Czechoslovak Mathematical Journal

Hugo H. Torriani

Profinite completions of the fundamental group of the Klein bottle

Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 4, 511–514

Persistent URL: http://dml.cz/dmlcz/102044

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of the Czechoslovak Academy of Sciences
V. *35* (*110*), PRAHA 31. 12. 1985, No 4

PROFINITE COMPLETIONS OF THE FUNDAMENTAL GROUP OF THE KLEIN BOTTLE

HUGO H. TORRIANI, Campinas

(Received May 17, 1983)

Let K be the fundamental group of the Klein bottle. In § 1 we compactify K by computing its profinite completion $\hat{K}$. It turns out that $\hat{K}$ is topologically isomorphic to a semidirect product of two copies of the profinite completion of the additive group of rational integers.

The analysis of the structure of $\hat{K}$ is begun in § 2. Our first step is to find all the normal subgroups of K of odd index. We then determine the p-profinite completions of K for each prime number p. In the last part of this section we obtain a decomposition of $\hat{K}$ in its p-Sylow subgroups, which are expressed in terms of those completions.

Finally, in § 3 we establish several finiteness, centrality and radical properties of $\hat{K}$.

The author wishes to thank K. O. Stöhr for several helpful and stimulating conversations.

1. The profinite completion. We recall that K has a presentation given by two generators a, b and the relation $abab^{-1} = 1$. If Z is the additive group of rational integers and σ the homomorphism of Z onto $\mathrm{Aut}(Z)$ that maps the integer 1 onto the automorphism -1 of Z, K is isomorphic to the semidirect product $Z \times_\sigma Z$.

Now let G be any group and Λ the family of all normal subgroups of G of finite index. Define a partial order $\leqq$ on Λ by setting $S \leqq T$ if and only if $T \subseteq S$ for S and T in Λ. For each such pair let ϕ_T^S be the canonical epimorphism of G/T onto G/S. Then $(\Lambda, (G/S), (\phi_T^S))$ is a projective family of groups. The projective limit $\varprojlim (G/S)$ of this family is called the *profinite completion* of G and is denoted by $\hat{G}$ or by $G\hat{}$.

If $G = Z$ we can take Λ to be the set of positive integers with the partial order $\leqq$ defined by $m \leqq n$ if and only if m divides n; and for any pair m, n of elements of Λ such that m divides n we let ϕ_n^m be the canonical epimorphism of Z/nZ onto Z/mZ. Then $\mathscr{F} = (\Lambda, (Z/mZ), (\phi_n^m))$ is a projective family and we have $\hat{Z} = \varprojlim (Z/mZ)$.

Let r be a fixed positive integer. Since the family $(\Lambda, (Z/rmZ), (\phi_{rn}^{rm}))$ is cofinal in $\mathscr{F}$ we also have $\hat{Z} = \varprojlim (Z/rmZ)$. Let $r\hat{Z}$ be the projective limit of the family $(\Lambda, (rZ/rmZ), (\phi_{rn}^{rm}))$, where we use the same symbol for the restriction of ϕ_{rn}^{rm} to

rZ/rnZ. It is clear that $r\hat{Z}$ is a subgroup of $\hat{Z}$, and as projective limits commute with the operation of passing to quotient groups we have an isomorphism $\hat{Z}/r\hat{Z} \cong Z/rZ$. (Here and in the sequel isomorphisms are topological.) Thus the subgroup $r\hat{Z}$ is open (and therefore closed) in $\hat{Z}$, and if we consider $\hat{Z}$ with its natural structure of commutative ring, $r\hat{Z}$ is the (open and closed) ideal in $\hat{Z}$ generated by r.

Now let $1^\wedge$ be the identity automorphism of $\hat{Z}$ and $\hat{\sigma}$ the homomorphism of $\hat{Z}$ into Aut $(\hat{Z})$ defined for all $\eta \in \hat{Z}$ by $\hat{\sigma}(\eta) = 1^\wedge$ if $\eta \in 2\hat{Z}$ and $\hat{\sigma}(\eta) = -1^\wedge$ if $\eta \notin 2\hat{Z}$. Let $\hat{Z} \times_{\hat{\sigma}} \hat{Z}$ be the semidirect product of two copies of $\hat{Z}$ with respect to $\hat{\sigma}$.

