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PERIODIC SOLUTIONS TO A ONE-DIMENSIONAL STRONGLY NONLINEAR WAVE EQUATION WITH STRONG DISSIPATION

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1. Introduction. In this paper we shall investigate the problem given by the equation

$$(1.1)\qquad u_{tt}(t, x) - u_{xx}(t, x) + au_t(t, x) - bu_{txx}(t, x) + \\ + f_1(t, x, u(t, x), u_t(t, x)) + (f_2(t, x, u(t, x), u_t(t, x)))_x = \\ = g(t, x), \quad (t, x) \in \mathbb{R} \times (0, l),$$

by the homogeneous Dirichlet boundary conditions

$$(1.2)\qquad u(t, 0) = u(t, l) = 0, \quad t \in \mathbb{R}$$

and by the condition of ω-periodicity with respect to the time variable t

$$(1.3)\qquad u(t + \omega, x) - u(t, x) = 0, \quad (t, x) \in \mathbb{R} \times (0, 1).$$

Here $a \geqq 0$, $b > 0$ are given constants and g, f_1 and f_2 given functions ω-periodic in t.

In the sequel we give some sufficient conditions on f_1 and f_2 to ensure the existence of a solution to the problem (1.1)–(1.3). The presence of the term $-bu_{txx}$ is essential. It is sometimes called the internal friction (cf. [21]) or (following Ebihara) the strong dissipation.

The corresponding IBV problem (if not stated otherwise we always have in mind the homogeneous Dirichlet boundary conditions) has been studied by many authors under various restrictions on the elliptic part and on the nonlinearities f_i.

Greenberg, MacCamy, and Mizel [16] studied the equation (1.1) with $a = 0$, $g = 0, f_i = 0$ and u_{xx} replaced by a more general term $(\sigma(u_x))_x$, where σ is a smooth increasing function, $\sigma(0) = 0$. Their third-order equation

$$(1.4)\qquad u_{tt}(t, x) - (\sigma(u_x(t, x)))_x - bu_{txx}(t, x) = 0$$

governs the purely longitudinal motion of a homogeneous thin bar which, in its original stress free state, is of uniform cross-section and length l. If x is the position of a cross-section in the original configuration and $u(t, x)$ and $\tau(t, x)$ are respectively the displacement of the section from its rest position and the stress on the section at time t, the equation of motion is

$$\varrho_0 u_{tt} = \tau_x\,, \tag{1.5}$$

where ϱ_0 is the constant density of the bar in the rest position. The equation studied in [16] is obtained from (1.5) by introducing a specific constitutive assumption on the dependence of the stress τ on the strain (deformation gradient) u_x and the time rate of change u_{xt} of strain, namely

$$\tau = \sigma(u_x) + b u_{xt}\,, \tag{1.6}$$

and setting $\varrho_0 = 1$. (For more general nonlinear stress-strain relation $\tau = \sigma(u_x, u_{xt})$ see Dafermos [7], Ebihara [10], [11], or Prestel [32].) The constitutive equation (1.6) corresponds to the simplest possible model of a visco-elastic material whose stress depends on the history of the motion. The visco-elastic term bu_{xt} reflects the past history of the strain u_x. The positive constant b is interpreted as a viscosity coefficient. These imperfectly elastic materials fall in the category of the so called materials with internal friction (see [21]).

If the effect of viscosity is not introduced, (1.6) contains only the elastic part $\sigma(u_x)$ of the stress (nonlinear Hooke's law), we get the nonlinear wave equation

$$u_{tt} = (\sigma(u_x))_x \tag{1.7}$$

and it is well-known (see e.g. [29]) that under various hypotheses on σ and for smooth initial data the equation (1.7) does not admit, in general, global smooth solutions since some second derivatives become infinite in a finite time, the solutions contain shocks. In [16] it is shown that if the stress depends on the past history of motion in an appropriate way these shocks cannot occur. The authors in [16] (followed by [14], [15]) prove the existence of a unique classical global stable solution under the condition that the initial data u_0 and u_1 belong to $C^4([0, l])$ and $C^2([0, l])$, respectively. The proof is based on some special properties of the Green function for the heat equation.

Initiated by [16] a series of similar results has appeared and generalizations in various directions have been given. The aim of Yamada [39] is to weaken the assumptions of [16] and give a simplified proof by means of the semigroup theory.

Several authors deal with the n-dimensional equation corresponding to (1.4),

$$u_{tt} - \sum_{i=1}^{n} (\sigma_i(u_{x_i}))_{x_i} - b\,\Delta u_t = g \tag{1.8}$$

with σ_i rather general or, on the other hand, quite special functions. Tsutsumi [36] applies his result for the abstract equation $u'' - Au + Bu' = g$ to the IBV problem for (1.8) with $\sigma_i(\xi) = \xi^{2p-1}$, $p > 1$. Yamada [38] treats the same abstract equation with $B = B(t)$ and generalizes Tsutsumi's results. In particular, he does not require the homogeneity of A and consequently his result can be applied to (1.8) with only "polynomial-like" functions σ_i.

Using compactness as well as monotonicity arguments Clements [5] shows the existence of a solution of the IBV problem and of a periodic solution under the monotonicity and coercivity conditions on functions $\sigma_i = \sigma_i(t, x, u_{x_i})$ which are allowed to be of polynomial growth in u_{x_i}. In all the three preceding papers weak solutions (solutions with a finite energy) are found. The existence of strong solutions to the IBV problem is established in Clements [6] provided σ_i are increasing functions in u_{x_i} of at most linear growth. The latter restriction makes it possible to use a mere compactness argument.

