD*), New Delhi, and S. M. MASKEY, Kathmandu

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1. INTRODUCTION

The concept of domination of sequences of vectors was given by Banach [3]. The study in this direction was carried out by several authors; for instance, one may refer to [1, 4, 11, 12, 15, 16]. Singer [12, 13] defined domination, strict domination, equivalence and strict equivalence of sequences of vectors in Banach spaces so that, if a sequence of vectors dominates another sequence of vectors and vice versa, the two sequences are called equivalent. Further, if a sequence of vectors strictly dominates another sequence of vectors and vice versa, the two sequences are called strictly equivalent. Very recently, Singer [14] generalised the concept of domination for sequences of subspaces of Banach spaces. He defined that if a sequence of subspaces of a Banach space dominates a sequence of subspaces of another Banach space and vice versa, the two sequences are called equivalent. Further, if a sequence of subspaces of a Banach space strictly dominates a sequence of subspaces of another Banach space and vice versa, the two sequences are not strictly equivalent. An example worked out by Singer [14] shows that strict equivalence of sequences of subspaces is not a direct generalisation of sequences of vectors. Motivated by this fact the authors in [8] obtained a number of results on equivalence and strict equivalence of sequences of subspaces in Banach spaces. In the present paper we continue the study of equivalence and strict equivalence of sequences of subspaces in duals of Banach spaces.

2. PRELIMINARIES

Throughout we shall assume that E and F are two Banach spaces. In general, we use $\{ \}$ to denote sets, $[\ ]$ to denote the closed linear spans of the sets indicated, and $(\ )$ to denote sequences.

Definition 2.1. A sequence (M_i) of subspaces of E is said to *dominate a sequence*

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(N_i) *of subspaces* of F if for each i, there exists a one-to-one linear transformation T_i from M_i onto N_i such that each sequence (x_i) in E, $x_i \in M_i$, $\sum_{i=1}^{\infty} x_i$ converges in E implies $\sum_{i=1}^{\infty} T_i(x_i)$ converges in F.

Definition 2.2. A sequence (M_i) of subspaces of E is said to *dominate strictly a sequence* (N_i) *of subspaces of* F if there exists a continuous linear transformation T from $[\bigcup_{i=1}^{\infty} M_i]$ into $[\bigcup_{i=1}^{\infty} N_i]$ such that $T(M_i) = N_i$ $(i = 1, 2, \ldots)$ and T restricted on M_i is one-to-one.

Definition 2.3. [14]. Sequences (M_i) of subspaces of E and (N_i) of subspaces of F are said to be *strictly equivalent* if there exists a linear homeomorphism T from $[\bigcup_{i=1}^{\infty} M_i]$ onto $[\bigcup_{i=1}^{\infty} N_i]$ such that $T(M_i) = N_i$ $(i = 1, 2, \ldots)$.

Now, we state some results in the form of lemmas which we will use in our work. For definitions of Schauder decomposition and unconditional Schauder decomposition one is referred to [7].

Lemma 2.4 [2]. *Let* (M_i, P_i) *be a Schauder decomposition of E. Then* (M_i, P_i) *is an unconditional Schauder decomposition of E if and only if for every increasing sequence* (m_i) *of positive integers, the subspaces* $[\bigcup_{i=1}^{\infty} M_{m_i}]$ *and* $[\bigcup_{j\in\omega\sim(m_i)} M_j]$ *are complementary to each other, i.e.* $E = [\bigcup_{i=1}^{\infty} M_{m_i}] \oplus [\bigcup_{j\in\omega\sim(m_i)} M_j]$.

