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CHARACTERIZATIONS OF CONFORMALLY FLAT HYPERSURFACES

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1. INTRODUCTION

In this paper we study hypersurfaces of Euclidean space satisfying one of the conditions $C . C = 0$, $C . R = 0$ or $Q . C = 0$, where R denotes the Riemann-Christoffel curvature tensor, Q the Ricci tensor and C the Weyl conformal curvature tensor of the hypersurface and where the first tensor acts on the second as a derivation.

Semi-symmetric spaces, i.e. Riemannian manifolds for which $R . R = 0$, have been studied by various authors. For references one can consult the recent work of Z. I. Szabó on this subject [12].

In [8] K. Nomizu studied semi-symmetric hypersurfaces of Euclidean space and P. J. Ryan investigated semi-symmetric hypersurfaces of space-forms [9]. Y. Matsuyama [6], I. Mogi and H. Nakagawa [7], P. J. Ryan [10], S. Tanno [13], S. Tanno and T. Takahashi [14] studied hypersurfaces of space forms satisfying one of the conditions $R . Q = 0$ or $\nabla Q = 0$. In [16] two of the authors characterized hypercylinders in Euclidean spaces by the condition $Q . R = 0$. For hypersurfaces in Euclidean space with $R . C = 0$ or $C . R = 0$, see [1]. Complex hypersurfaces in complex space forms satisfying similar conditions have been investigated by P. J. Ryan [11], T. Takahashi [15] and the authors [4].

We prove the following theorem.

Theorem. *Let M^n be a hypersurface in an $(n + 1)$-dimensional Euclidean space $(n > 3)$ and denote by R the Riemann-Christoffel curvature tensor, by Q the Ricci tensor and by C the Weyl conformal curvature tensor of M^n. Then the following assertions are equivalent:*

(i) $C . C = 0$,
(ii) $C . R = 0$,
(iii) $Q . C = 0$,
(iv) *M^n is conformally flat.*

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2. BASIC FORMULAS

Let M be a hypersurface of an $(n + 1)$-dimensional Euclidean space E^{n+1}. Let ξ be a local normal section on M. In the following X, Y, Z denote vector fields which are tangent to M. Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\,\xi$$

and

$$\tilde{\nabla}_X \xi = -AX\,,$$

where $\tilde{\nabla}$ is the Euclidean connection on E^{n+1} and ∇ is the Levi Civita connection on M. The second fundamental tensor A is related to the second fundamental form h by $h(X, Y) = g(AX, Y)$, where g is the Riemannian metric on M. Let $X \wedge Y$ denote the endomorphism $Z \mapsto g(Z, Y)\,X - g(Z, X)\,Y$. Then the curvature tensor R of M is given by the equation of Gauss:

$$R(X, Y) = AX \wedge AY\,.$$

Since A is symmetric there exists an orthonormal frame $e_1, e_2, \ldots, e_n$ consisting of eigenvectors, i.e. such that

$$Ae_i = \lambda_i e_i\,, \tag{2.1}$$

$(i \in \{1, 2, \ldots, n\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the principal curvatures of $M)$.

The hypersurface M is said to be *quasiumbilical* when M has a principal curvature with multiplicity $\geqq n - 1$. Let C denote the Weyl conformal curvature tensor of M. For $n \geqq 4$ M is *conformally flat* iff C vanishes identically. If $n \geqq 4$, E. Cartan proved that a hypersurface M of E^{n+1} is conformally flat if and only if it is quasiumbilical [2]. We recall that every surface is conformally flat and that for every 3-dimensional Riemannian manifold the Weyl conformal curvature tensor C vanishes identically. If $n = 3$ there exist nonquasiumbilical hypersurfaces M of E^{n+1} which are conformally flat [5]. By Theorem 1 in [3] M is conformally flat if and only if $(\lambda - \lambda_j)\,.\,(\lambda_k - \lambda_l) = 0$ for all mutually distinct i, j, k, l in $\{1, 2, \ldots, n\}$.

By (2.1) the equation of Gauss implies that

$$R(e_i, e_j) = c_{ij} e_i \wedge e_j\,,$$

where

$$c_{ij} = \lambda_i \lambda_j\,,$$

and consequently

$$C(e_i, e_j) = a_{ij} e_i \wedge e_j\,,$$

where

$$a_{ij} = c_{ij} - \frac{1}{n-2}\Big(\sum_{t \neq i} c_{ti} + \sum_{t \neq j} c_{tj}\Big) + \frac{2}{(n-1)(n-2)} \sum_{\substack{t,s \\ t<s}} c_{ts}\,,$$

(see also [3], $i, j \in \{1, \ldots, n\}$ and $i \neq j$).

