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ON THE RIEMANNIAN CURVATURE TENSOR OF AN ALMOST-PRODUCT MANIFOLD

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0. Introduction. Let E be an n-dimensional real vector space with a positive definite inner product, and consider the vector space $\mathscr{R}(E)$ of all the 4-covariant tensors on E satisfying the same symmetries as the Riemannian curvature tensor of a Riemannian manifold. Then, $\mathscr{R}(E)$ decomposes (see [1]) as a direct sum of subspaces invariant and irreducible under the action of the orthogonal group $O(n)$, the structure group of Riemannian manifolds. If we consider an almost-Hermitian structure on E, i.e., an automorphism J of E such that $J^2 = -$ identity and $g(JL, JM)$ structure on E, i.e., an automorphism J of E such that $J^2 = -$ identity and $g(JL, JM) = g(L, M)$ for all $L, M \in E$, then, Tricerri and Vanhecke ([4]) have given a decomposition of $\mathscr{R}(E)$ as a direct sum of subspaces invariant and irreducible under the action of $U(m)$ (assuming $n = 2m$), the structure group of almost Hermitian manifolds. In this paper we get a similar result for the structure group of almost-product manifolds, $O(p) \times O(q)$, where p and q, with $p + q = n$, are the dimensions of the vertical and horizontal subspaces determined by such a structure. Then we compute a system of generators of the space of invariant quadratic forms on $\mathscr{R}(E)$ from which we conclude the irreducibility of the decomposition. Finally, we prove that the projectors of $\mathscr{R}(E)$ onto some of the subspaces are conformal invariants.

1. The decomposition of $\mathscr{R}(E)$ under the action of $O(p) \times O(q)$. Let E be an n-dimensional real vector space with a positive definite inner product g, and let V and H be orthogonal subspaces of E of dimensions p and q, respectively, with $p + q = n$, and such that $E = V \oplus H$. (This is equivalent to giving an automorphism P of E such that $P^2 =$ identity and $g(PL, PM) = g(L, M)$ for all $L, M \in E$; i.e., an almost-product structure.) An orthonormal basis $\{E_i\}_{i=1,\dots,n}$ will be said to be *adapted* if $E_i \in V$ for $i = 1, \dots, p$ and $E_i \in H$ for $i = p + 1, \dots, n$. Next, we consider the space of 4-covariant tensors on E satisfying the same symmetries as the Riemannian curvature tensor of a Riemannian manifold,

$$\mathscr{R}(E) = \{R \in \otimes^4 E^* \mid R(L, M, N, U) = -R(M, L, N, U) = -R(L, M, U, N) \quad \text{and}$$
$$R(L, M, N, U) + R(M, N, L, U) + R(N, L, M, U) = 0 \quad \text{for all } L, M, N, U \in E\},$$

where E^* stands for the dual space of E. As is well-known, if $R \in \mathscr{R}(E)$, then $R(L, M, N, U) = R(N, U, L, M)$ for all $L, M, N, U \in E$, and also $\dim \mathscr{R}(E) = \frac{1}{12}n^2(n^2 - 1)$.

If $O(p)$ and $O(q)$ are the groups of orthogonal transformations of V and H, respectively, then $O(p) \times O(q)$ acts upon E in a natural way, so that the action preserves the subspaces V and H, and the inner product g. It induces an action on $\mathscr{R}(E)$ as follows:

$$(AR)(L, M, N, U) = R(A^{-1}L, A^{-1}M, A^{-1}N, A^{-1}U)$$

for all $A \in O(p) \times O(q)$, $R \in \mathscr{R}(E)$ and $L, M, N, U \in E$.

We also have a positive definite inner product $<, >$ in $\mathscr{R}(E)$, defined by

$$\langle R, R' \rangle = \sum_{i,j,k,l=1}^{n} R(E_i, E_j, E_k, E_l)\, R'(E_i, E_j, E_k, E_l)$$

where $\{E_i\}_{i=1,\cdots,n}$ is an adapted orthonormal basis of E.

First, we have a trivial decomposition of $\mathscr{R}(E)$ as a direct sum of subspaces invariant under the action of $O(p) \times O(q)$, namely,

$$\mathscr{R}(E) = \mathscr{R}_{40} \oplus \mathscr{R}_{04} \oplus \mathscr{R}_{31} \oplus \mathscr{R}_{13} \oplus \mathscr{R}_{22}$$

where, if $\{E_i\}_{i=1,\cdots,n}$ is an adapted orthonormal basis of E, $\mathscr{R}_{\alpha\beta}$ is the subspace of the $R \in \mathscr{R}(E)$ whose non-vanishing components $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ are exactly those having α arguments in V and β arguments in H. It is clear that these subspaces are invariant by $O(p) \times O(q)$ and mutually orthogonal with reespect to $<, >$.

In order to get a further decomposition of each of these subspaces, we define two 2-covariant tensors associated to each curvature tensor:

$$\varrho_V(R)(M, N) = \sum_{a=1}^{p} R(M, E_a, N, E_a),$$

$$\varrho_H(R)(M, N) = \sum_{u=p+1}^{n} R(M, E_u, N, E_u)$$

for all $M, N \in E$, where $\{E_i\}_{i=1,\cdots,n}$ is an adapted orthonormal basis.