Lemma 2.5 [14]. *Let* (M_i) *be a sequence of nontrivial closed subspaces of E such that* $[\bigcup_{i=1}^{\infty} M_i] = E$. *Then the following statements are equivalent*:

(i) (M_i) *is an unconditional Schauder decomposition of E*;

(ii) *Every permutation* $(M_{p(i)})$ *of* (M_i) *is a Schauder decomposition of E.*

Lemma 2.6 [14]. *Let* (M_i, P_i) *be a Schauder decomposition of E. Then* $(P_i^*(E^*))$ *is a Schauder decomposition of* $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ *and* $f = \sum_{i=1}^{\infty} P_i^*(f)$, $f \in [\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ *in the norm topology of* E^*.

Lemma 2.7 [14]. *Let* (M_i, P_i) *be a Schauder decomposition of* E, $V = [\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ *and* $v = \sup_{1 \leqq n < \infty} \|\sum_{i=1}^{n} P_i\|$.

Then

(a) $r(V) \geqq 1/v > 0$, *i.e. the unit ball of* $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ *is* σ-(E^*, E)*-dense in the* $(1/v)$*-ball of* E^*;

(b) *the canonical mapping u of E into* V^* *is a linear homeomorphism satisfying*

$$\|x\|_V \leqq \|x\| \leqq \frac{1}{r(V)} \|x\|_V, \quad x \in E,$$

where

$$\|x\|_V = \|u_x\| = \sup_{f\in V, \|f\| \leqq 1} |f(x)|, \quad x \in E.$$

Lemma 2.8 [14]. *If* (M_i) *is a Schauder decomposition of* E *and* (m_i) *is an increasing sequence of positive integers, then* (M_{m_i}) *is a Schauder decomposition of* $[\bigcup_{i=1}^{\infty} M_{m_i}]$.

Proposition 2.9. *Let* (M_i, P_i) *be an unconditional Schauder decomposition of* E. *Then* $(P_i^*(E^*))$ *is an unconditional Schauder decomposition of* $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$. *Conversely, if* $(P_i^*(E^*))$ *is an unconditional Schauder decomposition of* $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$, *then* (M_i, P_i) *is an unconditional Schauder decomposition of* E.

Proof. If (M_i) is an unconditional Schauder decomposition of E, then by Lemma 2.5, every permutation $(M_{p(i)})$ is a Schauder decomposition of E. Therefore, by Lemma 2.6, every permutation $(P_{p(i)}^*(E^*))$ is a Schauder decomposition of $[\bigcup_{i=1}^{\infty} P_{p(i)}^*(E^*)]$ and hence by Lemma 2.5, $(P_i^*(E^*))$ is an unconditional Schauder decomposition of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$.

Conversely, let $(P_i^*(E^*))$ be an unconditional Schauder decomposition of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$. If u is the canonical mapping of E into V^*, we have

$$u(M_i) \subset (P_i^*/V)^*(V^*).$$

Then the associated sequence of continuous projections $(u(M_i)) \subset [\bigcup_{i=1}^{\infty} P_i^*(E^*)]^*$ to the Schauder decomposition $(P_i^*(E^*))$ of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ is an unconditional Schauder decomposition of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]^*$. Therefore, by Lemma 2.7, the canonical mapping u is a linear homeomorphism. Hence (M_i) is an unconditional Schauder decomposition of E.

3. STRICT EQUIVALENCE OF SEQUENCES OF SUBSPACES

Theorem 3.1. *If a sequence* (M_i) *of closed subspaces of* E *is a Schauder decomposition of* $[\bigcup_{i=1}^{\infty} M_i]$ *and dominates a sequence* (N_i) *of subspaces of* F, *then* (M_i) *strictly dominates* (N_i).

