Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

# EMBEDDING IN GLOBALS OF FINITE SEMILATTICES

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(Received October 24, 1984)

## 1. INTRODUCTION

The *global*, or *power semigroup*, of a semigroup $S$ is the family $P(S)$ of all non-empty subsets of $S$, with the natural multiplication

$$AB = \{ab\colon a \in A,\ b \in B\} \quad \text{for all} \quad A, B \in P(S)\,.$$

$S$ is said to be *combinatorial* if all subgroups of $S$ are trivial. Among the combinatorial semigroups are those semigroups $S$ with zero such that $S^n = \{0\}$ for some $n$; such a semigroup is said to be *nil*.

The purpose of this paper is to establish the embeddability of various kinds of finite, commutative semigroups in globals of finite semilattices; by a result of the first two authors [3], any semigroup embeddable in the global of a finite semilattice is also embeddable in the global of a finite abelian group (indeed, an elementary 2-group). The semigroups considered here include all finite Rees quotients of free commutative semigroups, all combinatorial commutative semigroups of order less than 5, and certain extensions of zero semigroups. (To avoid confusion of the terms nil and null, we use the term *zero semigroup* to indicate that all products are equal.)

We are concerned only with combinatorial semigroups because, as noted by M. S. Putcha [5], the global of a finite semilattice is combinatorial. Our emphasis is on nil semigroups because, by results of Gould and Iskra [3] and A. Lau [4], the embeddability of all finite commutative nil semigroups in globals of finite semilattices would imply the embeddability of all finite commutative semigroups in globals of finite abelian groups. Finiteness is considered because, by a result of V. Trnková [8], every commutative semigroup is embeddable in the global of an infinite abelian group. Finally, commutativity is presumed for two reasons: S. G. Beršadskiĭ [1] has shown that every semigroup is embeddable in the global of an infinite non-abelian group; and Lau [4] noted the existence of finite non-commutative semigroups that are not embeddable in the global of any finite group.

Unless otherwise noted, all semigroups in this section are given in additive notation. Because the symbol 0 in this notation would represent an identity element, the symbol $\infty$ will be used for the additive counterpart of the multiplicative zero.

The *height* of a nil semigroup $S$ is the smallest positive integer $h$ such that $hS = \{\infty\}$. Given positive integers $n$ and $h$, the free commutative nil semigroup of height $h$ on $n$ free generators is discussed in T. Tamura [7]. We shall denote this semigroup by $F(n, h)$. It is free in the sense that every commutative nil semigroup of height at most $h$ and having a generating set of at most $n$ elements is a homomorphic image of $F(n, h)$.

To construct $F(n, h)$, first consider the free commutative semigroup $F(n)$ on $n$ generators. $F(n)$ is the subsemigroup of non-zero elements of the $n$-th direct power of the additive semigroup of non-negative integers. $F(n, h)$ is then $F(n)/I$, where $I$ is the ideal of $F(n)$ given by

$$I = \{(w_1, \ldots, w_n) \in F(n) : \sum_{i=1}^{n} w_i \geqq h\}.$$

The free generators of $F(n, h)$ are of course the unit vectors $x_i$ for $1 \leqq i \leqq n$, where $x_i$ is the $n$-tuple having 1 in its $i$-th component and 0 everywhere else. Although the vector $\mathbf{0} = (0, \ldots, 0)$ is not an element of $F(n, h)$, it is a linear combination of the unit vectors in the trivial sense.

In Gould and Iskra [3] it was shown that each $F(n, h)$ is embeddable in the global of a finite semilattice. Inasmuch as $F(n, h)$ is a Rees quotient of $F(n)$, the following theorem extends that result to Rees quotients of $F(n, h)$.

**Theorem 2.1.** *Every finite Rees quotient of a free commutative semigroup is embeddable in the global of a finite semilattice.*

Proof. Given an ideal $I$ of $F(n)$ such that $F(n)/I$ is finite, we seek a finite semilattice $Y$ and a homomorphism $f$ of $F(n)$ into $P(Y)$ such that $f(a) = f(b)$ if and only if $a$ and $b$ belong to $I$.

