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EDGE NEIGHBOURHOOD GRAPHS

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At the Symposium on Graph Theory in Smolenice [1] in 1963 A. A. Zykov proposed a problem concerning the neighbourhood graph of vertices of undirected graphs. This was a hint for many authors to study local properties of graphs. A survey of results concerning this topic is [2]. Here we shall study a problem analogous to that of Zykov, but concerning edge neighbourhood graphs.

Let G be an undirected graph, let e be an edge of G. By the symbol $N_G(e)$ we denote the subgraph of G induced by the set of all vertices of G which are not incident to e and are adjacent to at least one end vertex of e. The graph $N_G(e)$ will be called the edge neighbourhood graph of e in G.

The edge neighbourhood version of the problem of Zykov is the following:

Characterize the graphs H with the property that there exists a graph G such that $N_G(e) \cong H$ for each edge e of G.

We shall not solve this problem completely, but only study some special cases. The class of all graphs with the above property will be denoted by $\mathcal{N}_e$.

First we shall present some simple propositions.

Proposition 1. *A complete graph K_n belongs to $\mathcal{N}_e$ for any positive integer n.*

Proof. The graph K_{n+2} has the property that $N_{K_{n+2}}(e) \cong K_n$ for each edge e of K_{n+2}.

Proposition 2. *A complete bipartite graph $K_{m,n}$ belongs to N_e for any positive integers m, n.*

Proof. The graph $K_{m+1,n+1}$ has the property that $N_{K_{m+1,n+1}}(e) \cong K_{m,n}$ for each edge e of $K_{m+1,n+1}$.

The symbol C_n will denote a circuit of the length n, i.e., with n edges .

Proposition 3. *The circuits C_3, C_4, C_6, C_8 belong to $\mathcal{N}_e$.*

Proof. The assertion for C_3 follows from Proposition 1, because $C_3 \cong K_3$. The assertion for C_4 follows from Proposition 2, because $C_4 \cong K_{2,2}$. For C_6 the required graph is the graph of the regular icosahedron and for C_8 it is the graph of the covering of the plane by regular triangles.

Now we prove a theorem.

Theorem 1. *There exists no graph G with the property that $N_G(e) \cong C_5$ for each edge e of G.*

Proof. Suppose that a graph G with the required property exists. Let e be an edge of G, let u_1, u_2 be its end vertices. According to the assumption $N_G(e) \cong C_5$, thus it is a circuit of the length 5. Let the vertices of $N_G(e)$ be v_1, v_2, v_3, v_4, v_5 and the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$. If u_1 is adjacent to none of the vertices v_1, v_2, v_3, v_4, v_5, then it has the degree 1 in G; otherwise $N_G(e)$ would contain a vertex not belonging to the set $\{v_1, v_2, v_3, v_4, v_5\}$. The edge u_2v_1 exists and u_1 is an isolated vertex of $N_G(u_2v_1)$, thus $N_G(u_2v_1)$ is not isomorphic to C_5, which is a contradiction. Hence u_1 is adjacent to at least one of the vertices v_1, v_2, v_3, v_4, v_5 and, obviously, so is u_2. Now among the vertices v_1, v_2, v_3, v_4, v_5 there exists a pair of adjacent ones with the property that one of them is adjacent to u_1 and the other to u_2. Without loss of generality let v_1 be adjacent to u_1 and v_2 to u_2. Now we shall investigate which of the edges $u_1v_3, u_1v_5, u_2v_3, u_2v_5$ may exist simultaneously in G.

If v_3 is adjacent to both u_1, u_2, then $N_G(v_1v_2)$ contains a triangle with the vertices u_1, u_2, v_3 and is not isomorphic to C_5, which is a contradiction. Analogously if v_5 is adjacent to both u_1, u_2.

Suppose that both v_3, v_5 are adjacent to u_1. The vertex v_4 must be also adjacent to at least one of the vertices u_1, u_2. If it is adjacent to u_1, then $N_G(v_2v_3)$ contains a star with the centre u_1 and terminal vertices v_1, v_4, u_2 and is not isomorphic to C_5. If v_4 is adjacent to u_2, then $N_G(u_1v_5)$ contains a path of the length 2 with the vertices u_2, v_4, v_3 and a vertex v_1. As $N_G(u_1v_5)$ has to be isomorphic to C_5, the vertex v_1 must be adjacent to one of the vertices u_2, v_3. It cannot be adjacent to v_3, because then $N_G(e)$ would not be a circuit (it would contain a chord of C_5). Therefore v_1 is adjacent to u_2. But then $N_G(v_1v_5)$ contains a star with the centre u_2 and terminal vertices u_1, v_2, v_4, which is again a contradiction. Hence u_1 cannot be adjacent to both v_3, v_5 and analogously, neither can u_2.