Also, we consider for each $R \in \mathscr{R}(E)$ the scalars

$$\tau_V(R) = \sum_{b=1}^{p} \varrho_V(R)(E_b, E_b) = \sum_{a,b=1}^{p} R(E_a, E_b, E_a, E_b),$$

$$\tau_H(R) = \sum_{v=p+1}^{n} \varrho_H(R)(E_v, E_v) = \sum_{u,v=p+1}^{n} R(E_u, E_v, E_u, E_v),$$

$$\tau_{VH}(R) = \sum_{u=p+1}^{n} \varrho_V(R)(E_u, E_u) = \sum_{a=1}^{p} \varrho_H(R)(E_a, E_a) =$$

$$= \sum_{a=1}^{p} \sum_{u=p+1}^{n} R(E_a, E_u, E_a, E_u).$$

To begin with, it is clear that $\mathscr{R}_{40}$ is isomorphic to the space of curvature tensors $\mathscr{R}(V)$ on the vector space V, and $O(q)$, as a subgroup of $O(p) \times O(q)$ acts on $\mathscr{R}_{40}$ as the identity, so that a decomposition of $\mathscr{R}_{40}$ as a direct sum of irreducible subspaces under the action of $O(p) \times O(q)$ is given by the classical decomposition of $\mathscr{R}(V)$ under $O(p)$ (see, for instance, [1]). Then, we can write

$$\mathscr{R}_{40} = \mathscr{W}_V \oplus \mathscr{R}_V \oplus \mathbb{R} \, . \, R_{ov} ,$$

where

$$\mathscr{W}_V = \{R \in \mathscr{R}_{40} \mid \varrho_V(R) = 0\} ,$$

$$\mathscr{W}_V \oplus \mathscr{R}_V = \{R \in \mathscr{R}_{40} \mid \tau_V(R) = 0\} ,$$

$$\mathscr{R}_V = \mathscr{W}_V^{\perp} \text{ (the orthogonal complement of } \mathscr{W}_V \text{ in } \mathscr{W}_V \oplus \mathscr{R}_V) ,$$

and $\mathbb{R} \, . \, R_{ov} = (\mathscr{W}_V \oplus \mathscr{R}_V)^{\perp}$ (the orthogonal complement of $\mathscr{W}_V \oplus \mathscr{R}_V$ in $\mathscr{R}_{40}$).

The notation in the last case is due to the fact that $(\mathscr{W}_V \oplus \mathscr{R}_V)^{\perp}$ is the one-dimensional subspace of $\mathscr{R}_{40}$ spanned by the tensor R_{ov}, given by

$$R_{ov}(A, B, C, D) = g(A, C) \, g(B, D) - g(A, D) \, g(B, C)$$

for all $A, B, C, D \in V$.

Similarly,

$$\mathscr{R}_{04} = \mathscr{W}_H \oplus \mathscr{R}_H \oplus \mathbb{R} \, . \, R_{0h} ,$$

where

$$\mathscr{W}_H = \{R \in \mathscr{R}_{04} \mid \varrho_H(R) = 0\} ,$$

$$\mathscr{W}_H \oplus \mathscr{R}_H = \{R \in \mathscr{R}_{04} \mid \tau_H(R) = 0\} ,$$

$$\mathscr{R}_H = \mathscr{W}_H^{\perp} \text{ (the orthogonal complement of } \mathscr{W}_H \text{ in } \mathscr{W}_H \oplus \mathscr{R}_H) ,$$

and $\mathbb{R} \, . \, R_{oh} = (\mathscr{W}_H \oplus \mathscr{R}_H)^{\perp}$ (the orthogonal complement of $\mathscr{W}_H \oplus \mathscr{R}_H$ in $\mathscr{R}_{04}$), R_{oh} being the element of $\mathscr{R}_{04}$ determined by

$$R_{oh}(X, Y, Z, W) = g(X, Z) \, g(Y, W) - g(X, W) \, g(Y, Z)$$

for all $X, Y, Z, W \in H$.

On the other hand, $\mathscr{R}_{31}$ can be considered as the subspace of $\Lambda^2 V^* \otimes V^* \otimes H^*$ of all tensors R such that $R(A, B, C, X) + R(B, C, A, X) + R(C, A, B, X) = 0$ for all $A, B, C \in V$ and $X \in H$. Since the action of $O(q)$ upon V is trivial, the decomposition of $\mathscr{R}_{31}$ is given by that of the subspace of $\Lambda^2 V^* \otimes V^*$ formed by all tensors α such that

$$\alpha(A, B, C) + \alpha(B, C, A) + \alpha(C, A, B) = 0$$

for all $A, B, C \in V$, under the action of $O(p)$. The latter is well known (see for example [3]) and as a result we get

$$\mathscr{R}_{31} = \mathscr{G}_{v1} \oplus \mathscr{G}_{v2}$$

where

$$\mathcal{G}_{v1} = \{R \in \mathcal{R}_{31} \mid \varrho_V(R) = 0\}$$

$$\text{and } \mathcal{G}_{v2} = \{R \in \mathcal{R}_{31} \mid R(A, B, C, X) = \frac{1}{p-1} (g(A, C)\, \varrho_V(R)\, (B, X) - $$

$$- g(B, C)\, \varrho_V(R)\, (A, X)) \quad \text{for all} \quad A, B, C \in V \quad \text{and} \quad X \in H\}\,.$$

The fact that $\mathcal{G}_{v1}$ is orthogonal to $\mathcal{G}_{v2}$ follows by a straightforward computation.

Similarly,

$$\mathcal{R}_{13} = \mathcal{G}_{h1} \oplus \mathcal{G}_{h2}$$

with

$$\mathcal{G}_{h1} = \{R \in \mathcal{R}_{13} \mid \varrho_H(R) = 0\}\,,$$

$$\text{and } \mathcal{G}_{h2} = \{R \in \mathcal{R}_{13} \mid R(X, Y, Z, A) = \frac{1}{q-1} (g(X, Z)\, \varrho_H(R)\, (Y, A) - $$

$$- g(Y, Z)\, \varrho_H(R)\, (X, A)) \quad \text{for all} \quad A \in V \quad \text{and} \quad X, Y, Z \in H\}$$

and, as before, $\mathcal{G}_{h1}$ is orthogonal to $\mathcal{G}_{h2}$.