Caughey and Ellison [4] examine the equation (1.4) with $\sigma(\xi) = \xi$ and with an additional Lipschitz nonlinearity of the type $f(t, x, u, u_x, u_t, u_{xt}, u_{xx})$. Exploiting the Fourier series expansions, the Banach contraction principle and the Liapunov type stability techniques the authors are interested in the existence, uniqueness and stability of classical solutions. To get global solutions they assume small initial data and introduce the concept of the exponential Lipschitz function f. Similar questions concerning the strong solutions to the IBV problem for (1.8) with $\sigma_i(\xi) = \xi$ and $g = f(u)$ where $f'(u) \leqq c_0$ for all $u \in \mathbb{R}$ ($c_0 \geqq 0$), $\limsup_{|u| \to \infty} (f(u)/u) \leqq 0$ and $f(0) = 0$ are taken up in Webb [37] by means of some results on analytic semigroups. The stability analysis rests upon the Liapunov functional approach combined with some topological dynamics. The semigroup theory is also applied in Fitzgibbon [13] to the study of the abstract problem $u'' - Au - bAu' = F(t, u, u')$ where the linear operator A is supposed to be the infinitesimal generator of an analytic semigroup. Local solutions are obtained by means of the Schauder fixed point theorem. Then the conditions on the growth of F are given to ensure the existence of global solutions.

A great attention to the equations with strong dissipation has been paid by Ebihara. In [9] he treats the equation (1.8) with $\sigma_i(\xi) = \xi$ and g replaced by a nonlinear perturbation term of the form $f = u_t f_1(u, u_{x_i}, u_{x_i x_j}) + f_2(u_t, u_{x_i t}, u_{x_i x_j t})$ where f_1 and f_2 are polynomials with constant coefficients. Using the small energy method he proves the existence of a unique global classical solution assuming the initial data to be sufficiently smooth and small. The same results are obtained in [10] for the one-dimensional wave equation with the strong dissipation of the form $(\partial/\partial x) \,.\, [\varphi(u)(1 + \alpha_1 u^p + \alpha_2 u_x^q + \alpha_3 u_t^r) u_{xt}]$ where $\alpha_i \in \mathbb{R}$, $p, q, r \in \mathbb{N}$ and φ is a positive bounded function with some other properties.

The main objective of Andrews [1] is to study (1.4) with σ non-monotone. The detailed study of the Green function for the heat equation, the fixed-point theorem due to Krasnosel'skiĭ and the maximum principle technique are used to prove the existence of a unique weak solution. In addition to homogeneous Dirichlet boundary conditions the author treats also the boundary conditions $\sigma(u_x(t, 0)) = u(l, t) = 0$ which correspond to the case when one end of the bar is stress-free and the other fixed. Some further properties of solutions are examined in Andrews-Ball [2].

The initial-value problem for the equation (1.8) is dealt with in Pecher [31] and Yamada [40]. Pecher obtains a unique global classical solution by means of the Banach contraction principle and of à priori estimates under the assumptions that σ_i is increasing if $n \leqq 2$ and for $n = 2$ of at most cubic growth. Yamada gets local classical solutions for n arbitrary and σ_i increasing. In fact, he treats a somewhat more general equation

$$(1.9) \qquad u_t - \sum_{i,j=1}^{n} a_{ij}(u_{x_1}, \ldots, u_{x_n})\, u_{x_i x_j} + c u_t - b\, \Delta u_t = g$$

with $c \in \mathbb{R}$. He shows that there exists a unique global classical solution provided $c > 0$ and imposing the smallness condition on the initial data. In the proof he converts the original problem to the initial-value problem for the abstract evolution equation.

The equations of the type (1.8) are of importance by themselves because of their usefulness in applications. As we have seen, they arise in the theory of visco-elasticity (cf. [16], [12]). Further, we can encounter them e.g. in hydrodynamics (cf. [24]). There is still another reason (mathematical; however, physically motivated as well) why to investigate them. Namely, it turns out that they can be used to study the nonlinear wave equations. In [16] it was conjectured that if $b \to 0+$ the global solution of (1.4) approaches some preferred weak solution of the corresponding nonlinear wave equation. The idea to add an artificial higher order derivative (here it is $-u_{txx}$) multiplied by a small parameter b, solve the problem for $0 < b < b_0$ and then let $b \to 0+$ leads to the so called artificial viscosity method. This method is successfully performed by Davis in [8] and Yamada in [40]. Using the Fourier series technique Davis gets local classical solutions for the IBV problem for the nonlinear wave equation (1.7) with $\sigma(\xi) = a_1\xi + a_2\xi^3$ ($a_1 > 0$, $a_2 > 0$). Yamada passes to the limit $b \to 0+$ in his equation (1.9) and obtains a unique classical (in general, local) solution of the initial-value problem.

Before turning to the periodic problems let us note that there is a number of treatments dealing with equations containing the term $-\Delta u_t$ the elliptic part of which, however, is degenerate or not present at all. See e.g. [3], [9, II], [11], [22], [23], [31].

In addition to Clements [5] already mentioned some other papers have appeared dealing with periodic solutions to the equations of the above discussed types with various nonlinear perturbations. Sowunmi [33] looks for weak periodic solutions to the Dirichlet problem for the equation (1.4) with non-zero right-hand side $g(t, x)$ periodic in t and σ of "almost linear" behavior. He uses a topological method of continuation with respect to the parameter. Using compactness and monotonicity methods Kakita [20] deals with weak periodic solutions to the Dirichlet problem for the equation (1.8) with $\sigma_i(\xi) = |\xi|^q \xi$ ($q \geqq 2$) and with an additional nonlinear term of the form $p(|u|^2)\, u_t$ where $p(s^2)$ is a non-negative function on $\mathbb{R}$ of polynomial growth. (In fact, the study of somewhat more general equations is included.) The problem in an abstract setting is investigated (in linear and weakly nonlinear cases) by Straškraba-Vejvoda [34] and Herrmann [19]: in [34] by means of spectral integrals and in [19] by means of the time-space Fourier method. In the latter paper a special attention is paid to the regularity of solutions in the linear case.