Suppose that u_1 is adjacent to v_5 and u_2 is adjacent to v_3. The vertex v_4 is adjacent to one of the vertices u_1, u_2; without loss of generality let it be adjacent to u_1. The graph $N_G(u_1v_4)$ contains the edges u_2v_3 and v_1v_5; as it has to be isomorphic to C_5, one of the vertices u_2, v_3 must be adjacent to one of the vertices v_1, v_5. The vertex v_3 cannot be adjacent to any of them, because then $N_G(e)$ would not be isomorphic to C_5.

If u_2 is adjacent to v_5, then $N_G(v_1v_2)$ contains a triangle with the vertices u_1, u_2, v_5. Thus u_2 is adjacent to v_1. The graph $N_G(u_2v_1)$ contains the edges u_1v_5 and v_2v_3. Again one of the vertices u_1, v_5 must be adjacent to one of vertices v_2, v_3 and this is possible only in such a way that u_1 is adjacent to v_2. But then $N_G(v_1v_5)$ contains a triangle with the vertices u_1, u_2, v_2.

Thus the last case remains, when u_1 is adjacent to v_3 and u_2 to v_5. Again without loss of generality let v_4 be adjacent to u_1. Then $N_G(u_1v_4)$ contains a path of the length

2 with the vertices u_2, v_5, v_1 and the vertex v_3. Then v_3 must be adjacent to u_2 or v_1. It cannot be adjacent to v_1, because then $N_G(e)$ would not be isomorphic to C_5, and thus v_3 is adjacent to u_2. But then $N_G(v_1v_2)$ contains a triangle with the vertices u_1, u_2, v_3, which is a contradiction. All cases are exhausted and thus the assertion is proved.

Now we turn our attention to complements of circuits. The complement of a circuit of the length n will be denoted by $\bar{C}_n$.

Theorem 2. *A graph $\bar{C}_n$ belongs to $\mathcal{N}_e$ if and only if $n = 3$ or $n = 4$.*

Proof. For $n = 3$ the graph $\bar{C}_n$ consists of three isolated vertices. Take a regular graph of degree 3 without triangles and insert a vertex onto each of its edges (i.e., replace each edge by a path of the length 2). The graph G thus obtained has the property that $N_G(e) \cong \bar{C}_3$ for each edge e of G and thus $\bar{C}_3 \in \mathcal{N}_e$. For $n = 4$ the graph $\bar{C}_n$ consists of two connected components, each of which is a complete graph with two vertices. If G is the graph of the 3-dimensional cube, then $N_G(e) \cong \bar{C}_4$ for each edge e of G and thus $\bar{C}_4 \in \mathcal{N}_e$. For $n = 5$ we have $\bar{C}_5 \cong C_5$ and according to Theorem 1 the graph $\bar{C}_5 \notin \mathcal{N}_e$. Now let $n \geqq 6$ and suppose that there exists a graph G such that $N_G(e) \cong \bar{C}_n$ for each edge e of G. Let e be an edge of G, let u_1, u_2 be its end vertices. According to the assumption, $N_G(e) \cong \bar{C}_n$; let the vertices of $N_G(e)$ be $v_1, \ldots, v_n$ and let the edges of its complement be v_iv_{i+1} for $i = 1, \ldots, n-1$ and v_nv_1. Each of the vertices $v_1, \ldots, v_n$ is adjacent to at least one of the vertices u_1, u_2. Without loss of generality suppose that v_1 is adjacent to u_1. The graph $N_G(u_1v_1)$ contains the vertices $u_2, v_3, \ldots, v_{n-1}$. If some vertex v_i for $4 \leqq i \leqq n-2$ is not adjacent to u_2, then the complement of $N_G(u_1v_1)$ contains a star with the centre v_i and with the terminal vertices u_2, v_{i-1}, v_{i+1}; this is a contradiction with the assumption that the complement of $N_G(u_1v_1)$ is a circuit. Hence all the vertices $v_4, \ldots, v_{n-2}$ are adjacent to u_2.