Further,

$$Qe_i = \mu_i e_i$$

where

$$\mu_i = \lambda_i(\operatorname{tr} A - \lambda_i).$$

By $C \,.\, C = 0$ we mean that $C(X, Y) \,.\, C = 0$ for all vector fields X and Y tangent to M, where $C(X, Y)$ acts as a derivation on the algebra of tensor fields on M, i.e.

$$(C(X, Y) \,.\, C)(Z, W)\, V =$$
$$= [C(X, Y), C(Z, W)]\, V - C(C(X, Y)\, Z, W)\, V - C(Z, C(X, Y)\, W)\, V$$

for X, Y, Z, V, W tangent to M.

Because this derivation commutes with contractions the implication (ii) $\Rightarrow$ (i) holds good. Furthermore (iv) trivially implies (i), (ii) and (iii).

3. PROOF OF (i) $\Rightarrow$ (iv)

First we state that assertion (i) implies the following: $(*)$ for all mutually distinct i, j, k: $a_{ij}(a_{ik} - a_{jk}) = 0$. In fact, for i, j, k mutually distinct indices, we have

$$(C(e_i, e_j) \,.\, C)(e_i, e_k)\, e_k = a_{ij}(a_{jk} - a_{ik})\, e_j\,.$$

Let $\lambda_1, \ldots, \lambda_p$ be the (mutually) distinct eigenvalues of A with multiplicities $s_1, \ldots, s_p$ respectively. Denote by V_α the space of eigenvectors with eigenvalue λ_α. If $e_i, e_k \in V_\alpha$ and $e_j, e_l \in V_\beta$ for $i \neq j$ and $k \neq l$, then $a_{ij} = a_{kl}$. We define numbers $b_{\alpha\beta} = a_{ij}$, where $i \neq j$, $e_i \in V_\alpha$ and $e_j \in V_\beta$ $(\alpha, \beta \in \{1, \ldots, p\})$. To prove (iv) it is sufficient to show that $b_{\alpha\beta} = 0$ for all α, β such that $b_{\alpha\beta}$ is defined. In the following we will prove that the assumption $b_{\alpha\beta} \neq 0$ for some α and β in $\{1, \ldots, p\}$ always leads to a contradiction.

First we consider the case $p \geqq 4$. If there are distinct indices α and β such that $b_{\alpha\beta} \neq 0$, $(*)$ implies that there exist indices γ and δ such that $\alpha, \beta, \gamma, \delta$ are mutually distinct and

$$b_{\alpha\gamma} = b_{\beta\gamma} \tag{2.2}$$

and

$$b_{\alpha\delta} = b_{\beta\delta}\,. \tag{2.3}$$

This gives

$$(\lambda_\alpha - \lambda_\beta)\left(\lambda_\gamma - \frac{1}{n-2}(\operatorname{tr} A - \lambda_\alpha - \lambda_\beta)\right) = 0$$

and

$$(\lambda_\alpha - \lambda_\beta)\left(\lambda_\delta - \frac{1}{n-2}(\operatorname{tr} A - \lambda_\alpha - \lambda_\beta)\right) = 0\,.$$

Substraction yields

$$(\lambda_\alpha - \lambda_\beta)(\lambda_\gamma - \lambda_\delta) = 0\,,$$

which is a contradiction.

If $b_{\alpha\beta} = 0$ for all distinct indices α and β in $\{1, \ldots, p\}$, we obtain (2.2) and (2.3) in a trivial way. As before, this leads to a contradiction.

Next, we treat the case $p = 3$. Suppose first that there are distinct indices α, β in $\{1, 2, 3\}$, such that $b_{\alpha\beta} \neq 0$, say $b_{12} \neq 0$. If $s_1 \geqq 2$ then $(*)$ implies

$$b_{11} = b_{12}$$

and

$$b_{31} = b_{32}\,.$$

As in the first paragraph this yields

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) = 0\,,$$

which gives a contradiction. The case $s_2 \geqq 2$ can be handled analogously. If $s_1 = 1$ and $s_2 = 1$, $(*)$ gives

$$(b_{12} - b_{13})\, b_{23} = 0\,,$$

from which we find that

$$b_{23} = 0 \tag{2.4}$$

or

$$b_{13} = b_{12}\,. \tag{2.5}$$

(2.4) yields

$$b_{23} = \frac{-(n-3)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{(n-1)(n-2)} = 0\,,$$

which gives a contradiction.

From (2.5) we obtain

$$(\lambda_2 - \lambda_3)\left(\lambda_1 - \frac{1}{n-2}(\operatorname{tr} A - \lambda_2 - \lambda_3)\right) = 0\,,$$

and thus

$$(n-3)(\lambda_1 - \lambda_3) = 0\,,$$

which again contradicts $\lambda_1 \neq \lambda_3$.