Let $Y$ be the semilattice of all subsets of the set $(F(n) \cup \{\mathbf{0}\}) \setminus I$, under the operation of union. For $A \in Y$ and $j \in \{1, \ldots, n\}$ we define the "$j$-th coefficient count" of $A$ to be the number of distinct coefficients of the generator $x_j$ that occur in the representation of members of $A$. That is, we define $Q_j(A) = |\{a_j : a \in A\}|$, where each $a \in F(n) \cup \{\mathbf{0}\}$ has the unique representation $a = \sum_{i=1}^{n} a_i x_i$ for non-negative $a_i$.

We now define $f$ to be the unique homomorphism of $F(n)$ into $P(Y)$ having the property that $f(x_i) = \{A \in Y : Q_i(A) = 1\}$ for all $i$. Specifically, for $b \in F(n)$ and $A \in Y$, we have $A \in f(b)$ if and only if there exist $A_1, \ldots, A_n \in Y$ such that $A = \bigcup_{i=1}^{n} A_i$ and $Q_i(A_i) \leqq b_i$ for all $i$.

We first show that all elements of $I$ have the same image under $f$, namely $P(Y)$. Indeed, fix $t \in I$ and let $A \in Y$. For each $i \in \{1, \ldots, n\}$ set $A_i = \{a \in A : a_i < t_i\}$.

Clearly $Q_i(A_i) \leqq t_i$, and, since $A$ is disjoint from the ideal $I$, every element of $A$ belongs to some $A_i$. It follows that $A \in f(t)$, whence $f(t) = P(Y)$.

For each $a \in F(n) \setminus I$ define $C_a = \{x \in F(n): x_i \leqq a_i \text{ for all } i\} \in Y$. If $C_a \supseteq \bigcup_{i=1}^{n} A_i$ with $Q_i(A_i) \leqq a_i$ then there are numbers $0 \leqq c_i \leqq a_i$ such that $c_i$ is not the $i$-th component of any member of $A_i$. Then $c = \sum_{i=1}^{n} c_i x_i \notin \bigcup_{i=1}^{n} A_i$. This shows that $C_a \notin f(a)$, hence $f(a) \neq P(Y)$ if $a \in F(n) \setminus I$. Moreover, if $a$ and $b$ are distinct elements of $F(n) \setminus I$ then we may suppose $a_j < b_j$ for some $j$, whereupon $C_a \notin f(a)$ but $C_a \in f(b)$ as $Q_j(C_a) = a_j + 1 \leqq b_j$. Thus $f$ is one-to-one on $F(n) \setminus I$ and the theorem is proved.

Before generalizing the above theorem to a wider class of quotients, we define the concept of *height* for elements of $F(n, h)$. Given $a \in F(n, h)$ other than $\infty$, define $H(a) = \sum_{i=1}^{n} a_i$, where $a = \sum_{i=1}^{n} a_i x_i$. Finally, $H(\infty)$ is defined to be $h$. (This definition is consistent with the more general definition of Tamura [7].)

A congruence $\theta$ on $F(n, h)$ is said to be *height-preserving* if $H(a) = H(b)$ whenever $a \; \theta \; b$. For such $\theta$ we define the *height of a $\theta$-class* to be the height of each of its elements.

We omit the very easy proof of the following statement.

**Lemma 2.2.** *For all positive integers n and h the following hold*:
(i) *For all $a, b, c \in F(n, h)$, if $a + b = c + b \neq \infty$, then $a = c$.*
(ii) *A congruence $\theta$ on $F(n, h)$ is height-preserving if and only if $\{\infty\}$ is a $\theta$-class.*

**Theorem 2.3.** *Let $\theta$ be a height-preserving congruence on $F(n, h)$ such that there is at most one non-singleton $\theta$-class of each height. Then $F(n, h)/\theta$ can be embedded in the global of a finite semilattice.*

Proof. Set $S = F(n, h)$. It is obviously sufficient to express $S/\theta$ as a subdirect product of semigroups each of which is embeddable in the global of a finite semilattice. Thus we seek to express $\theta$ as the intersection of congruences $\eta$ and $\varrho$ such that $S/\eta$ and $S/\varrho$ can be so embedded.

Let $\eta$ be the congruence on $S$ given by: $a \; \eta \; b$ if and only if $H(a) = H(b)$. Then $S/\eta$ is isomorphic to $F(1, h)$, hence is embeddable in the global of a finite semilattice by Theorem 2.1.