Now we shall distinguish the cases $n = 6$ and $n \geqq 7$. We begin with the case $n \geqq 7$ which is simpler. In this case we continue doing the same consideration for other vertices than v_1. As v_{n-2} is adjacent to u_2, we prove that $v_1, \ldots, v_{n-5}$ are adjacent to u_1. Then we proceed in the same way with u_1 and v_{n-5}; we continue until we obtain the result that each of the vertices $v_1, \ldots, v_n$ is adjacent to both u_1 and u_2. Now consider $N_G(v_1v_4)$; this graph contains the vertices u_1, u_2 and all the vertices v_i for $i \neq 1$ and $i \neq 4$. As $N_G(v_1v_4)$ has to be isomorphic to $\bar{C}_n$, it cannot contain other vertices than those just mentioned. But then in the complement of $N_G(v_1v_4)$ the vertices u_1, u_2 are isolated, which is a contradiction. Hence $\bar{C}_n \notin \mathcal{N}_e$ for $n \geqq 7$.

Now the case $n = 6$ remains. The consideration at the beginning of the proof implies that if v_1 is adjacent to u_1, then v_4 is adjacent to u_2. In general, if a vertex v_i is adjacent to u_1, then v_{i+3} (the subscript $i + 3$ being taken modulo 6) is adjacent to u_2. Consider the graph $N_G(v_1v_4)$; it contains the vertices $u_1, u_2, v_2, v_3, v_5, v_6$ and, as it has to be isomorphic to $\bar{C}_6$, it contains no other vertices than those. The graph

$N_G(v_1v_4)$ can be isomorphic to $\bar{C}_6$ only if exactly one of the vertices v_2, v_3 is non-adjacent to u_1 and exactly one of them is non-adjacent to u_2, and if the same holds for the vertices v_5, v_6. Therefore none of the vertices v_2, v_3, v_5, v_6 can be adjacent to both u_1, u_2; analogously this can be proved also for v_1 and v_4. Thus we may distinguish two cases: either v_1, v_2, v_6 are adjacent to u_1 and v_3, v_4, v_5 to u_2, or v_1, v_3, v_5 are adjacent to u_1 and v_2, v_4, v_6 to u_2; any other case can be transferred to one of them by an isomorphism. In the first case consider the graph $N_G(u_1v_6)$; it contains the vertices u_2, v_1, v_2, v_3, v_4 and thus its complement contains the star with the centre v_2 and the terminal vertices u_2, v_1, v_3; this is a contradiction with the assumption that this complement is a circuit. In the second case consider the graph $N_G(u_1v_1)$; its complement contains a circuit of the length 4 with the vertices u_2, v_3, v_4, v_5 and cannot be a circuit of the length 6, which is a contradiction. Hence also $\bar{C}_6 \notin \mathcal{N}_e$.

Remark. The assertion of Theorem 1 is in fact part of the assertion of Theorem 2. But in spite of it, we present Theorem 1 separately, because $\bar{C}_5$ is not only a complement of a circuit, but also a circuit, and the proof of this case is very different from the proof for $n \geqq 6$.

In the end we shall add propositions concerning a certain special class of graphs.

By the symbol $K^*_{n,n}$ we shall denote the graph obtained from the complete bipartite graph $K_{n,n}$ by deleting edges of the maximal matching. The graph $K^*_{n,n}$ is the complement of the graph $K_2 \times K_n$.

Proposition 4. *The graph $K^*_{n,n}$ belongs to $\mathcal{N}_e$ to for any positive integer n.*

Proof. The graph $K^*_{n+2,n+2}$ has the property that $N_{K^*_{n+2,n+2}}(e) \cong K^*_{n,n}$ for any positive integer n and an each edge e of $K^*_{n+2,n+2}$.

Now we can state a proposition concerning the graphs of cubes of dimensions 1, 2 and 3. If Q_n denotes the graph of the cube of the dimension n, then $Q_1 \cong K_2$, $Q_2 \cong K_{2,2}$, $Q_3 \cong K^*_{4,4}$ and hence Propositions 1, 2, 4 yield the following proposition.

Proposition 5. *The graphs of the cubes of dimensions* 1, 2 *and* 3 *belong to* $\mathcal{N}_e$.

References

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