In the present paper we investigate (generalizing [18]) periodic solutions with finite energy to (1.1) and (1.2) with $f_i = f_i(t, x, y, z)$ $(i = 1, 2)$ ω-periodic in t. The restriction will be imposed on the growth at $\pm\infty$ of f_i with respect to the variable z which is allowed to be at most linear. Further restrictions on f_i refer to the rate of the growth with respect to y at $\pm\infty$ (1^{st} set of conditions) or at the origin (2^{nd} set of conditions). In the first case we get solutions for any right-hand side g ω-periodic in t (even for first-order distributions over $(0, l)$ such as the Dirac delta function $\delta(x - x_0)$, $x_0 \in (0, l)$) with no restriction on its size. Since the only tool in the proof is a compactness argument the functions f_i may be rather general in the sense that no monotonicity assumptions are made. In particular, the theory yields the existence of an ω-periodic solution for any right-hand side g ω-periodic in t under the condition that f_1 and f_2 are any continuous and bounded functions on $\mathbb{R} \times [0, l] \times \mathbb{R}^2$, ω-periodic in t.

Our plan in this paper is as follows. In the next section we recall some notations and results, in particular two embedding theorems. In Section 3 we begin the study of the corresponding IBV problem for the corresponding linear case, i.e. we assume $f_1 = f_2 \equiv 0$. To prove the existence and correctness of a solution with finite energy (weak solution) the standard Faedo-Galerkin method is used. In Section 4 we convert the equation (again in the linear case) to a system of first-order equations and using the method from [28] we show the exponential decay of the associated semigroup (for a similar result see also [37]). Once we have this basic result we are in a position to prove the existence of a unique periodic solution in the linear case. This can be done, for instance, by a simple application of the Banach contraction principle as we show in Section 5 (for another simple approach see [17], p. 154).

The result on isomorphism (Theorem 2) shows a great difference between our equation in the linear case and the similar one in which $b = 0$, i.e. the telegraph $(a \neq 0)$ or the wave equation. For the latter two equations such a result is no longer valid (see [19]). In this sense our equation is more closely related to parabolic equations than to hyperbolic ones. (Another feature which relates our problem to the parabolic problems is the *analyticity* of the semigroup S associated with the operator A in (4.1), cf. [37].)

In the final Section 6 we first recall two results on compact embeddings. Then we apply the Schauder fixed-point theorem to the nonlinear case. Finally, two sets of conditions on functions f_1 and f_2 are given ensuring the existence of ω-periodic solutions to (1.1) and (1.2).

2. Preliminaries. In what follows let l, T and ω be arbitrary but fixed real positive numbers. We use the standard notation. In particular, the symbol $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ the set of real numbers, $L_p(0, l)$ (for each p, $1 \leqq p \leqq \infty$) the (real) Lebesgue space, $H^m(0, l)$ $(= W_2^m(0, l))$ $(m \in \mathbb{N})$ the Sobolev space, $H_0^1(0, l) = \{u;\ u \in H^1(0, l),\ u(0) = u(l) = 0\}$, $H^{-1}(0, l)$ the dual to $H_0^1(0, l)$. The inner product in $L_2(0, l)$ (as well as the duality between $H_0^1(0, l)$ and $H^{-1}(0, l)$) is denoted by $(\cdot, \cdot)$ and the corresponding norm by $|\cdot|$. The inner product in $H_0^1(0, l)$ is denoted by $((\cdot, \cdot))$, i.e. $((u, v)) = (u_x, v_x)$ for all $u, v \in H_0^1(0, l)$, and the corresponding norm by $\|\cdot\|$. The norm in $H^{-1}(0, l)$ is denoted by $\|\cdot\|_*$.

Let X be a Banach space. We denote by $L_p(\omega, X)$ $(1 \leqq p \leqq \infty)$ the space of (classes of) strongly measurable functions $u: \mathbb{R} \to X$ ω-periodic and such that

$$\|u\|_{L_p(\omega;X)} = \left(\int \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty \quad \text{if} \quad 1 \leqq p < \infty\,,$$

where the integration with respect to t is taken over any interval of periodicity, and

$$\|u\|_{L_\infty(\omega;X)} = \operatorname*{supess}_t \|u(t)\|_X < \infty\,,$$

The spaces $L_p(0, T; X)$ $(1 \leqq p \leqq \infty)$ of functions $u: (0, T) \to X$ are defined analogously.

If X is separable and reflexive then $L_p(0, T; X)$ for $1 < p < \infty$ is a reflexive Banach space with the dual space topologically isomorphic to $L_q(0, T; X^*)$, where $1/p + 1/q = 1$, X^* being the dual space to X. The dual space $(L_1(0, T; X))^*$ is isomorphic to the Banach space $L_\infty(0, T; X^*)$.

By $H^m(\omega, X)$ $(m \in \mathbb{N})$ we denote the space of (classes of) functions $u: \mathbb{R} \to X$ for which all derivatives in the distributional sense up to the order m belong to $L_2(\omega, X)$. It is a Banach space (or a Hilbert space if X is such) with the norm

$$\|u\|_{H^m(\omega;X)} = \left(\sum_{j=0}^{m} \|u^{(j)}\|_{L_2(\omega;X)}^2\right)^{1/2}.$$

The space $H^m(0, T; X)$ is defined analogously.

By $C(\omega; X)$ we denote the space of continuous ω-periodic functions $u: \mathbb{R} \to X$ with the norm

$$\|u\|_{C(\omega;X)} = \max_t \|u(t)\|_X\,,$$

and analogously for $C([0, T]; X)$.

If $u \in L_1(0, T; L_1(0, l))$ then there exists a function $\tilde{u}(t, \cdot)$ measurable on $(0, T) \times \times (0, l)$ such that $u(t) = \tilde{u}(t, \cdot)$ a.e. on $(0, T)$ and $\tilde{u} \in L_1((0, T) \times (0, l))$. The function $\tilde{u}$ is determined uniquely on $(0, T) \times (0, l)$ up to a subset of measure zero. If $u \in C([0, T]; L_p(0, l))$ then there exists $\tilde{u}(t, \cdot)$ measurable on $(0, T) \times (0, l)$ such that $u(t) = \tilde{u}(t, \cdot)$ for every $t \in [0, T]$. Using the usual licence we shall identify the "abstract" function u and the corresponding "two argument" function $\tilde{u}$. Adopting this identification we can write $L_2(0, T; L_2(0, l)) = L_2((0, T) \times (0, l))$ and so on.