As for $\mathcal{R}_{22}$, the defining conditions of $\mathcal{R}(E)$ imply that the components of a tensor $R \in \mathcal{R}_{22}$ are determined by those of the form $R(E_a, E_u, E_b, E_v)$, for $1 \leqq a,\ b \leqq p$ and $p + 1 \leqq u,\ v \leqq n$. As a consequence, $\mathcal{R}_{22}$ can be considered as the space

$$(V^* \otimes H^*) \vee (V^* \otimes H^*)\,;$$

($\vee$ means the symmetric tensor product). Actually, if we identify this space with the space of 4-linear maps $\alpha: V \times H \times V \times H \to \mathbb{R}$ such that for all $A, B \in V$ and all $X, Y \in H$, $\alpha(A, X, B, Y) = \alpha(B, Y, A, X)$, then the map $\Phi: \mathcal{R}_{22} \to (V^* \otimes H^*) \vee \vee (V^* \otimes H^*)$ given by $\Phi(R)\,(A, X, B, Y) = R(A, X, B, Y)$ for all $R \in \mathcal{R}_{22}$, $A, B \in V$, $X, Y \in H$ is a vector space isomorphism, whose inverse is

$$\Psi: (V^* \otimes H^*) \vee (V^* \otimes H^*) \to \mathcal{R}_{22}$$

defined by

$$\Psi(\alpha)\,(L, M, N, W) = -\alpha(vM, hN, vL, hW) + \alpha(vL, hN, vM, hW) +$$

$$+ \alpha(vL, hM, vN, hW) - \alpha(vL, hM, vW, hN) - \alpha(vM, hL, vN, hW) +$$

$$+ \alpha(vM, hL, vW, hN) - \alpha(vW, hL, vN, hM) + \alpha(vN, hL, vW, hM)$$

for all $\alpha \in (V^* \otimes H^*) \vee (V^* \otimes H^*)$ and all $L, M, N, W \in E$.

Now, having in mind that

$$V^* \otimes V^* = \Lambda^2 V^* \oplus \vee_0^2 V^* \oplus \{1\}_v\,,$$

where $\Lambda^2 V^*$ is the space of skewsymmetric convariant 2-tensors on V, $\vee_0^2 V^*$ the space of traceless symmetric covariant 2-tensors on V, and $\{1\}_v$ the orthogonal complement of $\Lambda^2 V^* \oplus \vee_0^2 V^*$ in $V^* \otimes V^*$ (with regard to the inner product induced by the restriction of g to V), and that, similarly,

$$H^* \otimes H^* = \Lambda^2 H^* \oplus \vee_0^2 H^* \oplus \{1\}_h ,$$

we get

$$\mathcal{R}_{22} = S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus S_5$$

where

$$S_1 = \{R \in \mathcal{R}_{22} \mid R(A, X, B, Y) = -R(B, X, A, Y) = -R(A, Y, B, X) \text{ for all } A, B \in V \text{ and } X, Y \in H\} ,$$

$$S_2 = \{R \in \mathcal{R}_{22} \mid R(A, X, B, Y) = R(B, X, A, Y) = R(A, Y, B, X) \text{ for all } A, B \in V \text{ and } X, Y \in H, \text{ and } \varrho_H(R) = \varrho_V(R) = 0\} ,$$

$$S_3 = \{R \in \mathcal{R}_{22} \mid R(A, X, B, Y) = \frac{1}{q} g(X, Y)\, \varrho_H(R)(A, B) \text{ for all } A, B \in V \text{ and } X, Y \in H, \text{ and } \varrho_V(R) = 0\} ,$$

$$S_4 = \{R \in \mathcal{R}_{22} \mid R(A, X, B, Y) = \frac{1}{p} g(A, B)\, \varrho_V(R)(X, Y) \text{ for all } A, B \in V \text{ and } X, Y \in H, \text{ and } \varrho_H(R) = 0\} \text{ and}$$

$$S_5 = \{R \in \mathcal{R}_{22} \mid R(A, X, B, Y) = \frac{\tau_{VH}}{pq} g(A, B)\, g(X, Y) \text{ for all } A, B \in V \text{ and } X, Y \in H\} .$$

It is easy to see that these five subspaces are mutually orthogonal, and hence, we have proved

Theorem 1. *The space $\mathcal{R}(E)$ is isomorphic to the direct sum of the following fifteen subspaces invariant by $O(p) \times O(q)$:*

$$\mathcal{W}_V, \mathcal{R}_V, \mathbb{R} \,.\, R_{ov}, \mathcal{W}_H, \mathcal{R}_H, \mathbb{R} \,.\, R_{oh}, \mathcal{G}_{v1}, \mathcal{G}_{v2}, \mathcal{G}_{h1}, \mathcal{G}_{h2}, S_1, S_2, S_3, S_4, S_5 .$$

The dimensions of these subspaces are given in Table I, in terms of the dimensions p and q, of V and H.