If $b_{12} = b_{23} = b_{13} = 0$, we obtain from $b_{12} - b_{13} = 0$ and $b_{23} - b_{13} = 0$ that

$$(n-2)\,\lambda_1 - \operatorname{tr} A + \lambda_2 + \lambda_3 = 0$$

and

$$(n-2)\,\lambda_3 - \operatorname{tr} A + \lambda_1 + \lambda_2 = 0\,.$$

Substraction yields $(n-3)(\lambda_1 - \lambda_3) = 0$, which gives a contradiction.

Suppose $p = 2$. If $s_1 = 1$ or $s_2 = 1$, [3] gives that all $b_{\alpha\beta} = 0$, which contradicts the initial assumption. Thus we may suppose $s_1 \geqq 2$ and $s_2 \geqq 2$. If $b_{12} \neq 0$, $(*)$ gives

$$b_{11} = b_{12}$$

and

$$b_{22} = b_{12}\,.$$

This leads to a contradiction $b_{12} = 0$ gives

$$b_{12} = -\frac{(s_1 - 1)(s_2 - 1)(\lambda_1 - \lambda_2)^2}{(n - 1)(n - 2)} = 0\,,$$

which is in contradiction with $\lambda_1 \neq \lambda_2$.

If $p = 1$, then all $b_{\alpha\beta} = 0$, which gives again a contradiction.

4. PROOF OF (iii) ⇒ (iv)

The condition $Q \,.\, C = 0$ implies that: $(**)$ for all distinct $i, j \in \{1, \ldots, n\}$: $\lambda_i(\operatorname{tr} A - \lambda_i)\, a_{ij} = 0$. Indeed we have

$$(Q \,.\, C)(e_i, e_j)\, e_i = 2\mu_i a_{ij} e_j\,,$$

where $i, j \in \{1, \ldots, n\}$ and $i \neq j$.

We use the same conventions concerning the eigenvalues of A as in Sec. 3 and we define numbers $b_{\alpha\beta}$ in the same way.

First we consider the case that $\lambda_\alpha \neq 0$ and $\lambda_\alpha \neq \operatorname{tr} A$ for all α in $\{1, \ldots, p\}$. $(**)$ implies that all $b_{\alpha\beta} = 0$.

Suppose that there are distinct α and β in $\{1, \ldots, p\}$ such that $\lambda_\alpha = 0$ and $\lambda_\beta = \operatorname{tr} A$, say $\lambda_1 = 0$ and $\lambda_2 = \operatorname{tr} A$. If $p \geqq 3$, then $(**)$ yields $a_{13} = a_{23} = 0$. From $a_{13} - a_{23} = 0$, we obtain

$$(\lambda_2 - \lambda_1)\left((n - 2)\,\lambda_3 - \operatorname{tr} A + \lambda_2 + \lambda_1\right) = 0\,,$$

which gives

$$(n - 2)\,\lambda_3 = 0\,.$$

This is in contradiction with $\lambda_1 \neq \lambda_3$. If $p = 2$, we have $\lambda_2 = \operatorname{tr} A = s_2\lambda_2$. Since $\lambda_2 \neq 0$, this yields $s_2 = 1$. This implies that all $b_{\alpha\beta} = 0$.

If there is an α in $\{1, \ldots, p\}$ such that $\lambda_\alpha = 0$ and $\lambda_\beta \neq \operatorname{tr} A$ for all $\beta \neq \alpha$ in $\{1, \ldots, p\}$ or if there is an α in $\{1, \ldots, p\}$ such that $\lambda_\alpha = \operatorname{tr} A$ and $\lambda_\beta \neq 0$ for all $\beta \neq \alpha$ in $\{1, \ldots, p\}$, then $b_{\alpha\beta} = 0$ for all α in $\{1, \ldots, p\}$ and all β in $\{2, \ldots, p\}$ for which $b_{\alpha\beta}$ exists. If $p \geqq 3$, then we obtain from $b_{12} - b_{13} = 0$ and $b_{12} - b_{23} = 0$ that

$$(n - 2)\,\lambda_1 - \operatorname{tr} A + \lambda_2 + \lambda_3 = 0$$

and

$$(n - 2)\,\lambda_2 - \operatorname{tr} A + \lambda_1 + \lambda_3 = 0\,.$$

Substraction yields

$$(n - 3)(\lambda_1 - \lambda_2) = 0\,,$$

which gives a contradiction.

We consider the case $p = 2$. If $s_2 = 1$ [3] implies that all $b_{\alpha\beta} = 0$. If $s_2 \geqq 2$, then $(**)$ yields $b_{12} = b_{22}$. This gives

$$(\lambda_2 - \lambda_1)\left(\lambda_2(n - 2) - \operatorname{tr} A + \lambda_1 + \lambda_2\right) = 0\,,$$

and thus

$$(s_1 - 1)(\lambda_2 - \lambda_1) = 0.$$

This gives a contradiction.

The case $p = 1$ is trivial. This completes the proof of the theorem.

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