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SUMMABLE SUBSEQUENCES IN CONVERGENCE GROUPS

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In this note, we present two examples which are related to solutions of problems concerning convergence groups which were put forth by J. Novák in [5].

In Problem 14 of [5], Novák asks for an example of a convergence group which has a sequence $\{x_i\}$ which converges to o but for which the sum $\sum_{i=1}^{\infty} x_{k_i}$ fails to exist for any subsequence $\{x_{k_i}\}$ of $\{x_i\}$. This problem has been solved by F. Zanolin ([6]) and by R. Frič and V. Koutník ([2]), but in both cases, the spaces involved in the solution are convergence groups. In Example 1, we present a normed space which contains such a sequence.

In Problem 15 of [5], Novák asks for an example of a convergence group which contains a sequence such that each subsequence contains one subsequence which is summable and another subsequence which is not summable. (A sequence $\{x_i\}$ is summable if the series $\sum_{i=1}^{\infty} x_i$ converges.) In [3], C. Klis has given an example of a normed space which contains such a sequence. (C. Ryll-Nardzewski had previously given such an example using the Continuum Hypothesis.) Following Example 1 we give a discussion concerning the types of convergent sequences which arise in this problem.

Henceforth, let m_0 be the subspace of l^∞, the vector space of all bounded real sequences, which consists of those sequences $x = \{t_j\}$ with finite range, i.e., $Rx = \{t_j: 1 \leqq j < \infty\}$ is finite. Equip both l^∞ and m_0 with the sup-norm, $\|x\| = \|\{t_j\}\| = \sup |t_j|$. Under the sup-norm, m_0 is a dense, proper subspace of l^∞.

Conserning Problem 14 of [5], we have

Example 1. Let e_j be the sequence in m_0 which has a 1 in the jth coordinate and a zero in the other coordinates. Then the sequence $\{(1/j)\, e_j\}$ converges to o in m_0 but no sequence is summable to an element of m_0. (If $\{(1/k_j)\, e_{k_j}\}$ is any subsequence of $\{(1/j)\, e_j\}$, then $\sum_{j=1}^{\infty} (1/k_j)\, e_{k_j}$ will not converge to an element of m_0 since each element of m_0 has finite range.)

Concerning Problem 15 of [5], we introduce the following terminology. Let G be an Abelian topological group (or convergence group). A sequence $\{x_i\}$ in G is said to be $\mathscr{K}$ *convergent* if each subsequence of $\{x_i\}$ has a subsequence $\{x_{k_i}\}$ such that the series $\sum_{i=1}^{\infty} x_{k_i}$ converges to an element of G. (These sequences are named in honor of Katowice, Poland, where the members of the Katowice Branch of the Mathematics Institute have introduced and studied many such properties of convergent sequences; an equivalent definition in metric linear spaces was introduced by S. Mazur and W. Orlicz in their study of the uniform boundedness principle, [4], Axiom II, p. 169.) A sequence $\{x_i\}$ in G is said to be $\mathscr{N}$ *convergent* if $\{x_i\}$ has a subsequence $\{x_{k_i}\}$ such that each subsequence of $\{x_{k_i}\}$ is summable to an element of G. (Such sequences were introduced by L. S. Sobolev in Novisibirsk.)

Clearly, any sequence which is $\mathscr{N}$ convergent is also $\mathscr{K}$ convergent. Problem 15 of [5] asks for an example of a sequence which is $\mathscr{K}$ convergent but not $\mathscr{N}$ convergent. Klis has given an example of such a sequence in a normed space ([3]).

If G is a complete normed group, then a sequence $\{x_i\}$ in G is $\mathscr{K}$ convergent iff it is $\mathscr{N}$ convergent. (If $\{x_i\}$ is $\mathscr{K}$ convergent, then $x_i \to o$ so there is a subsequence $\{x_{x_i}\}$ of $\{x_i\}$ such that $\sum |x_{k_i}| < \infty$, where $| \;|$ is the quasi-norm in G. The completeness of G then implies that any subseries of $\sum x_{k_i}$ is summable to an element of G. In fact, this argument shows that any sequence which converges to o is $\mathscr{K}$ convergent.) The following question then naturally arises concerning the converse of the statement above.

Problem 2. Suppose the Abelian topological group (or convergence group) G has the property that a sequence in G is $\mathscr{K}$ convergent iff it is $\mathscr{N}$ convergent. Then is G necessarily complete? That is, does the converse of the statement in the paragraph above hold?

We show in Corollary 4 below that this conjecture is false.

Again, the example involves the space m_0.

We first state a lemma concerning $\mathscr{K}$ convergent sequences in m_0. The lemma in both proof and content is modeled after Theorem 1 of [1] so we omit the proof. We introduce the following notation. If A is a subset of m_0, $[A]$ denotes the linear span of A.

Lemma 3. *If $\{x_n\}$ is $\mathscr{K}$ convergent in m_0 (in sup-norm), then $[x_n\colon n \in \mathbb{N}]$ is finite dimensional.*

Concerning the problem posed above, we have

Corollary 4. *A sequence in m_0 is $\mathscr{K}$ convergent iff it is $\mathscr{N}$ convergent .*

Proof. Let $\{x_i\}$ be $\mathscr{K}$ convergent in m_0. Then $\{x_i\}$ is contained in a finite dimensional subspace F of m_0 by Lemma 3. Thus, if a subsequence $\{x_{k_i}\}$ of $\{x_i\}$ is chosen such that $\sum_{i=1}^{\infty} \|x_{k_i}\| < \infty$, then $\sum x_{k_i}$ is subseries convergent in F by the completeness of F. That is, $\{x_i\}$ is $\mathscr{N}$ convergent.

Since $\mathscr{N}$ convergence always implies $\mathscr{K}$ convergence, the result follows.

Thus, m_0 provides an example of an incomplete normed space with the property that any sequence in m_0 is $\mathscr{K}$ convergent iff it is $\mathscr{N}$ convergent. This gives a negative solution to Problem 2.

References

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