Proof. Since (M_i) is a Schauder decomposition of $[\bigcup_{i=1}^{\infty} M_i]$ and dominates the sequence (N_i) of subspaces of F, the series $\sum_{i=1}^{\infty} y_i$, $y_i \in N_i$ for each i, converges in F

so that

$$T(x) = \sum_{i=1}^{\infty} y_i , \quad x = \sum_{i=1}^{\infty} x_i \in [\bigcup_{i=1}^{\infty} M_i] .$$

The mapping T: $[\bigcup_{i=1}^{\infty} M_i] \to [\bigcup_{i=1}^{\infty} N_i]$ defined above is a linear transformation satisfying $T(M_i) = N_i$, $i = 1, 2, \ldots$. Further,

$$T(x) = \lim_{n\to\infty} T_n(x) , \quad x \in E ,$$

where

$$T_n(x) = \sum_{i=1}^{n} y_i \quad (x = \sum_{i=1}^{\infty} x_i \in [\bigcup_{i=1}^{\infty} M_i]) ,$$

$n = 1, 2, \ldots$. Each T_n being continuous, by Banach-Steinhaus theorem, T is continuous as well. This completes the proof.

Theorem 3.2. *If a sequence (M_i) of closed subspaces of E is a Schauder decomposition of $[\bigcup_{i=1}^{\infty} M_i]$ and a sequence (N_i) of closed subspaces of F is a Schauder decomposition of $[\bigcup_{i=1}^{\infty} N_i]$ such that (M_i) is equivalent to (N_i), then (M_i) is strictly equivalent to (N_i).*

Proof. Since the Schauder decomposition (M_i) of $[\bigcup_{i=1}^{\infty} M_i]$ is equivalent to the Schauder decomposition (N_i) of $[\bigcup_{i=1}^{\infty} N_i]$, for each $x = \sum_{i=1}^{\infty} x_i \in [\bigcup_{i=1}^{\infty} M_i]$, $x_i \in M_i$ $(i = 1, 2, \ldots)$, there exists a linear homeomorphism T of $[\bigcup_{i=1}^{\infty} M_i]$ onto $[\bigcup_{i=1}^{\infty} N_i]$ such that

$$T(x) = \sum_{i=1}^{\infty} y_i , \quad y_i \in N_i \quad (i = 1, 2, \ldots) .$$

This gives $T(M_i) = N_i$ $(i = 1, 2, \ldots)$. Hence (M_i) is strictly equivalent to (N_i).

Theorem 3.3. *Let (M_i, P_i) be an unconditional Schauder decomposition of E, (M_{i_j}) a subsequence of (M_i), $E_0 = [\bigcup_{j=1}^{\infty} M_{i_j}]$ and $\phi_{i_j} = P_{i_j}|_{E_0}$ $(j = 1, 2, \ldots)$. Then $(\phi_{i_j}^*(E_0^*))$ is a Schauder decomposition of $[\bigcup_{j=1}^{\infty} \phi_{i_j}^*(E_0^*)]$. Moreover, $(\phi_{i_j}^*(E_0^*))$ is strictly equivalent to the Schauder decomposition $(P_{i_j}^*(E^*))$ of $[\bigcup_{j=1}^{\infty} P_{i_j}^*(E^*)]$.*

Proof. By Lemma 2.6 and Lemma 2.8, $(\phi_{i_j}^*(E_0^*))$ and $(P_{i_j}^*(E^*))$ are Schauder decompositions of $[\bigcup_{j=1}^{\infty} \phi_{i_j}^*(E_0^*)]$ and $[\bigcup_{j=1}^{\infty} P_{i_j}^*(E^*)]$, respectively. Since (M_i, P_i) is an unconditional Schauder decomposition of E, by Proposition 2.9, $(P_i^*(E^*))$ is an unconditional Schauder decomposition of $[\bigcup_{j=1}^{\infty} \phi_{i_j}^*(E_0^*)]$ and $[\bigcup_{j=1}^{\infty} P_{i_j}^*(E^*)]$, respectively. Since (M_i, P_i) is an unconditional Schauder decomposition of E, by Proposition 2.9, $(P_i^*(E^*))$ is an unconditional Schauder decomposition of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ and by Lemma 2.4, the mapping

$$(1) \qquad u(f) = \sum_{j=1}^{\infty} P_{i_j}^*(f), \quad f \in [\bigcup_{i=1}^{\infty} P_i^*(E^*)],$$