Theorem 1. *The groups $\hat{K}$ and $\hat{Z} \times_{\hat{\sigma}} \hat{Z}$ are isomorphic.*

Proof. Since any normal subgroup of K of finite index contains $2nZ \times_\sigma 2nZ$ for some integer n, we have an isomorphism

$$(Z \times_\sigma Z)^\wedge \cong \varprojlim ((Z \times_\sigma Z)/(2nZ \times_\sigma 2nZ)) .$$

Now let 1_{2n} be the identity automorphism of $Z/2nZ$ and τ_{2n} the homomorphism of $Z/2nZ$ into Aut $(Z/2nZ)$ defined for all $\bar{x} \in Z/2nZ$ by $\tau_{2n}(\bar{x}) = 1_{2n}$ if $\bar{x} \in 2Z/2nZ$ and $\tau_{2n}(\bar{x}) = -1_{2n}$ if $\bar{x} \notin 2Z/2nZ$. The canonical mapping of $Z \times_\sigma Z$ onto $(Z/2nZ) \times_{\tau_{2n}} (Z/2nZ)$ induces an isomorphism of $(Z \times_\sigma Z)/(2nZ \times_\sigma 2nZ)$ onto $(Z/2nZ) \times_{\tau_{2n}} (Z/2nZ)$ for all n which is compatible with all the connecting epimorphisms. Therefore, we have an isomorphism

$$\varprojlim ((Z \times_\sigma Z)/(2nZ \times_\sigma 2nZ)) \cong \varprojlim ((Z/2nZ) \times_{\tau_{2n}} (Z/2nZ)) .$$

As a topological space, the right-hand side is the cartesian product $\hat{Z} \times \hat{Z}$. The naturally induced composition law on this space is continuous, as is the operation of taking inverses, and we obtain an isomorphism

$$\varprojlim ((Z/2nZ) \times_{\tau_{2n}} (Z/2nZ)) \cong \hat{Z} \times_{\hat{\sigma}} \hat{Z} .$$

2. The p-profinite completions. Let G be any group, p a fixed prime number and Λ_p the family of all normal subgroups of G of index a finite power of p. Define a partial order on Λ_p and connecting epimorphisms ϕ_T^S just as in § 1. Then $(\Lambda_p, (G/S), (\phi_T^S))$ is a projective family of groups. The projective limit $\varprojlim (G/S)$ of this family is called the *p-profinite completion* of G and is denoted here by G_p.

If $G = Z$ we can take Λ_p to be the set of positive integers of the form p^n with the usual partial order; and for any pair p^m, p^n of elements of Λ_p such that $m \leqq n$ we let ψ_n^m be the canonical epimorphism of Z/p^nZ onto Z/p^mZ. Then $(\Lambda_p, (Z/p^mZ), (\psi_n^m))$ is a projective family and we have $Z_p = \varprojlim (Z/p^mZ)$. (This is the group of p-adic integers.)

The following result will allow us to compute the p-profinite completion of K when p is an odd prime.

Lemma. *If H is a normal subgroup of K of finite odd index i, then $H = Z \times_\sigma iZ$.*

Proof. By hypothesis H contains an element $a^m b^n$ with n odd and positive. Conjugation by a then shows that $a^2 \in H$. Now the non-empty set $S = \{a^p b^q \in H : p$ odd,

$q \geqq 0\}$ may be preordered by $a^p b^q \leqq a^r b^s$ if and only if $q \leqq s$, and if $a^p b^{q_0}$ is a minimal element we obtain $ab^{q_0} \in H$. Consideration of the images of a and b in K/H shows that $a \in H$ and $q_0 = i$, hence H is the subgroup of K generated by a and b^i.

Remark. In [1], Exer. 6.5(iv), p. 141, it is asserted that the subgroup $K_{i,0,1}$ (which coincides with our $Z \times_o iZ$) cannot be normal.

Now let 1_2 be the identity automorphism of Z_2 and σ_2 the homomorphism of Z_2 into $\mathrm{Aut}(Z_2)$ defined for all $\eta_2 \in Z_2$ by $\sigma_2(\eta_2) = 1_2$ if $\eta_2 \in 2Z_2$ and $\sigma_2(\eta_2) = -1_2$ if $\eta_2 \notin 2Z_2$. Let $Z_2 \times_{\sigma_2} Z_2$ be the semidirect product of two copies of Z_2 with respect to σ_2.

Theorem 2. *The 2-profinite completion K_2 of K is isomorphic to $Z_2 \times_{\sigma_2} Z_2$. For any odd prime p the p-profinite completion K_p of K is isomorphic to Z_p.*

Proof. The first assertion is proved by making simple modifications in the proof of Theorem 1; and the second assertion follows readily from the lemma above.

Remark. Since finite groups are required to be Hausdorff, the p-profinite completion of a topological (but not necessarily Hausdorff) group G must coincide with that of the Hausdorff group associated to G. A non-abelian group may therefore turn out to have an abelian p-profinite completion. A concrete example of that situation is given by the second assertion of Theorem 2.

Let P^* be the set of odd primes and for each $p \in P^*$ let 1_p be the identity automorphism of Z_p. Let υ be the homomorphism of $Z_2 \times_{\sigma_2} Z_2$ into $\mathrm{Aut}\,(\prod_{p \in P^*} (Z_p \times Z_p))$ defined for all $(\xi_2, \eta_2) \in Z_2 \times_{\sigma_2} Z_2$ by $\upsilon(\xi_2, \eta_2) = (1_p, 1_p)_{p \in P^*}$ if $\eta_2 \in 2Z_2$ and $\upsilon(\xi_2, \eta_2) = (-1_p, 1_p)_{p \in P^*}$ if $\eta_2 \notin 2Z_2$. Our next result gives a Sylow decomposition of the profinite group $\hat{K}$.