If $R \in \mathcal{R}(E)$, then its orthogonal projections into each of the invariant subspaces (in the same order as they appear in Table I) are determined as follows, for all $A, B, C, D \in V$; $X, Y, Z, W \in H$:

$$p_1(R)(A, B, C, D) = R(A, B, C, D) - \frac{1}{p-2} (g(A, C)\, \varrho_V(R)(B, D) - g(B, C)\, \varrho_V(R)(A, D) - g(A, D)\, \varrho_V(R)(B, C) + g(B, D)\, \varrho_V(R)(A, C)) + \frac{\tau_V(R)}{(p-1)(p-2)} (g(A, C)\, g(B, D) - g(A, D)\, g(B, C)) ,$$

Table I

	$p, q \geqq 3$	$p = 2,\ q \geqq 3$	$p = 2$, $q = 2$	$p = 1,\ q \geqq 3$	$p = 1$, $q = 2$	$p = q = 1$
$\mathscr{W}_V$	$\frac{p(p+1)(p+2)(p-3)}{12}$	0	0	0	0	0
$\mathscr{R}_V$	$\frac{(p-1)(p+2)}{2}$	0	0	0	0	0
$\mathbb{R} \, . \, R_{ov}$	1	1	1	0	0	0
$\mathscr{W}_H$	$\frac{q(q+1)(q+2)(q-3)}{12}$	$\frac{q(q+1)(q+2)(q-3)}{12}$	0	$\frac{q(q+1)(q+2)(q+3)}{12}$	0	0
$\mathscr{R}_H$	$\frac{(q-1)(q+2)}{2}$	$\frac{(q-1)(q+2)}{2}$	0	$\frac{(q-1)(q+2)}{2}$	0	0
$\mathbb{R} \, . \, R_{oh}$	1	1	1	1	1	0
$\mathscr{G}_{v1}$	$\frac{pq(p+2)(p-2)}{3}$	0	0	0	0	0
$\mathscr{G}_{v2}$	pq	$2q$	4	0	0	0
$\mathscr{G}_{h1}$	$\frac{pq(q+2)(q-2)}{3}$	$\frac{2q(q+2)(q-2)}{3}$	0	$\frac{q(q+2)(q-2)}{3}$	0	0
$\mathscr{G}_{h2}$	pq	$2q$	4	q	2	0
S_1	$\frac{pq(p-1)(q-1)}{4}$	$\frac{q(q-1)}{2}$	1	0	0	0
S_2	$\frac{(p-1)(p+2)(q-1)(q+2)}{4}$	$(q-1)(q+2)$	4	0	0	0
S_3	$\frac{(p-1)(p+2)}{2}$	2	2	0	0	0
S_4	$\frac{(q-1)(q+2)}{2}$	$\frac{(q-1)(q+2)}{2}$	2	$\frac{(q-1)(q+2)}{2}$	2	0
S_5	1	1	1	1	1	1

$$p_2(R)(A, B, C, D) = \frac{1}{p-2}(g(A, C)\,\varrho_V(R)(B, D) - g(B, C)\,\varrho_V(R)(A, D) -$$

$$- g(A, D)\,\varrho_V(R)(B, C) + g(B, D)\,\varrho_V(R)(A, C)) - \frac{2\tau_V(R)}{p(p-2)}$$

$$(g(A, C)\,g(B, D) - g(A, D)\,g(B, C)),$$

$$p_3(R)(A, B, C, D) = \frac{\tau_V(R)}{p(p-1)}(g(A, C)\,g(B, D) - g(A, D)\,g(B, C)),$$

$$p_4(R)(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{q-2}(g(X, Z)\,\varrho_H(R)(Y, W) -$$

$$- g(Y, Z)\,\varrho_H(R)(X, w) - g(X, W)\,\varrho_H(R)(Y, Z) +$$

$$+ g(Y, W)\,\varrho_H(R)(X, Z)) + \frac{\tau_H(R)}{(q-1)(q-2)}(g(X, Z)\,g(Y, W) -$$

$$- g(X, W)\,g(Y, Z)),$$

$$p_5(R)(X, Y, Z, W) = \frac{1}{q-2}(g(X, Z)\,\varrho_H(R)(Y, W) - g(Y, Z)\,\varrho_H(R)(X, W) -$$

$$- g(X, W)\,\varrho_H(R)(Y, Z) + g(Y, W)\,\varrho_H(R)(X, Z)) - \frac{2\tau_H(R)}{q(q-2)}$$

$$(g(X, Z)\,g(Y, W) - g(X, W)\,g(Y, Z)),$$

$$p_6(R)(X, Y, Z, W) = \frac{\tau_H(R)}{q(q-1)}(g(X, Z)\,g(Y, W) - g(X, W)\,g(Y, Z)),$$

$$p_7(R)(A, B, C, X) = R(A, B, C, X) - \frac{1}{p-1}(g(B, C)\,\varrho_V(R)(A, X) -$$

$$- g(A, C)\,\varrho_V(R)(B, X)),$$

$$p_8(R)(A, B, C, X) = \frac{1}{p-1}(g(B, C)\,\varrho_V(R)(A, X) - g(A, C)\,\varrho_V(R)(B, X)),$$

$$p_9(R)(X, Y, Z, A) = R(X, Y, Z, A) - \frac{1}{q-1}(g(Y, Z)\,\varrho_H(R)(X, A) -$$

$$- g(X, Z)\,\varrho_H(R)(Y, A)),$$

$$p_{10}(R)(X, Y, Z, A) = \frac{1}{q-1}(g(Y, Z)\,\varrho_H(R)(X, A) - g(X, Z)\,\varrho_H(R)(Y, A)),$$

$$p_{11}(R)(A, X, B, Y) = \tfrac{1}{2}(R(A, X, B, Y) - R(B, X, A, Y)),$$

$$p_{12}(R)(A, X, B, Y) = \tfrac{1}{2}(R(A, X, B, Y) + R(B, X, A, Y)) -$$

$$- \frac{1}{q} g(X, Y)\,\varrho_H(R)(A, B) - \frac{1}{p} g(A, B)\,\varrho_V(R)(X, Y) + \frac{\tau_{VH}(R)}{pq} g(A, B)\, g(X, Y),$$

$$p_{13}(R)(A, X, B, Y) = \frac{1}{q} g(X, Y)\,\varrho_H(R)(A, B) - \frac{\tau_{VH}(R)}{pq} g(A, B)\, g(X, Y),$$

$$p_{14}(R)(A, X, B, Y) = \frac{1}{p} g(A, B)\,\varrho_V(R)(X, Y) - \frac{\tau_{VH}(R)}{pq} g(A, B)\, g(X, Y),$$