In the sequel the derivative $\partial/\partial t$ is denoted by the prime $'$, the derivative $\partial/\partial x$ by ∇ and $\partial^2/\partial x^2$ by Δ.

At the end of this section let us recall two embedding theorems. The proofs may be found e.g. in [27].

Lemma 1. $H^1(0, T; H_0^1(0, l)) \hookrightarrow C([0, T]; H_0^1(0, l))$.

Corollary 1. $H^1(0, T; H_0^1(0, l)) \hookrightarrow C([0, T) \times [0, l])$.

Lemma 2. $L_2(0, T; H_0^1(0, l)) \cap H^1(0, T; H^{-1}(0, l)) \hookrightarrow C([0, T]; L_2(0, l))$.

3. Linear initial-boundary value problem. We shall start with the study of the corresponding linear IBV problem which is given by the equation (1.1) with $f_1 = f_2 \equiv 0$, the boundary conditions (1.2) and the initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in (0, l).$$

We rewrite the equation and the initial conditions in the abstract form

$$u''(t) - \Delta u(t) + a\, u'(t) - b\, \Delta u'(t) = g(t) \quad \text{a.e.} \quad t \in (0, T), \tag{3.1}$$

$$u(0) = u_0, \tag{3.2}$$

$$u'(0) = u_1. \tag{3.3}$$

Theorem 1. *Let a, b, g, u_0 and u_1 be given, satisfying*

$$a \geqq 0, \quad b > 0, \tag{3.4}$$

$$g \in L_2(0, T; H^{-1}(0, l)), \tag{3.5}$$

$$u_0 \in H_0^1(0, l), \tag{3.6}$$

$$u_1 \in L_2(0, l). \tag{3.7}$$

Then there exists exactly one function $u \in H^1(0, T; H_0^1(0, l))$ which satisfies (3.1)–(3.3).

If we denote $u = \mathscr{U}(u_0, u_1, g)$, $\mathscr{U}$ is a linear continuous mapping of $H_0^1(0, l) \times L_2(0, l) \times L_2(0, T; H^{-1}(0, l))$ into $H^1(0, T; H_0^1(0, l)) \cap H^2(0, T; H^{-1}(0, l))$.

If $u \in H^1(0\ T; H_0^1(0, l))$ satisfies (3.1) then in view of (3.5) we get $u'' \in L_2(0, T; H^{-1}(0, l))$. By Lemmas 1 and 2 we have

$$u \in C([0, T]; H_0^1(0, l))$$

and

$$u' \in C([0, T]; L_2(0, l))$$

so that the conditions (3.2) and (3.3) make sense.

The precise meaning of the equation (3.1) being fulfilled is the following:

$$\int_0^T [-(u', \varphi') + ((u, \varphi)) + a(u', \varphi) + b((u', \varphi))]\, \mathrm{d}t = \int_0^T (g, \varphi)\, \mathrm{d}t + (u_1, \varphi(0))$$

for all functions φ such that $\varphi \in L_2(0, T; H_0^1(0, 1))$, $\varphi' \in L_2(0, T; L_2(0, l))$ and $\varphi(T) = 0$.

We have just specified the sense of the equation (3.1) but let us rewrite it once again in an equivalent form (see [25]), which is more appropriate when using the Faedo-Galerkin method:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u'(t), w) + ((u(t), w)) + a(u'(t), w) + b((u'(t), w)) = (g(t), w) \tag{3.8}$$

(in the sense of distributions over $(0, T)$) for all $w \in H_0^1(0, l)$.

Proof of the uniqueness. Let $u \in H^1(0, T; H_0^1(0, l))$ satisfy (3.1)–(3.3) with $g = 0$, $u_0 = 0$, and $u_1 = 0$. Let us apply a functional from $L_2(0, t; H^{-1}(0, l))$ given by the left-hand side of (3.1) to $u' \in L_2(0, t; H_0^1(0, l))$, $t \in [0, T]$ arbitrary. Since

$$\tfrac{1}{2}|u'(t)|^2 = \int_0^t (u''(\tau), u'(\tau))\, d\tau$$

and

$$\tfrac{1}{2}\|u(t)\|^2 = \int_0^t ((u'(\tau), u(\tau)))\, d\tau$$

we get

$$\tfrac{1}{2}(|u'(t)|^2 + \|u(t)\|^2) + a\int_0^t |u'(\tau)|^2\, d\tau + b\int_0^t \|u'(\tau)\|^2\, d\tau = 0\,, \quad t \in [0, T]\,,$$

and this implies $u \equiv 0$.

Proof of the existence will be carried out by means of the standard Faedo-Galerkin method.

Let us choose a basis (i.e. a linearly independent set whose finite linear combinations are dense) $\{w_k\}_{k=1}^{\infty}$ in the space $H_0^1(0, l)$. (In particular, we may take

$$w_k(x) = \frac{\sqrt{(2l)}}{k\pi} \sin\frac{k\pi x}{l}\,, \quad k \in \mathbb{N}\,.$$

These functions form an orthonormal basis in $H_0^1(0, l)$ and the relation

$$((w_k, w)) = \frac{k^2\pi^2}{l^2}(w_k, w)$$

holds for any $w \in H_0^1(0, l)$ and $k \in \mathbb{N}$.)