Set $I = K \cup \{\infty\}$, where $K$ is the union of the non-singleton $\theta$-classes. We note that $I$ is an ideal, as follows. Let $a \in I$ and $b \in S$ such that $a + b \neq \infty$. Then there exists $c \in S \setminus \{a\}$ such that $a \; \theta \; c$, hence $a + b$ and $c + b$ are $\theta$-related, and the above lemma gives $a + b \neq c + b$. Hence $a + b \in I$, and so $I$ is an ideal.

Now let $\varrho$ be the Rees congruence associated with $I$. By Theorem 2.1 $S/\varrho$ is embeddable in the global of a finite semilattice. It thus remains to show that $\eta \cap \varrho = \theta$.

As $\theta$ is height-preserving, we have $\theta \subseteq \eta$, and the definition of $I$ then gives $\theta \subseteq \eta \cap \varrho$. Now let $(a, b) \in \eta \cap \varrho$ such that $a \neq b$. Then $a$ and $b$ are elements of $I$

having the same height, and each belongs to a non-singleton $\theta$-class. Since there is only one such class of a given height, we have $a \,\theta\, b$. Thus $\theta = \eta \cap \varrho$, concluding the proof.

## 3. SEMIGROUPS OF ORDER $<5$ AND EXTENSIONS OF ZERO SEMIGROUPS

Let $S$ be a combinatorial commutative semigroup of finite order $n$. In the proof of Theorem 2.4 of Gould and Iskra [3] it is shown that $S$ is embeddable in the global of a finite semilattice if every subdirectly irreducible commutative *nil* semigroup of order $\leqq n$ is so embeddable. By a result of B. M. Schein [6], a commutative nil semigroup $T$ is subdirectly irreducible if and only if the map $t \to \{x \in T\colon tx = 0\}$ is one-to-one for all non-zero $t \in T$. (It follows that the semigroups $F(1, h)$ are subdirectly irreducible.) Clearly, no zero semigroup of more than two elements is subdirectly irreducible.

The following proof utilizes the fact (noted by Tamura [7]) that every non-trivial finite nil semigroup $S$ has a unique prime generating set, that is, a set of generators disjoint from $S^2$.

**Theorem 3.1.** *Each combinatorial commutative semigroup of order less than 5 is embeddable in the global of a finite semilattice.*

Proof. In light of the above considerations, let $S$ be a subdirectly irreducible commutative nil semigroup of order $<5$. Let $h$ denote the height of $S$, and and let $n$ be the number of elements in the prime generating set of $S$.

We assume that $n$ is not 1, for otherwise $S$ would be isomorphic to $F(1, h)$ and Theorem 2.1 would apply. Clearly $n$ cannot be 4 since 0 is not prime. Moreover, if $n = 3$ then $S$ is a zero semigroup and therefore not subdirectly irreducible. Thus $n = 2$, and $h$ is at least 2. If $h$ were 2 then $S$ would be a zero semigroup, while $h = 4$ would imply $n = 1$. Hence $h = 3$, and $|S|$ is at least 3. But then $|S| = 4$, since otherwise $S$ would be a zero semigroup.

We may now take $S = F/\theta$, where $F = F(2, 3)$ and $\theta$ is a congruence on $F$. Let $I$ denote the $\theta$-class of $\infty$; obviously $I$ is an ideal. Because $F$ does not contain more than three elements of a given height, Theorem 2.3 would give the desired embedding if $\theta$ were height-preserving, so we assume this is not the case. By Lemma 2.2, it then follows that $|I| > 1$. Since $|F| = 6$ and $|S| = 4$, we cannot have $|I| > 3$, and $|I| = 3$ would imply that $\theta$ is the Rees congruence associated with $I$, whereupon Theorem 2.1 would apply. We therefore now assume that $|I| = 2$. It follows that there is only one other non-singleton $\theta$-class $K$, and $|K| = 2$.

Suppose the elements of $K$ differ in height. Then one of the free generators of $F$ would have to be a member of $K$, inasmuch as $\infty \notin K$. Assuming $x_1 \in K$, we then have $K = \{x_1, b\}$ for some $b \in \{x_1 + x_2, 2x_1, 2x_2\}$. If $b = x_1 + x_2$ then addition of $x_2$ to each element of $K$ would show that $\infty$ is $\theta$-related to $b$, contradicting the disjointness of $I$ and $K$. Likewise the assumption that $b = 2x_1$ yields a contradiction. If $b = 2x_2$ the same method forces $\infty$ to be $\theta$-related to both $2x_1$ and $x_1 + x_2$, which is impossible because $|I| = 2$.