$$p_{15}(R)(A, X, B, Y) = \frac{\tau_{VH}(R)}{pq} g(A, B)\, g(X, Y).$$

2. Invariant quadratic forms on $\mathscr{R}(E)$. Let $\mathscr{V}$ be the space $\otimes^k E^*$; then $O(p) \times O(q)$ acts upon $\mathscr{V}$ as follows:

$$(A \,.\, \alpha)(X_1, \ldots, X_k) = \alpha(A^{-1} \,.\, X_1, \ldots, A^{-1} \,.\, X_k)$$

for all $A \in O(p) \times O(q)$, $\alpha \in \mathscr{V}$, and $X_1, \ldots, X_k \in E$. In a way similar to the case of $O(n)$, if $F\colon \mathscr{V} \to \mathbb{R}$ is a homogeneous polynomial of degree h, we say that F is a product of traces if the following holds:

– $k \times h$ is even, equal to $2s$, and

– there exist a permutation σ of $\{1, \ldots, 2s\}$ and an adapted orthonormal basis $\{E_1, \ldots, E_n\}$ of E, such that, for all $\alpha \in \mathscr{V}$,

$$F(\alpha) = \sum_{a_1,\ldots,a_r=1}^{p} \; \sum_{u_1,\ldots,u_{s-r}=p+1}^{n}$$

$$\sigma(\otimes^k \alpha)(E_{a_1}, E_{a_1}, \ldots, E_{a_r}, E_{a_r}, E_{u_1}, E_{u_1}, \ldots, E_{u_{s-r}}, E_{u_{s-r}}),$$

where r is an integer such that $0 \leqq r \leqq s$, $\otimes^k\alpha$ is the element of $\otimes^{kh} E^*$ taking $(X_1, \ldots, X_{kh})$ into $\alpha(X_1, \ldots, X_k) \times \alpha(X_{k+1}, \ldots, X_{2k}) \times \ldots \times \alpha(X_{k(h-1)+1}, \ldots, X_{kh})$, and $\sigma(\otimes^k \alpha)$ takes $(X_1, \ldots, X_{kh})$ into $(\otimes^h \alpha)(X_{\sigma(1)}, \ldots, X_{\sigma(kh)})$. It is clear that the expression of $F(\alpha)$ is independent of the choice of the adapted orthonormal basis, and in particular, F is invariant by $O(p) \times O(q)$. As a consequence of the corresponding theorem for $O(n)$ [5] (see also [1], [2]), we have

Theorem 2. *The vector space of real homogeneous polynomials on $\mathscr{V}$, invariant by $O(p) \times O(q)$, is spanned by the products of traces (as defined above).*

Now, for $h = 2$ and $k = 4$ we get $h \times k = 8$ and $s = 4$. Then, the products of traces, in this case, are the quadratic forms

$$R \to \sum_{a_1,\ldots,a_r=1}^{p} \; \sum_{u_1,\ldots,u_{4-r}=p+1}^{n}$$

$$\sigma(R \otimes R)(E_{a_1}, E_{a_1}, \ldots, E_{a_r}, E_{a_r}, E_{u_1}, E_{u_1}, \ldots, E_{u_{4-r}}, E_{u_{4-r}})$$

where σ is a permutation of the set $\{1, \ldots, 8\}$, and r is an integer with $1 \leqq r \leqq 4$.

Having in mind the defining symmetries of the curvature tensor R, and denoting by $R_{\alpha\beta}$ the component of R in $\mathscr{R}_{\alpha\beta}$, we get the following products of traces:

For $r = 4$,

$$I_1 = \sum_{a_1,\dots,a_4} R(E_{a_1}, E_{a_2}, E_{a_3}, E_{a_4})\, R(E_{a_1}, E_{a_2}, E_{a_3}, E_{a_4}) = \|R_{40}\|^2 ,$$

$$I_2 = \sum_{a_1,\dots,a_4} R(E_{a_1}, E_{a_2}, E_{a_3}, E_{a_2})\, R(E_{a_1}, E_{a_4}, E_{a_3}, E_{a_4}) = \|\varrho(R_{40})\|^2 = \|\varrho_V(R)\,|_{V\times V}\|^2 ,$$

$$I_3 = \sum_{a_1,\dots,a_4} R(E_{a_1}, E_{a_2}, E_{a_1}, E_{a_2})\, R(E_{a_3}, E_{a_4}, E_{a_3}, E_{a_4}) = \tau_V(R)^2 .$$

For $r = 0$,

$$I_4 = \sum_{u_1,\dots,u_4} R(E_{u_1}, E_{u_2}, E_{u_3}, E_{u_4})\, R(E_{u_1}, E_{u_2}, E_{u_3}, E_{u_4}) = \|R_{04}\|^2 ,$$

$$I_5 = \sum_{u_1,\dots,u_4} R(E_{u_1}, E_{u_2}, E_{u_3}, E_{u_2})\, R(E_{u_1}, E_{u_4}, E_{u_3}, E_{u_4}) = \|\varrho(R_{04})\|^2 = \|\varrho_H(R)|_{H\times H}\|^2 ,$$

$$I_6 = \sum_{u_1,\dots,u_4} R(E_{u_1}, E_{u_2}, E_{u_1}, E_{u_2})\, R(E_{u_3}, E_{u_4}, E_{u_3}, E_{u_4}) = \tau_H(R)^2 .$$