is a unique continuous projection of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ onto $[\bigcup_{j=1}^{\infty} P_{i_j}^*(E^*)]$. Further, $\phi_{i_j}^*$ is the restriction of $P_{i_j}^*$, $[\bigcup_{j=1}^{\infty} \phi_{i_j}^*(E_0^*)] \subset [\bigcup_{j=1}^{\infty} P_{i_j}^*(E^*)]$; the convergence of a series in the first space implies the convergence of the series in the second space. Also relation (1) shows that the convergence of a series in $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$ implies the convergence of a series in $[\bigcup_{j=1}^{\infty} P_{i_j}^*(E^*)]$, hence in $[\bigcup_{j=1}^{\infty} \phi_{i_j}^*(E_0^*)]$, and conversely. Thus the sequence $(P_{i_j}^*(E^*))$ of subspaces of the conjugate space is equivalent to the sequence $(\phi_{i_j}^*(E_0^*))$ of subspaces of the conjugate space. By Theorem 3.2 the two sequences are strictly equivalent.

Theorem 3.4. *Let (M_i, P_i) and (N_i, Q_i) be Schauder decompositions of E and F, respectively. Then (M_i, P_i) dominates (N_i, Q_i) if and only if $(Q_i^*(F^*))$ dominates $(P_i^*(E^*))$.*

Proof. If (M_i) dominates (N_i), by Theorem 3.1 (M_i) dominates (N_i) strictly. Therefore there exists a continuous linear transformation $T: E \to F$ such that $T(M_i) = N_i$ $(i = 1, 2, \ldots)$. Now for the adjoint mapping $T^*: F^* \to E^*$ and for each $h_i \in Q_i^*(F^*)$, $y_j \in N_j$ and $x_j \in M_j$, we have

$$(T^*(h_i))\, x_j = h_i(T(x_j)) = h_i(y_j) = \delta_{ij} = f_i(x_j),$$

where $T(x_j) \in T(M_j)$, $i, j = 1, 2, \ldots$. Since $[\bigcup_{i=1}^{\infty} M_i] = E$, we have $T^*(h_i) = f_i$ $(i = 1, 2, \ldots)$. This implies that $T^*(Q_i^*(F^*)) = P_i^*(E^*)$. Hence $(Q_i^*(F^*))$ dominates $(P_i^*(E^*))$ strictly and thus $(Q_i^*(F^*))$ dominates $(P_i^*(E^*))$.

Conversely, let $(Q_i^*(F^*))$ dominate $(P_i^*(E^*))$. If u_F is the canonical mapping of F into $[\bigcup_{i=1}^{\infty} Q_i^*(F^*)]^*$, it is obvious that $u_F(N_i) \subset (Q_i^*|_{[\bigcup_{i=1}^{\infty} Q_i^*(F^*)]})^* [\bigcup_{i=1}^{\infty} Q_i^*(F^*)]^*$ and by [14, Proposition 15.7], the latter subspace is linearly homeomorphic to $(Q_i^*(F^*))^*$. Then the associated sequence of continuous projections to the Schauder decomposition $(Q_i^*(F^*))$ of $[\bigcup_{i=1}^{\infty} Q_i^*(F^*)]$ is $(u_F(N_i))$. Similarly, $(u_E(M_i))$ is the associated sequence of continuous projections to the Schauder decomposition $(P_i^*(E^*))$ of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$. Thus, by the statement proved above, $(u_E(M_i))$ dominates $(u_F(N_i))$. Also by Lemma 2.7, both u_E and u_F are linear homeomorphisms. Hence (M_i) dominates (N_i).

