Lamberto Cesari; Heinz W. Engl

Existence and uniqueness of solutions for nonlinear alternative problems in a Banach space

Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 4, 670–678

Persistent URL: http://dml.cz/dmlcz/101779

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR ALTERNATIVE PROBLEMS IN A BANACH SPACE

LAMBERTO CESARI, Ann Arbor and HEINZ W. ENGL*), Linz

(Received September 17, 1980)

Dedicated to the memory of SVATOPLUK FUČÍK

A major success in nonlinear analysis was achieved when Landesman and Lazer proved their celebrated existence theorem for elliptic nonlinear Dirichlet problems ([9]). Their theorem has since been extended in various directions by Williams [13], Nečas, Hess, de Figueiredo, Fučík [6] and others; for a detailed account of these developments see e.g. Cesari [1].

In [2], Cesari and Kannan gave an abstract existence theorem for nonlinear alternative problems in a Hilbert space X from which the Landesman-Lazer theorem, most of its extensions, and similar results like the Lazer-Leach theorem [10] can be derived. A different proof of essentially the same abstract result appeared in Kannan-McKenna [7], again for nonlinear problems in a Hilbert space X. (A version in some sense more general of such abstract results was proved afterwards by Cesari in [3, p. 46] for nonlinear alternative problems concerning two Banach spaces X and Y). (See Cesari [1] for some of these versions).

In this paper we generalize the Cesari-Kannan-McKenna form of the result as it appears in [2] and [7] for nonlinear alternative problems in an arbitrary Banach space X. We illustrate our abstract result by applying it to the problem of existence of periodic solutions for a simple nonlinear ordinary differential equation with lag, and by taking for X the space C of continuous functions.

Let X be a real Banach space, $L: D(L) \to X$ be a closed linear operator with dense domain $D(L)$, $N : X \to X$ continuous and bounded (not necessarily linear). We will make the following assumptions:

(1) $\dim \ker (L) < +\infty$,

(2) range (L) closed, $X = \ker (L) \oplus \text{range} (L)$.

Let $P : X \to X$ denote the projector onto $\ker (L)$ parallel to range (L). Then,

*) The paper was written while this author was at the Department of Mathematical Sciences, University of Delaware, Newark, Delaware, USA.

$L: D(L) \cap \text{range}(L) \to \text{range}(L)$ is one to one and onto, and admits, therefore, a linear inverse $H : \text{range}(L) \to D(L) \cap \text{range}(L)$, which is often denoted as the partial inverse of L. For this concept, as well as for some views on the "alternative method", see Cesari [4].

About the nonlinearity we will assume that

(4) $H(I - P)N$ is completely continuous.

Assumption (4) is fulfilled if N is completely continuous, since it follows from (2) that H is bounded. For differential operators L and suitable spaces, H is frequently compact, so that the nonlinearity need not be completely continuous, but only bounded and continuous.

One approach used in the "alternative method" for treating the equation

$$Lx = Nx \tag{5}$$

is to replace (5) by the system

$$x = Px + H(I - P)Nx\,, \tag{6}$$

$$PNx = 0\,, \tag{7}$$

which is equivalent to (5) under assumption (2) (cf. [1]). In various ways system (6), (7) can be replaced by a fixed point problem, say, $x = Tx$. For instance, as pointed out in [1], Williams [12] (1966) and later Mawhin (1972) noted that system (6), (7) can be replaced by the sole equation

$$x = Px + H(I - P)Nx - PNx\,, \tag{8}$$

which is equivalent to (6), (7), and therefore to (5), under assumption (2). In [1] Cesari has given a number of applications of this and other processes of reduction of (6), (7) to a fixed point problem.

In the present paper we shall use the abstract form (8). Our main result is the following "existence theorem at resonance":

Theorem 1. *Let L, P, H, and N be as above and assume that (1), (2), and (4) are fulfilled. Let $\bar{x}$ be a fixed element in X. We assume the existence of R, $r > 0$ such that*

(9) *For $x \in X$ with $\|P(x - \bar{x})\| \leqq R$ and $\|(I - P)(x - \bar{x})\| = r$ we have $\|Nx - L\bar{x}\| \leqq \|H(I - P)\|^{-1} r$.*

(10) *For $x \in X$ with $\|P(x - \bar{x})\| = R$ and $\|(I - P)(x - \bar{x})\| \leqq r$ we have $\|PNx\|^2 \geqq \|P(x - \bar{x} - Nx)\|^2 - R^2$.*