For $r = 3$,

$$I_7 = \sum_{a_1,a_2,a_3,u_1} R(E_{a_1}, E_{a_2}, E_{a_1}, E_{a_2})\, R(E_{a_3}, E_{u_1}, E_{a_3}, E_{u_1}) = \tau_V(R)\,\tau_{VH}(R) ,$$

$$I_8 = \sum_{a_1,a_2,a_3,u_1} R(E_{a_1}, E_{a_2}, E_{a_3}, E_{a_2})\, R(E_{a_1}, E_{u_1}, E_{a_3}, E_{u_1}) = \langle \varrho_V(R)\,|_{V\times V}, \varrho_H(R)|_{V\times V}\rangle ,$$

$$I_9 = \sum_{a_1,a_2,a_3,u_1} R(E_{a_1}, E_{a_2}, E_{a_3}, E_{u_1})\, R(E_{a_1}, E_{a_2}, E_{a_3}, E_{u_1}) = \tfrac{1}{4}\|R_{31}\|^2 ,$$

$$I_{10} = \sum_{a_1,a_2,a_3,u_1} R(E_{a_1}, E_{a_2}, E_{u_1}, E_{a_2})\, R(E_{a_1}, E_{a_3}, E_{u_1}, E_{a_3}) = \|\varrho_V(R)\,|_{V\times H}\|^2 =$$
$$= \|\varrho_V(R)\,|_{H\times V}\|^2 .$$

For $r = 1$,

$$I_{11} = \sum_{a_1,u_1,u_2,u_3} R(E_{u_1}, E_{u_2}, E_{u_1}, E_{u_2})\, R(E_{u_3}, E_{a_1}, E_{u_3}, E_{a_1}) = \tau_H(R)\,\tau_{VH}(R) ,$$

$$I_{12} = \sum_{a_1,u_1,u_2,u_3} R(E_{u_1}, E_{u_2}, E_{u_3}, E_{u_2})\, R(E_{u_1}, E_{a_1}, E_{u_3}, E_{a_1}) = \langle \varrho_H(R)\,|_{H\times H}, \varrho_V(R)|_{H\times H}\rangle ,$$

$$I_{13} = \sum_{a_1,u_1,u_2,u_3} R(E_{u_1}, E_{u_2}, E_{u_3}, E_{a_1})\, R(E_{u_1}, E_{u_2}, E_{u_3}, E_{a_1}) = \tfrac{1}{4}\|R_{13}\|^2 ,$$

$$I_{14} = \sum_{a_1,u_1,u_2,u_3} R(E_{u_1}, E_{u_2}, E_{a_1}, E_{u_2})\, R(E_{u_1}, E_{u_3}, E_{a_1}, E_{u_3}) = \|\varrho_H(R)|_{H\times V}\|^2 .$$

For $r = 2$,

$$I_{15} = \sum_{a_1,a_2,u_1,u_2} R(E_{a_1}, E_{a_2}, E_{a_1}, E_{a_2})\, R(E_{u_1}, E_{u_2}, E_{u_1}, E_{u_2}) = \tau_V(R)\,\tau_H(R) ,$$

$$I_{16} = \sum_{a_1,a_2,u_1,u_2} R(E_{a_1}, E_{a_2}, E_{a_1}, E_{u_1})\, R(E_{a_2}, E_{u_2}, E_{u_1}, E_{u_2}) =$$
$$= \langle \varrho_V(R)|_{V\times H}, \varrho_H(R)|_{V\times H}\rangle ,$$

$$I_{17} = \sum_{a_1,a_2,u_1,u_2} R(E_{a_1}, E_{u_1}, E_{a_2}, E_{u_2})\, R(E_{a_2}, E_{u_1}, E_{a_1}, E_{u_2}) ,$$

$$I_{18} = \sum_{a_1,a_2,u_1,u_2} R(E_{a_1}, E_{u_1}, E_{a_2}, E_{u_2})\, R(E_{a_1}, E_{u_1}, E_{a_2}, E_{u_2}),$$

$$I_{19} = \sum_{a_1,a_2,u_1,u_2} R(E_{a_1}, E_{u_1}, E_{a_1}, E_{u_2})\, R(E_{a_2}, E_{u_1}, E_{a_2}, E_{u_2}) = \|\varrho_V(R)|_{H\times H}\|^2,$$

$$I_{20} = \sum_{a_1,a_2,u_1,u_2} R(E_{a_1}, E_{u_1}, E_{a_2}, E_{u_1})\, R(E_{a_1}, E_{u_2}, E_{a_2}, E_{u_2}) = \|\varrho_H(R)|_{V\times V}\|^2,$$

$$I_{21} = \sum_{a_1,a_2,u_1,u_2} R(E_{a_1}, E_{u_1}, E_{a_1}, E_{u_1})\, R(E_{a_2}, E_{u_2}, E_{a_2}, E_{u_2}) = \tau_{VH}(R)^2.$$

So, from Theorem 2, we have

Theorem 3. *The vector space of quadratic forms on $\mathscr{R}(E)$, invariant by $O(p) \times O(q)$, is spanned by $I_1, \ldots, I_{21}$.*

Now, in order to prove that the invariant subspaces of Theorem 1 are irreducible we make use of the following theorem:

Theorem 4. [4]. *Let G be a subgroup of $O(n)$ and let T be a finite dimensional real vector space acted upon by G. Let $<,>$ be a positive definite inner product on T, invariant by G. Then, T is irreducible if and only if the space of quadratic invariants on T is one-dimensional.*

As a consequence, to prove that one of the fifteen subspaces is irreducible, it suffices to prove that the restrictions to it of the twenty-one products of traces vanish or are multiples of just one of them. In Table II we list the non-vanishing invariants on each subspace (we treat only the case $p, q \geqq 3$, the others being similar).