The approximate solutions are looked for in the form

$$u_m(t) = \sum_{k=1}^{m} g_{km}(t)\, w_k\,,$$

where g_{km} are defined by the system of linear ordinary equations

$$(3.9) \qquad (u_m''(t), w_j) + ((u_m(t), w_j)) + a(u_m'(t), w_j) + b((u_m'(t), w_j)) = (g(t), w_j)$$
$$(j = 1, \ldots, m)$$

and by the initial conditions

$$(3.10) \qquad u_m(0) = u_{0m}\,, \quad u_m'(0) = u_{1m}\,,$$

where u_{0m} and u_{1m} are such that

$$(3.11) \qquad u_{0m} = \sum_{k=1}^{m} \alpha_{km} w_k \to u_0 \quad \text{in} \quad H_0^1(0, l)$$

and

$$(3.12) \qquad u_{1m} = \sum_{k=1}^{m} \beta_{km} w_k \to u_1 \quad \text{in} \quad L_2(0, l)\,.$$

The initial problem (3.9)–(3.10) has for any $m \in \mathbb{N}$ exactly one solution $(g_{1m}, \dots, g_{mm}) \in (H^2(0, T))^m$.

To obtain à priori (energy) estimates let us multiply (3.9) by g'_{jm} and sum on j. Using then the Schwarz inequality, the inequality

$$ab \leqq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad (a, b \in \mathbb{R},\ \varepsilon > 0) \tag{3.13}$$

and integrating from 0 to t we arrive at the following estimate:

$$|u'_m(t)|^2 + \|u_m(t)\|^2 + 2a \int_0^t |u'_m(\tau)|^2 \,\mathrm{d}\tau + b \int_0^t \|u'_m(\tau)\|^2 \,\mathrm{d}\tau \leqq \tag{3.14}$$

$$\leqq |u_{1m}|^2 + \|u_{0m}\|^2 + \frac{1}{b} \int_0^t \|g(\tau)\|_*^2 \,\mathrm{d}\tau\,, \quad t \in [0, T]\,.$$

The sequences $\|u_{0m}\|$ and $\|u_{1m}\|$ are bounded owing to (3.11) and (3.12). Thus, taking into account the assumption (3.5) we get the à priori estimates

$$|u'_m(t)| \leqq C\,, \quad \|u_m(t)\| \leqq C\,, \quad \int_0^T \|u'_m(t)\|^2 \,\mathrm{d}t \leqq C\,,$$

where the constant C is independent of m and t.

Now we pass to the limit. The usual weak sequential compactness argument (see e.g. Taylor [35]) yields the existence of a subsequence $\{u_\mu\}$ of the sequence $\{u_m\}$ and an element $u \in L_\infty(0, T; H_0^1(0, l))$ such that $u' \in L_2(0, T; H_0^1(0, l)) \cap L_\infty(0, T; L_2(0, l))$ and

$$u_\mu \to u \quad \textit{weakly star in} \quad L_\infty(0, T; H_0^1(0, l))\,, \tag{3.15}$$

$$u'_\mu \to u' \quad \textit{weakly in} \quad L_2(0, T; H_0^1(0, l))\,, \tag{3.16}$$

$$u'_\mu \to u' \quad \textit{weakly star in} \quad L_\infty(0, T; L_2(0, l))\,. \tag{3.17}$$

It is a matter of routine to check that u satisfies the relation (3.8). The initial conditions are satisfied as well. In fact, since $u_\mu \to u$ and $u'_\mu \to u'$ both weakly in $L_2(0, T; H_0^1(0, l))$ we get by using the embedding stated in Lemma 1 that $u_\mu(0) \to u(0)$ weakly in $H_0^1(0, l)$. But $u_\mu(0) = u_{0\mu}$ and $u_{0\mu} \to u_0$ in $H_0^1(0, l)$ so that $u(0) = u_0$. Likewise, by means of (3.16), (3.9) and (3.12) we get $u'(0) = u_1$.

Let us observe that the uniqueness of the solution implies that all convergent subsequences of $\{u_m\}$ converge (in the above described sense) to the same element u and this in turn implies that the sequence $\{u_m\}$ itself is convergent.

It remains to show the continuous dependence of the solution on the data u_0, u_1, and g. But this is a consequence of the inequality (3.14). This completes the proof of the theorem.

4. Exponential decay of the energy. In this section we derive an important consequence of the presence of the strongly dissipative term $-bu_{txx}$ in the equation (3.1).

Let us denote $X = L_2(0, l) \times H^{-1}(0, l)$ and define an operator $A: D_A \subset X \to X$ by

$$D_A = H_0^1(0, l) \times H_0^1(0, l)$$

and

$$A[u, v] = [-v, -\Delta u - b\,\Delta v + av]\,.$$

Equation (3.1) can be converted to a first-order system in X of the form

$$U_t + AU = G\,, \tag{4.1}$$

where $U = [u, v]$ and $G = [0, g]$.

It follows from the results of the previous section that for any $[u_0, u_1] \in H_0^1(0, l) \times L_2(0, l)$ there exists a unique solution

$$U = \left[\mathscr{U}(u_0, u_1, 0), \frac{\partial}{\partial t}\mathscr{U}(u_0, u_1, 0)\right]$$

of the equation (4.1) with $g \equiv 0$ satisfying $U(0) = [u_0, u_1]$ and bounded on $(0, \infty)$. Moreover, setting

$$U(t) = S(t)\,[u_0, u_1] \tag{4.2}$$

and denoting by $|||\cdot|||$ the norm in the space $\mathscr{L}(H_0^1(0, l) \times L_2(0, l))$ of linear bounded operators on $H_0^1(0, l) \times L_2(0, l)$, there exists a positive constant M_1 such that

$$|||S(t)||| \leqq M_1 \quad \text{for any} \quad t \geqq 0\,.$$

We shall actually demonstrate that this bound can be strengthened to

$$|||S(t)||| \leqq M e^{-\alpha t} \quad \text{for any} \quad t \geqq 0\,,$$

where M and α are positive constants.