Theorem 3. *There exists an isomorphism*

$$\hat{K} \cong \left(\prod_{p \in P^*} (Z_p \times Z_p)\right) \times_\upsilon (Z_2 \times_{\sigma_2} Z_2)\,.$$

Moreover, $Z_2 \times_{\sigma_2} Z_2$ is a 2-Sylow subgroup of $\hat{K}$ and $Z_p \times Z_p$ is the unique p-Sylow subgroup of $\hat{K}$ for each $p \in P^*$.

Proof. Clearly there is an isomorphism of $\hat{K}$ onto the group on the right-hand side. As $Z_2 \times_{\sigma_2} Z_2$ is a 2-group, there exists a 2-Sylow subgroup S of $\hat{K}$ containing it (cf. [3], Theor. 4(2), p. 13), and it follows that $S = Z_2 \times_{\sigma_2} Z_2$. The same argument shows that $Z_p \times Z_p$ is a p-Sylow subgroup of $\hat{K}$ for each $p \in P^*$, which is normal in $\hat{K}$.

3. Further properties of the profinite completion. It is clear that $\hat{K}$ is locally infinite, non-noetherian and residually finite. If $|X|$ denotes the cardinality of a set X and G is a given finite group we have $|\mathrm{Hom}\,(\hat{K}, G)| \leqq |G|^2$, so that $\hat{K}$ is hopfian (but not co-hopfian).

Some centrality properties of K have counterparts in $\hat{K}$. For example, for each integer $i \geqq 2$ the i-th term of the lower central series of $\hat{K}$ is $2^{i-1}\hat{Z} \times_{\hat{\sigma}} \{0\}$. In particular, the commutator subgroup of $\hat{K}$ is isomorphic to $\hat{Z}$, and the abelianized group of $\hat{K}$ is isomorphic to $(Z/2Z) \oplus \hat{Z}$. On the other hand, although K is residually nilpotent, $\hat{K}$ is not. Since $\hat{K}$ is residually central (but not Baer-nilpotent) we have a description of the position of this solvable group among generalized nilpotent groups. (Cf. [2], Part 2, p. 13.)

We shall now state some other centrality properties of $\hat{K}$ that carry over from those of K. (Cf. [4].) For each $(\xi, \eta) \in \hat{K}$ let $Z(\xi, \eta)$ be its centralizer in $\hat{K}$. Let $\hat{K}_0 = \hat{Z} \times_{\hat{\theta}} 2\hat{Z}$, and for each $\xi \in \hat{Z}$ let A_ξ be the subgroup of $\hat{K}$ generated by $\{(\xi, \omega): \omega \in \hat{Z}, \omega \notin 2\hat{Z}\}$. Obviously, $\hat{K}_0$ and A_ξ are respectively isomorphic to $\hat{Z} \oplus \hat{Z}$ and $\hat{Z}$. If $\eta \in 2\hat{Z}$ and $\xi = 0$, then $Z(\xi, \eta) = \hat{K}$; if $\eta \in 2\hat{Z}$ and $\xi \neq 0$, then $Z(\xi, \eta) = \hat{K}_0$; and if $\eta \notin 2\hat{Z}$, then $Z(\xi, \eta) = A_\xi$. The center of $\hat{K}$ equals $\{0\} \times_{\hat{\theta}} 2\hat{Z}$.

As regards radical subgroups, the Frattini subgroup of $\hat{K}$ is evidently abelian. Since any nilpotent subgroup of $\hat{K}$ is abelian, the Fitting subgroup and the Hirsch-Plotkin radical of $\hat{K}$ are equal to $\hat{K}_0$. This group is the unique maximal abelian normal subgroup of $\hat{K}$. In addition, there exists a continuous family of maximal abelian subgroups of $\hat{K}$, namely the subgroups A_ξ ($\xi \in \hat{Z}$) introduced above.

References

[1] *C. Godbillon:* Éléments de Topologie Algébrique, Hermann, Paris, 1971.

[2] *D. J. S. Robinson:* Finiteness Conditions and Generalized Soluble Groups, Parts 1—2, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vols. 62—63, Springer-Verlag, Berlin—Heidelberg—New York, 1972.

[3] *S. S. Shatz:* Profinite Groups, Arithmetic, and Geometry, Annals of Mathematics Studies, No. 67, Princeton Univ. Press, Princeton, N.J., 1972.

[4] *H. H. Torriani:* Subgroups of the Klein bottle group, submitted.

Author's address: IMECC, UNICAMP, Caixa Postal 1170, 13100 Campinas (São Paulo), Brasil.