Thus the elements of $K$ have the same height, which we denote $k$. If $k = 1$ then $2x_1, 2x_2$, and $x_1 + x_2$ would be three $\theta$-related elements, a contradiction. Thus $k = 2$, and a similar argument shows that the noninfinity element of $I$, which we shall denote $r$, also has height 2.

There are three cases to consider, depending on the value of $r$. However, we need only deal with two cases, since the semigroups obtained for $r = 2x_1$ and $r = 2x_2$ are isomorphic. In both cases $S$ will be embedded into $P(Y)$, where $Y$ is the semilattice of all subsets of the set $\{1, \ldots, 5\}$, under the operation of union.

Case 1. Suppose $r = x_1 + x_2$. Then addition in $S$ is given by: $\{x_1\} + \{x_1\} = \{x_2\} + \{x_2\} = K$ and all other sums are equal to $I$. It follows that an embedding of $S$ into $P(Y)$ is given by:

$$\begin{aligned}
\{x_1\} &\to \{A \in Y\colon |A| < 3\}\,,\\
\{x_2\} &\to \{A \in Y\colon |A| < 3 \text{ and } A \neq \{1, 2\}\} \cup \{\{3, 4, 5\}\}\,,\\
K &\to \{A \in Y\colon |A| < 5\}\,,\\
I &\to Y\,.
\end{aligned}$$

Case 2. Suppose $r = 2x_1$. Then addition in $S$ is given by: $\{x_1\} + \{x_2\} = \{x_2\} + \{x_1\} = \{x_2\} + \{x_2\} = K$ and all other sums are equal to $I$. It follows that an embedding of $S$ into $P(Y)$ is given by:

$$\begin{aligned}
\{x_1\} &\to \{A \in Y\colon |A| < 3 \text{ and } A \neq \{3, 5\}\} \cup \{\{1, 2, 3\}\}\,,\\
\{x_2\} &\to \{A \in Y\colon |A| < 3 \text{ and } A \neq \{4, 5\}\}\,,\\
K &\to \{A \in Y\colon |A| < 5\}\,,\\
I &\to Y\,,
\end{aligned}$$

and the theorem is proved.

M. Yamada and T. Tamura [10] noted that a non-trivial finite commutative semigroup is nil if and only if it is an ideal extension of a nil semigroup by a two-element zero semigroup. For such extensions of zero semigroups, we have the desired embeddability.

**Theorem 3.2.** *Every commutative ideal extension of a finite zero semigroup by a two-element zero semigroup is embeddable in the global of a finite semilattice.*

Proof. Let $\mathscr{K}$ denote the class of all such extensions. As in the proof of Theorem 2.3, it is sufficient to express each semigroup in $\mathscr{K}$ as a subdirect product of semigroups each of which is embeddable in the global of a finite semilattice.

It is easy to verify that any non-trivial homomorphic image of a member of $\mathscr{K}$ must again belong to $\mathscr{K}$. It then follows from G. Birkhoff's [2] subdirect representation theorem that every semigroup in $\mathscr{K}$ is a subdirect product of subdirectly irreducible members of $\mathscr{K}$. By Theorem 3.1, it remains only to show that every subdirectly irreducible semigroup in $\mathscr{K}$ has order less than 5.

Let $T$ be a subdirectly irreducible member of $\mathscr{K}$, and let $S$ be an ideal of $T$ such that $S^2 = \{0\}$ and $T/S$ is a two-element zero semigroup. Set $A = \{x \in S: xt = 0\}$, where $t$ is the only member of $T$ that does not belong to $S$. Every $a \in A$ satisfies $\{x \in T: ax = 0\} = T$, and every $b \in S \setminus A$ satisfies $\{x \in T: bx = 0\} = S$. The criterion of Schein [6] cited above now implies that $|A| < 3$ and $|S \setminus A| < 2$, whence $|T| < 5$ and the theorem is proved.

We note in conclusion that methods for constructing all finite commutative nil semigroups were developed in Yamada and Tamura [10] and Yamada [9]. Our initial proofs of Theorems 3.1 and 3.2 made extensive use of those methods.

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