Let us multiply (in the sense of $(\cdot, \cdot)$) the second equation of the system (4.1) (with $g \equiv 0$) by $v + \delta u$ with $\delta > 0$. We get

$$\frac{1}{2}\frac{d}{dt}(|v(t)|^2 + \|u(t)\|^2 + 2\delta(v(t), u(t))) +$$

$$+ \delta(|v(t)|^2 + \|u(t)\|^2 + 2\delta(v(t), u(t))) =$$

$$= -b((v(t), v(t) + \delta\, u(t))) - (a - 2\delta)\,(v(t), v(t) + \delta\, u(t))\,.$$

Thus,

$$|v(t)|^2 + \|u(t)\|^2 + 2\delta(v(t), u(t)) =$$

$$= e^{-2\delta t}(|u_1|^2 + \|u_0\|^2 + 2\delta(u_1, u_0)) + 2\int_0^t e^{-2\delta(t-\tau)}[-b((v(\tau), v(\tau) + \delta\, u(\tau))) -$$

$$- (a - 2\delta)\,(v(\tau), v(\tau) + \delta\, u(\tau))]\, d\tau\,, \quad t \geqq 0\,.$$

Using the inequalities

$$|w| \leqq \pi^{-1}\, l\|w\|\,, \quad w \in H_0^1(0, l)$$

and (3.13) we get from the preceding formula the following inequality that holds for $0 < \delta < \min(\pi l^{-1}, 2^{-1}\pi^2 l^{-2} b)$:

$$(1 - \pi^{-1} l\delta)(|v(t)|^2 + \|u(t)\|^2) \leqq (1 + \pi^{-1} l\delta)\, e^{-2\delta t}(|u_1|^2 + \|u_0\|^2) +$$
$$+ 2\delta^2 \int_0^t e^{-2\delta(t-\tau)} \left[\left(\frac{b^2\pi^2}{4(b\pi^2 - 2l^2\delta)} + \frac{(4 + a)\, l^2}{4\pi^2}\right) \|u(\tau)\|^2 + |v(\tau)|^2\right] d\tau .$$

Denote

$$\mathscr{E}(t) = e^{2\delta t}(|v(t)|^2 + \|u(t)\|^2),$$
$$c_1 = (1 + \pi^{-1} l\delta)(1 - \pi^{-1} l\delta)^{-1},$$
$$c_2 = \delta\left[\frac{b^2\pi^2}{4(b\pi^2 - 2l^2\delta)} + \frac{(4 + a)\, l^2}{4\pi^2} + 1\right](1 - \pi^{-1} l\delta)^{-1}$$

and pick out $\delta > 0$ so small that $c_1 > 0$ and $0 < c_2 < 1$. Then

$$\mathscr{E}(t) \leqq c_1\, \mathscr{E}(0) + 2\delta c_2 \int_0^t \mathscr{E}(\tau)\, d\tau$$

and the Gronwall lemma yields

$$\mathscr{E}(t) \leqq c_1 e^{2\delta c_2 t}\, \mathscr{E}(0) .$$

Hence

$$|v(t)|^2 + \|u(t)\|^2 \leqq c_1 e^{-2\alpha t}(|u_1|^2 + \|u_0\|^2)$$

where $\alpha = \delta(1 - c_2) > 0$. In other words, we have proved the following assertion.

Lemma 3. *There exist constants $M > 1$ and $\alpha > 0$ such that*

(4.3) $$|||\mathsf{S}(t)||| \leqq M e^{-\alpha t} \quad \textit{for all} \quad t \geqq 0 .$$

5. Periodic solutions — the linear case. In addition to the assumptions (3.4)—(3.7) let us assume that g is ω-periodic, i.e.

(5.1) $$g \in L_2(\omega; H^{-1}(0, l)) .$$

According to Theorem 1 the solution of (3.1)–(3.3) will be ω-periodic if and only if the initial data $[u_0, u_1]$ will be reproduced at the time $t = \omega$, i.e. if and only if

(5.2) $$u(\omega) = \mathscr{U}(u_0, u_1, g)(\omega) = u_0 ,$$
$$u'(\omega) = \frac{\partial}{\partial t} \mathscr{U}(u_0, u_1, g)(\omega) = u_1 .$$

Putting

$$\mathsf{s} = \left[\mathscr{U}(0, 0, g)(\omega), \frac{\partial}{\partial t} \mathscr{U}(0, 0, g)(\omega)\right]$$

and using the definition (4.2) of $\mathsf{S}(t)$ we easily see that the system (5.2) is equivalent to

(5.3) $$\mathsf{U}_0 = \mathsf{S}(\omega)\, \mathsf{U}_0 + \mathsf{s}$$

for $U_0 = [u_0, u_1] \in H_0^1(0, l) \times L_2(0, l)$. Let us note that the element s belongs to the space $H_0^1(0, l) \times L_2(0, l)$ (by Lemmas 1 and 2).

The existence and uniqueness of a solution to (5.3) can be proved by means of (4.3) and of the following lemma which is an easy consequence of the Banach contraction principle.

Lemma 4. *Let B be a Banach space and $L \in \mathscr{L}(B)$. Let there exist a positive integer n such that $|||L^n|||_{\mathscr{L}(B)} < 1$. Then for any $y \in B$ the equation $x = Lx + y$ has exactly one solution x. Moreover, there exists a constant $C > 0$ (independent of y) such that $\|x\|_B \leqq C\|y\|_B$.*

We apply this lemma with $B = H_0^1(0, l) \times L_2(0, l)$ and $L = S(\omega)$. Since $S^n(\omega) = S(n\omega)$ $(n \in \mathbb{N})$ we see from (4.3) that there is a positive integer n such that

$$|||S^n(\omega)||| < 1 .$$

Thus, there exists a unique couple $[u_0, u_1] \in H_0^1(0, l) \times L_2(0, l)$ satisfying (5.3). Moreover, by Lemma 4 and Theorem 1 we have

$$\|[u_0, u_1]\|_{H_0^1(0,l) \times L_2(0,l)} \leqq c\|g\|_{L_2(\omega; H^{-1}(0,l))}$$

where $c > 0$ is independent of g.

Let us summarize our results.

Theorem 2. *Let*

$$a \geqq 0 , \quad b > 0 , \tag{5.4}$$

$$g \in L_2(\omega; H^{-1}(0, l)) . \tag{5.5}$$

Then there exists exactly one function

$$u \in H^1(\omega; H_0^1(0, l))$$

satisfying (3.1) (for a.e. t).

