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NEARLY ACYCLIC DIGRAPHS

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Nearly acyclic undirected graphs were studied in [1] and [2]. Such a graph was defined as a connected graph which has a vertex u contained in every circuit. In [1] the nearly acyclic undirected graphs which are homeomorphically irreducible blocks were characterized. Here we shall present a similar characterization for nearly acyclic directed graphs.

A directed graph G is called *nearly acyclic*, if it is strongly connected and there exists a vertex u of G contained in every cycle of G. (Then we can also say that G is nearly acyclic at u.)

Similarly as in [1], we restrict our attention to digraphs which are blocks (considered regardless of the orientation). If a digraph G has a cut vertex u, then evidently it is nearly acyclic if and only if u is its unique cut vertex and all of its blocks are nearly acyclic at u.

However, we shall not require the homeomorphic irreducibility and we shall admit neither multiple edges of the same direction nor loops. By $G - u$ we shall mean the graph obtained from G by deleting the vertex u.

Theorem 1. *Let G be a finite strongly connected digraph which is a block. The graph G is nearly acyclic at a vertex u if and only if the graph $G - u$ is a connected acyclic digraph and there are edges from u to all sources of $G - u$ and edges from all sinks of $G - u$ to u.*

Proof. The graph $G - u$ must be acyclic; otherwise it would contain a cycle and this cycle would be a cycle in G not containing u. If there were no edge from u into a source of $G - u$, then this source would also be a source in G and G would not be strongly connected; analogously for sinks. Thus the necessity of the conditions is proved. Suppose that the conditions are fulfilled. If x is an arbitrary vertex of $G - u$, then there exists a directed path from x to a sink of $G - u$ and an edge from this sink to u; hence there exists a directed path from x to u. Further, there exists a directed path from a source of $G - u$ to x and an edge from u to this source; hence there exists a directed path from u to x. This also implies that there exists a directed path from x to any other vertex y of G and G is strongly connected. Evidently G is a block.

Each cycle in G contains u; otherwise it would be contained in $G - u$ and this is not possible, because $G - u$ is acyclic. Hence G is nearly acyclic.

Theorem 2. *A digraph G is nearly acyclic at k vertices, where $k \geqq 3$, if and only if these vertices can be denoted by $u_1, \ldots, u_k$ so that G is the union of acyclic digraphs $G_1, \ldots, G_k$ with the property that G_i, G_{i+1} (where the sum $i + 1$ is taken modulo k) have a unique common vertex u_{i+1} for $i = 1, \ldots, k$, the graphs G_i, G_j for $i \neq j$, $|i - j| \not\equiv 1 \pmod{k}$ have no common vertex and any G_i has a unique source u_i and a unique sink u_{i+1}.*

Proof. Suppose that G is nearly acyclic at all vertices of a set U and $|U| \geqq 3$. As G is strongly connected, it contains a cycle C. This cycle contains all vertices of U; we shall denote them by $u_1, \ldots, u_k$ when going around C. Now let u_i, u_j be two vertices of U. Suppose that $j \not\equiv i + 1 \pmod{k}$ and there exists a directed path from u_i to u_j not containing u_{i+1} (the subscripts are taken modulo k). Then the union of this path with the directed path from u_j to u_i, being part of C, is a cycle in G which does not contain u_{i+1}, which is a contradiction. For $i = 1, \ldots, k$ let G_i be the subgraph of G formed by the vertices and edges of all directed paths from u_i to u_{i+1}. The above proved assertion implies that G_i, G_j for $i \neq j$, $|i - j| \not\equiv 1 \pmod{k}$ have no common vertex and any edge of G belongs to some G_i. Further, G_i and G_{i+1} have a unique common vertex u_{i+1}; otherwise there would be a cycle in G not containing u_{i+1}. No graph G_i contains a source distinct from u_i, because this source would be a source also in G and G would not be strongly connected; analogously for sinks. Thus the necessity of the conditions is proved. The sufficiency is obvious.

Quite analogously the following theorem can be proved.

Theorem 3. *A digraph G is nearly acyclic at two vertices u_1, u_2 if and only if G is the union of two acyclic digraphs G_1, G_2 with the property that G_1, G_2 have unique common vertices u_1, u_2, the graph G_1 has a unique source u_1 and a unique sink u_2 and the graph G_2 has a unique source u_2 and a unique sink u_1.*

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