Then, we have

Theorem 5. *The fifteen subspaces given in Theorem 1 are irreducible for the action of $O(p) \times O(q)$.*

The norms of the projectors of $\mathscr{R}(E)$ onto each of these subspaces can be expressed in terms of the quadratic invariants as follows

$$\|p_1(R)\|^2 = I_1 - \frac{4}{p-2} I_2 + \frac{2}{(p-1)(p-2)} I_3,$$

$$\|p_2(R)\|^2 = \frac{4}{p-2} I_2 - \frac{4}{p(p-2)} I_3,$$

$$\|p_3(R)\|^2 = \frac{2}{p(p-1)} I_3,$$

$$\|p_4(R)\|^2 = I_4 - \frac{4}{q-2} I_5 + \frac{2}{(q-1)(q-2)} I_6,$$

$$\|p_5(R)\|^2 = \frac{4}{q-2} I_5 - \frac{4}{q(q-2)} I_6,$$

Table II

$\mathscr{W}_V$	I_1
$\mathscr{R}_V$	$I_1 = \frac{4}{p(p-1)} I_2$
$\mathbb{R} . R_{ov}$	$I_1 = \frac{2}{p-1} I_2 = \frac{2}{p(p-1)} I_3$
$\mathscr{W}_H$	I_4
$\mathscr{R}_H$	$I_4 = \frac{4}{q(q-1)} I_5$
$\mathbb{R} . R_{oh}$	$I_4 = \frac{2}{q-1} I_5 = \frac{2}{q(q-1)} I_6$
$\mathscr{G}_{v1}$	I_9
$\mathscr{G}_{v2}$	$I_9 = \frac{1}{p-1} I_{10}$
$\mathscr{G}_{h1}$	I_{13}
$\mathscr{G}_{h2}$	$I_{13} = \frac{1}{q-1} I_{14}$
S_1	$I_{17} = -I_{18}$
S_2	$I_{17} = I_{18}$
S_3	$I_{17} = I_{18} = \frac{1}{p} I_{20}$
S_4	$I_{17} = I_{18} = \frac{1}{q} I_{19}$
S_5	$I_{17} = I_{18} = \frac{1}{pq} I_{21} = \frac{1}{p} I_{19} = \frac{1}{q} I_{20}$

$$\|p_6(R)\|^2 = \frac{2}{q(q-1)} I_6 ,$$

$$\|p_7(R)\|^2 = 4I_9 - \frac{4}{p-1} I_{10} ,$$

$$\|p_8(R)\|^2 = \frac{4}{p-1} I_{10} ,$$

$$\|p_9(R)\|^2 = 4I_{13} - \frac{4}{q-1} I_{14},$$

$$\|p_{10}(R)\|^2 = \frac{4}{q-1} I_{14},$$

$$\|p_{11}(R)\|^2 = 6I_{18} - 6I_{17},$$

$$\|p_{12}(R)\|^2 = 2I_{18} + 2I_{17} - \frac{4}{q} I_{20} - \frac{4}{p} I_{19} + \frac{4}{pq} I_{21},$$

$$\|p_{13}(R)\|^2 = \frac{4}{q} I_{20} - \frac{4}{pq} I_{21},$$

$$\|p_{14}(R)\|^2 = \frac{4}{q} I_{19} - \frac{4}{pq} I_{21},$$

$$\|p_{15}(R)\|^2 = \frac{4}{pq} I_{21}.$$

3. Conformally invariant projectors. It is a classical result that the space $\mathscr{R}(E)$ decomposes as a direct sum of three irreducible invariant subspaces under the action of $O(n)$, namely

$$\mathscr{R}(E) = \mathscr{W} \oplus \mathscr{R} \oplus \mathbb{R} \,.\, R_0,$$

where

$\mathscr{W} = \{R \in \mathscr{R}(E) \mid \varrho(R) = 0\}$,

$\mathscr{W} \oplus \mathscr{R} = \{R \in \mathscr{R}(E) \mid \tau(R) = 0\}$,

$\mathscr{R} = \mathscr{W}^{\perp}$ (orthogonal complement in $\mathscr{W} \oplus \mathscr{R}$), and $\mathbb{R} \,.\, R_0 = (\mathscr{W} \oplus \mathscr{R})^{\perp}$.

If p is the projector of $\mathscr{R}(E)$ onto $\mathscr{W}$, then p is a conformal invariant, in the sense that if $(\mathscr{M}, g)$ is a Riemannian manifold, R its curvature tensor, g' a Riemannian metric in $\mathscr{M}$, conformally related with g, and R' the corresponding curvature tensor, then

$$p(R) = p(R'),$$

up to multiplication by a scalar, due to the contraction with g. In this context, we have

Theorem 6. *The projectors $p_1, p_4, p_7, p_9, p_{11}$ and p_{12} are conformally invariant.*

Proof. Let $(\mathscr{M}, g)$ be a Riemannian manifold and let g' be a Riemannian metric on $\mathscr{M}$ such that $g' = e^{2f} g$, where f is a real function on $\mathscr{M}$. Then, the curvature tensor R' of g' is related with that of g, R, by the formula ([4])

$$(1) \qquad \begin{aligned} R'(L, M, N, W) = e^{2f}(R(L, M, N, W) + \lambda(M, N)\, g(L, W) - \\ - \lambda(M, W)\, g(L, N) - \lambda(L, N)\, g(M, W) + \lambda(L, W)\, g(M, N) + \\ + \|\omega\|^2 (g(L, W)\, g(M, N) - g(L, N)\, g(M, W)) \end{aligned}$$

for all $L, M, N, W \in \mathscr{X}(\mathscr{M})$, where

$$\lambda(M, N) = (\nabla_M \omega)\, N - \omega(M)\, \omega(N)$$

and $\omega = \mathrm{d}f$. It can be easily seen that λ is symmetric.