Moreover, denoting

$$u = \mathscr{K} g ,$$

$\mathscr{K}$ is a linear continuous mapping of $L_2(\omega; H^{-1}(0, l))$ into $H^1(\omega; H_0^1(0, l))$. More precisely, $\mathscr{K}$ is a topological isomorphism of

$$L_2(\omega; H^{-1}(0, l)) \quad \textit{onto} \quad H^1(\omega; H_0^1(0, l)) \cap H^2(\omega; H^{-1}(0, l)) .$$

Remark. The bijectivity property stated in the preceding theorem makes a substantial distinction between our equation and the hyperbolic equations with $b = 0$. For the latter ones such a result is not true (cf. e.g. [19]).

6. Periodic solutions — the nonlinear case. Let the assumptions (5.4) and (5.5) be satisfied. *Let f_1 and f_2 be two functions*

$$f_i(t, x, y, z): \mathbb{R} \times [0, l] \times \mathbb{R}^2 \to \mathbb{R} \quad (i = 1, 2)$$

with the following properties:

(6.1) $f_i(t, x, \cdot, \cdot)$ *is continuous on* $\mathbb{R}^2$ *for almost all* $(t, x) \in \mathbb{R} \times [0, l]$,

(6.2) $f_i(\cdot, \cdot, y, z)$ *is measurable on* $\mathbb{R} \times [0, l]$ *for all fixed* $(y, z) \in \mathbb{R}^2$,

(6.3) f_i *is* ω*-periodic with respect to* t.

The objective of this section is to set forth some sets of sufficient conditions on f_1 and f_2 in order that the problem (1.1) and (1.2) may have an ω-periodic solution u. The solution is meant in the sense that $u \in H^1(\omega; H_0^1(0, l))$, both $f_1(\cdot, \cdot, u, u')$ and $\nabla f_2(\cdot, \cdot, u, u')$ belong to $L_2(\omega; H^{-1}(0, l))$ and

$$\int_0^\omega [-(u', \varphi') + ((u, \varphi)) + a(u', \varphi) + b((u', \varphi)) + \\ + (f_1, \varphi) - (f_2, \nabla\varphi) - (g, \varphi)] \, dt = 0$$

for all $\varphi \in L_2(\omega; H_0^1(0, l)) \cap H^1(\omega; L_2(0, l))$.

We rewrite the equation (1.1) in the form

(6.4) $$u'' - \Delta u + au' - \Delta b\, u' = g - F(u),$$

where

$$F(u) = F_1(u) + \nabla F_2(u)$$

and

$$F_i(u) = f_i(\cdot, \cdot, u, u').$$

Then we use the results of the preceding section. By Theorem 2 a function u is a solution if and only if u is a fixed point of the operator

(6.5) $$\mathscr{K}(g - F(u))$$

in the space

$$W = H^1(\omega; H_0^1(0, l)) \cap H^2(\omega; H^{-1}(0, l)).$$

We apply the Schauder fixed-point theorem. To this end, under suitable assumptions on f_1 and f_2 we derive that

1° F is a completely continuous mapping of W into $L_2(\omega; H^{-1}(0, l))$ which ensures by Theorem 2 that (6.5) is a completely continuous mapping of W into itself,

2° there exists a non-empty bounded closed convex subset of W which is invariant under the mapping (6.5). This subset will be a ball with its center at the origin and with a suitably chosen radius.

We shall need the following two lemmas.

Lemma 5. *The embedding*

$$H^1(\omega; H_0^1(0, l)) \hookrightarrow C([0, \omega] \times [0, l])$$

is compact.

Proof follows from Arzela's theorem, the relation

$$u(t_1, x_1) - u(t_2, x_2) = \int_{t_2}^{t_1} \int_0^{x_1} u_{tx}(\tau, \xi) \, d\xi \, d\tau + \int_{x_2}^{x_1} u_x(t_2, \xi) \, d\xi$$

and Lemma 1.

Lemma 6. *The embedding*

$$L_2(\omega; H_0^1(0, l)) \cap H^1(\omega; H^{-1}(0, l)) \subsetneq L_2(\omega; L_2(0, l))$$

is compact.

Proof see [26], p. 58.

Let us make the following assumption:

(6.6) *there exist constants σ_i, $0 \leqq \sigma_i \leqq 1$ and functions g_i, h_i, p_i and q_i $(i = 1, 2)$ such that $g_i \in L_2(Q)$, $Q = (0, \omega) \times (0, l)$, $h_i \in L_{2/(1-\sigma_i)}(Q)$ $(L_\infty(Q)$ if $\sigma_i = 1)$, p_i, q_i: $[0, \infty) \to [0, \infty)$ are non-decreasing and*

$$|f_i(t, x, y, z)| \leqq p_i(|y|) \, |g_i(t, x)| + q_i(|y|) \, |h_i(t, x)| \, |z|^{\sigma_i}$$

for almost all $(t, x) \in Q$ and all $(y, z) \in \mathbb{R}^2$.

Now, let $u_n \in W$, $\|u_n\| \leqq r_0$ and $u_n \to u$ in W weakly. In virtue of Lemmas 5 and 6, $u_n \to u$ in $C(\bar{Q})$ and $u_n' \to u'$ in $L_2(Q)$. Taking into account the assumptions (6.1) and (6.2) we infer that

$$F_i(u_n) \to F_i(u) \quad \text{in measure on } Q \, .$$

Using the assumption (6.6) and Young's inequality

$$ab \leqq \frac{1}{p} a^p + \frac{p-1}{p} b^{p/(p-1)} \quad (a \geqq 0, \; b \geqq 0, \; p > 1)$$

we get in the case $\sigma_i \in [0, 1)$

$$\begin{aligned} |F_i(u_n)(t, x) - F_i(u)(t, x)|^2 \leqq {} & 3(4p_i^2(vr_0) \, |g_i(t, x)|^2 + \\ & + 2(1 - \sigma_i) \, q_i^2(vr_0) \, |h_i(t, x)|^{2/(1-\sigma_i)} + \\ & + \sigma_i q_i^2(vr_0) \, (|u_n'(t, x)|^2 + |u'(t, x)|^2)) \end{aligned}$$