Corollary 3.5. *Let (M_i, P_i) and (N_i, Q_i) be Schauder decompositions of E and F, respectively. Then (M_i) is equivalent to (N_i) if and only if $(P_i^*(E^*))$ is equivalent to $(Q_i^*(F^*))$.*

Definition 4.1. A Schauder decomposition (M_i) of E is said to be *subsymmetric* if it is an unconditional Schauder decomposition and for every increasing sequence of positive integers (m_i) the Schauder decomposition (M_{m_i}) of $[\bigcup_{i=1}^{\infty} M_{m_i}]$ is equivalent to the Schauder decomposition (M_i).

Examples. Let χ be a Banach space with the norm $\| \ \|$. Let us consider the following Banach spaces:

(i) $c_0(\chi) = \{(x_i): x_i \in \chi, \lim_{i \to \infty} x_i = 0$ in the norm of $\chi\}$, the norm on $c_0(\chi)$ being given by

$$|(x_i)| = \sup_i \|x_i\| .$$

(ii) $l_1(\chi) = \{(x_i): x_i \in \chi, \sum_{i=1}^{\infty} \|x_i\| < \infty\}$,

the norm on $l_1(\chi)$ being given by

$$\|(x_i)\|^* = \sum_{i=1}^{\infty} \|x_i\| .$$

Since χ is a Banach space, it follows that the topological duals of the spaces $c_0(\chi)$ and $l_1(\chi)$ coincide with their respective cross duals (cf. Table 3.29 for the scalar case, Kamthan and Gupta [9], and Prop. 4.8 and Cor. 4.9 of Gupta et al. [6] for vector valued sequence spaces).

Now, we observe that (N_i) with

$$N_i = \{\delta_i^{x_i}: x_i \in \chi\} ,$$

where $\delta_i^{x_i}$ stands for the sequence $(0, 0, \ldots, x_i, 0, \ldots\}$, i.e., the i-th entry in $\delta_i^{x_i}$ is x_i and all the others are zero, forms an unconditional Schauder decomposition (see Gupta et al. [5, p. 291] and Marti [10, p. 95]) of each of the Banach spaces $c_0(\chi)$ and $l_1(\chi)$.

Now, one may easily verify that the Schauder decomposition (N_i) of $c_0(\chi)$ and $l_1(\chi)$ is subsymmetric.

Theorem 4.2. *If* (M_i, P_i) *is a subsymmetric Schauder decomposition of E, then* $(P_i^*(E^*))$ *is a subsymmetric Schauder decomposition of* $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$.

Proof. Let (i_j) be an increasing sequence of positive integers and (M_{i_j}) the subsequence of (M_i). Since (M_i, P_i) is a subsymmetric Schauder decomposition of E, it is unconditional and the subsequence (M_{i_j}) is equivalent to (M_i). Also, by Proposition 2.9, $(P_i^*(E^*))$ is an unconditional Schauder decomposition of $[\bigcup_{i=1}^{\infty} P_i^*(E^*)]$. Further, by Lemma 2.8, (M_{i_j}) is a Schauder decomposition of

$[\bigcup_{j=1}^{\infty} M_{i_j}] = E_0$, and by Corollary 3.5, $(P_i^*(E^*))$ is equivalent to the Schauder decomposition $(\phi_{i_j}^*(E_0^*))$ of $[\bigcup_{j=1}^{\infty} \phi_{i_j}^*(E_0^*)]$. Moreover, (M_i) is an unconditional Schauder decomposition of E, by Theorem 3.3, $(\phi_{i_j}^*(E_0^*))$ is equivalent to the Schauder decomposition $(P_{i_j}^*(E^*))$ of $[\bigcup_{j=1}^{\infty} P_{i_j}^*(E^*)]$. Thus $(P_{i_j}^*(E^*))$ is equivalent to $(P_i^*(E^*))$. This completes the proof.

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Authors' addresses: P. K. Jain, Department of Mathematics, Kuwait University, Kuwait; K. Ahmad, Department of Mathematics, Jamia Millia Islamia, New Delhi, India; S. M. Maskey, Department of Mathematics, Institute of Education, Tribhuwan University, Kathmandu, Nepal.