From (1) we get

$$(2) \qquad \varrho_V(R')\,(A, B) = \varrho_V(R)\,(A, B) - (p-2)\,\lambda(A, B) - \\ - g(A, B) \sum_{a=1}^{p} \lambda(E_a, E_a) - (p-1)\,\|\omega\|^2\, g(A, B)$$

for all $A, B \in V$, $\{E_a\}_{a=1,\cdots,p}$ being an orthonormal basis of V, and from (2) we have

$$(3) \qquad e^{2f}\, \tau_V(R') = \tau_V(R) - 2(p-1) \sum_{a=1}^{p} \lambda(E_a, E_a) - p(p-1)\,\|\omega\|^2 .$$

(2) and (3) yield

$$(4) \qquad \frac{\tau_V(R')\, g'(A, B)}{2(p-1)\,(p-2)} - \frac{1}{p-2}\, \varrho_V(R')\,(A, B) =$$

$$= \frac{\tau_V(R)\, g(A, B)}{2(p-1)\,(p-2)} - \frac{1}{p-2}\, \varrho_V(R)\,(A, B) + \tfrac{1}{2}\|\omega\|^2\, g(A, B) + \lambda(A, B) .$$

Now, (1) and (4) imply that

$$R'(A, B, C, D) + \frac{1}{p-2}\,(\varrho_V(R')\,(B, C)\, g'(A, D) - \varrho_V(R')\,(B, D)\, g'(A, C) - \\ - \varrho_V(R')\,(A, C)\, g'(B, D) + \varrho_V(R')\,(A, D)\, g'(B, C) - \\ - \frac{\tau_V(R')}{(p-1)\,(p-2)}\,(g'(B, C)\, g'(A, D) - g'(B, D)\, g'(A, C)) =$$

$$= e^{2f}(R(A, B, C, D) + \frac{1}{p-2}\,(\varrho_V(R)\,(B, C)\, g(A, D) - \varrho_V(R)\,(B, D)\, g(A, C) - \\ - \varrho_V(R)\,(A, C)\, g(B, D) + \varrho_V(R)\,(A, D)\, g(B, C) - \\ - \frac{\tau_V(R)}{(p-1)\,(p-2)}\,(g(B, C)\, g(A, D) - g(B, D)\, g(A, C)) .$$

The case of p_4 is similar.

As for p_7, we consider (1) for the arguments $A, B, C \in V$ and $X \in H$:

$$(5) \qquad R'(A, B, C, X) = e^{2f}(R(A, B, C, X) - \lambda(B, X)\, g(A, C) + \lambda(A, X)\, g(B, C) .$$

Then

$$\varrho_V(R') = \varrho_V(R)\,(A, X) - (p-1)\,\lambda(A, X)$$

whence

$$\lambda(A, X) = \frac{1}{p-1} (\varrho_V(R)(A, X) - \varrho_V(R')(A, X)),$$

and substituting $\lambda(A, X)$ and $\lambda(B, X)$ in (5) we get the result for p_7. That of p_9 is similar.

Next, let us consider (1) for the arguments A, X, B, Y and B, X, A, Y, with $A, B \in V$ and $X, Y \in H$. By subtraction we get

$$(6) \qquad R'(A, X, B, Y) - R'(B, X, A, Y) = e^{2f}(R(A, X, B, Y) - R(B, X, A, Y)),$$

which gives the result for p_{11}.

Finally, from (1) we get, for $A, B \in V$ and $X, Y \in H$,

$$(7) \qquad R'(A, X, B, Y) = e^{2f}(R(A, X, B, Y) - \lambda(X, Y)\, g(A, B) - \\ - \lambda(A, B)\, g(X, Y) - \|\omega\|^2\, g(A, B)\, g(X, Y).$$

Hence,

$$(8) \qquad \varrho_V(R')(X, Y) = \varrho_V(R)(X, Y) - p\lambda(X, Y) - \\ - g(X, Y) \sum_{a=1}^{p} \lambda(E_a, E_a) - p\|\omega\|^2\, g(X, Y),$$

where $\{E_a\}_{a=1,\dots,p}$ is an orthonormal basis of V.

Similarly,

$$(9) \qquad \varrho_H(R')(A, B) = \varrho_H(R)(A, B) - g(A, B) \sum_{u=p+1}^{n} \lambda(E_u, E_u) - \\ - q\lambda(A, B) - q\|\omega\|^2\, g(A, B).$$

From (6) or (7) we get

$$(10) \qquad e^{2f}\, \tau_{VH}(R') = \tau_{VH}(R) - p \sum_{u=p+1}^{n} \lambda(E_u, E_u) - q \sum_{a=1}^{p} \lambda(E_a, E_a) - pq\|\omega\|^2.$$

Now, from (7), (8), (9) and (10),

$$(11) \quad R'(A, X, B, Y) - \frac{1}{p} \varrho_V(R')(X, Y)\, g'(A, B) - \frac{1}{q} \varrho_H(R')(A, B)\, g'(X, Y) + \\ + \frac{1}{pq} \tau_{VH}(R')\, g'(A, B)\, g'(X, Y) = \\ = e^{2f}(R(A, X, B, Y) - \frac{1}{p} \varrho_V(R)(X, Y)\, g(A, B) - \frac{1}{q} \varrho_H(R)(A, B)\, g(X, Y) + \\ + \frac{1}{pq} \tau_{VH}(R)\, g(A, B)\, g(X, Y).$$

The result for p_{12} follows from (11) and (6).

We also have

Theorem 7.

$$e^{2f}\left(\frac{\tau_V(R')}{p(p-1)}+\frac{\tau_H(R')}{q(q-1)}-2\,\frac{\tau_{VH}(R')}{pq}\right)=$$

$$=\frac{\tau_V(R)}{p(p-1)}+\frac{\tau_H(R)}{q(q-1)}-2\,\frac{\tau_{VH}(R)}{pq}.$$

The proof is straightforward from (10), (3) and the analogue of (3) for τ_H.

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