where

$$\sup_{(t,x) \in Q} |u(t, x)| \leqq v \|u\|_W \, , \quad u \in W \, ,$$

which shows that the functions $|F_i(u_n) - F_i(u)|^2$ have uniformly absolutely continuous integrals (it is immediate that the same is true if $\sigma_i = 1$) and by the classical theorem of Vitali (see [30]),

$$\int_0^\omega \int_0^l |F_i(u_n)(t, x) - F_i(u)(t, x)|^2 \, dx \, dt \to 0$$

as $n \to \infty$. It is now easy to see that $F(u_n) \to F(u)$ in $L_2(\omega; H^{-1}(0, l))$. So, (6.5) is a completely continuous mapping of W into itself.

Let us next assume $\|u\|_W \leqq r$. With regard to (6.6) and using Hölder's inequality (dropping the space index in norms if no confusion seems possible) we get

$$\|\mathscr{K}(g - F(u))\|_W \leqq |||\mathscr{K}|||_{\mathscr{L}(L_2(\omega; H^{-1}(0,l)); W)} \left\{ \|g\|_{L_2(\omega; H^{-1}(0,l))} + \right.$$

$$+ \pi^{-1} l \left[\int_0^\omega \int_0^l (p_1(vr) |g_1| + q_1(vr) |h_1| |u'|^{\sigma_1})^2 \, dx \, dt \right]^{1/2} +$$

$$\left. + \left[\int_0^\omega \int_0^l (p_2(vr) |g_2| + q_2(vr) |h_2| |u'|^{\sigma_2})^2 \, dx \, dt \right]^{1/2} \right\} \leqq$$

$$\leqq |||\mathscr{K}||| \left\{ \|g\| + \pi^{-1} l \sqrt{(2)}\, p_1(vr) \|g_1\|_{L_2(Q)} + \sqrt{(2)}\, p_2(vr) \|g_2\|_{L_2(Q)} + \right.$$

$$+ \pi^{-1} l \sqrt{(2)}\, q_1(vr) \left(\int_0^\omega \int_0^l |h_1|^2 |u'|^{2\sigma_1} \, dx \, dt \right)^{1/2} +$$

$$\left. + \sqrt{(2)}\, q_2(vr) \left(\int_0^\omega \int_0^l |h_2|^2 |u'|^{2\sigma_2} \, dx \, dt \right)^{1/2} \right\} \leqq$$

$$\leqq |||\mathscr{K}||| \{ \|g\| + \pi^{-1} l \sqrt{(2)}\, p_1(vr) \|g_1\|_{L_2(Q)} + \sqrt{(2)}\, p_2(vr) \|g_2\|_{L_2(Q)} +$$

$$+ \pi^{-1} l \sqrt{(2)}\, q_1(vr) \|h_1\|_{L_{2/(1-\sigma_1)}(Q)} r^{\sigma_1} + \sqrt{(2)}\, q_2(vr) \|h_2\|_{L_{2/(1-\sigma_2)}(Q)} r^{\sigma_2} \} \, .$$

By applying the Schauder fixed-point theorem we obtain the following

Assertion. *If there exists $r_0 > 0$ such that*

$$\text{(6.7)} \qquad |||\mathscr{K}||| \{ \|g\| + \pi^{-1} l \sqrt{(2)} \|g_1\| \, p_1(vr_0) + \sqrt{(2)} \|g_2\| \, p_2(vr_0) +$$

$$+ \pi^{-1} l \sqrt{(2)} \|h_1\| \, q_1(vr_0) \, r_0^{\sigma_1} + \sqrt{(2)} \|h_2\| \, q_2(vr_0) \, r_0^{\sigma_2} \} \leqq r_0$$

then

(6.8) *there exists $u \in W = H^1(\omega; H_0^1(0, l)) \cap H^2(\omega; H^{-1}(0, l))$ satisfying* (6.4).

In the next theorem we introduce two sets of conditions on functions p_i and q_i under which (6.7) takes place.

Theorem 3. *Let the assumptions* (5.4), (5.5), (6.1)–(6.3) *and* (6.6) *be satisfied. Then each of the following conditions* I *and* II *implies* (6.8):

I. $p_i(r) = o(r)$ $r \to \infty$,
$q_i(r) = o(r^{1-\sigma_i})$ $r \to \infty$
(if $\sigma_i = 1$ for $i = 1$ and/or $i = 2$ then in the latter condition replace o by O, assuming moreover that $\sup_{r \in J} q_i(r)$ *is sufficiently small, $J = (0, \infty)$);*

II. $p_i(r) = o(r)$ $r \to 0+$,
$q_i(r) = o(r^{1-\sigma_i})$ $r \to 0+$
and $\|g\|$ is sufficiently small (with the same modification as in I *if $\sigma_i = 1$; J is now a right neighbourhood of* 0)

Proof follows from the preceding Assertion when choosing the radius r_0 sufficiently large (cond. I) or sufficiently small (cond. II).

In the end we give a few simple examples:

a) $F(u)(t, x) = f(t, x, u, u_t)$
and f is continuous and bounded (or, more generally, sublinear in the variable u),

b) $F(u)(t, x) = g_1(t, x) u^p$, $p > 1$, $g_1 \in L_2(\omega; L_2(0, l))$, $g = 0$ (cond. II),

c) $F(u)(t, x) = \lambda u^p$, $p > 1$, $\|g\|$ sufficiently small, $\lambda \in \mathbb{R}$,

d) $F(u)(t, x) = \lambda h_1(t, x) u_t \sin u$, $h_1 \in L_\infty(Q)$, $|\lambda|$ sufficiently small,

e) $F(u)(t, x) = g_2(t) |u|^{p-1} u_x$, $p > 1$, $g_2 \in L_2(\omega; \mathbb{R})$, $g = 0$.

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