Then $Lx = Nx$ has at least one solution x with $\|P(x - \bar{x})\| \leqq R$ and

$$\|(I - P)(x - \bar{x})\| \leqq r, \quad \text{i.e.,} \quad \|x - \bar{x}\| \leqq R + r\,.$$

Proof. Let $\tilde{N} : X \to X$ be defined by $\tilde{N}z := N(\bar{x} + z) - L\bar{x}$. To prove the theorem, we have to show that there exists a $z \in B(R, r) := \{z \in X / \|Pz\| \leqq R, \|(I-P)z\| \leqq r\}$ with $Lz = \tilde{N}z$, which is equivalent with the existence of a fixed point of T_0 in $B(R, r)$, where $T_0 : X \to X$ is defined by $T_0 z := Pz + H(I-P)\tilde{N}z - P\tilde{N}z$. By the properties of Leray-Schauder degree ([5], [11]), it suffices to show that

(11) $T(s, z) \neq 0$ for $z \in \partial B(R, r)$, $0 < s < 1$, where $T: [0, 1] \times X \to X$ is defined by $T(s, z) := z - sT_0 z$.

We assume the contrary; let $0 < s < 1$ and $z \in \partial B(R, r)$ be such that $T(s, z) = 0$. We distinguish between two cases:

a) $\|Pz\| \leqq R$, $\|(I-P)z\| = r$. By assumption, $0 = (I-P)(z - sT_0 z) = (I-P)z - sH(I-P)\tilde{N}z$, which implies $r \leqq s\|H(I-P)\| \, . \, \|N(\bar{x} + z) - L\bar{x}\| \leqq rs$ by assumption (9) with $x := \bar{x} + z$.

Since $r > 0$ and $s < 1$, this is a contradiction.

b) $\|Pz\| = R$, $\|(I-P)z\| \leqq r$. Then $0 = P(z - sT_0 z) = (1-s)Pz + sP\tilde{N}z = (1-s)Pz + sPN(\bar{x} + z)$, so that $\|PN(\bar{x} + z)\|^2 = s^{-2}(1-s)^2 R^2$. On the other hand it follows from $0 = Pz - sP(z - N(\bar{x} + z))$ that $\|P(z - N(\bar{x} + z))\|^2 = s^{-2}R^2$. Together with assumption (10), applied to $x := \bar{x} + z$, this implies $s^{-2}(1-s)^2 R^2 \geqq s^{-2}R^2 - R^2$, i.e., $0 \leqq s(s-1)$, which is a contradiction.

Thus, (11) must be true, which proves the theorem.

Remark 2. The estimates of Theorem 1 can be viewed as conditions about the existence of an exact solution in the neighborhood of an approximate solution $\bar{x}$ (note that assumption (9) implies for those x for which it is valid that $\|Lx - Nx\| \leqq \|H(I-P)\|^{-1} r + \|L\| (R + r)$).

To simplify the comparison with previous results in [2] and [7], we give the following corollary, which is nothing but Theorem 1 with $\bar{x} = 0$.

Corollary 3. *Let L, P, H, and N be as above and assume that* (1), (2), *and* (4) *are fulfilled. Let* $X_0 := \ker(L)$, $X_1 := \operatorname{range}(L)$. *Assume that there exist* $R, r > 0$ *such that*

(12) *For* $x^* \in X_0$, $x_1 \in X_1$ *with* $\|x^*\| \leqq R$, $\|x_1\| = r$ *we have* $\|N(x^* + x_1)\| \leqq \|H(I-P)\|^{-1} r$.

(13) *For* $x^* \in X_0$, $x_1 \in X_1$ *with* $\|x^*\| = R$, $\|x_1\| \leqq r$ *we have* $\|PN(x^* + x_1)\|^2 \geqq \|x^* - PN(x^* + x_1)\|^2 - R^2$.

Then $Lx = Nx$ *has at least one solution*

$$x = x^* + x_1 \quad \textit{with} \quad \|x^*\| \leqq R, \quad \|x_1\| \leqq r.$$

Remark 4. If X is a Hilbert space, then the inequality in assumption (13) can be written as

$$\langle PN(x^* + x_1), x^* \rangle \geqq 0 , \tag{14}$$

or, if $\ker(L)$ and range (L) are orthogonal, $\langle N(x^* + x_1), x^* \rangle \geqq 0$, which is the condition used in [2], [3], [7].

It is also of interest to consider two extreme cases. If $L = 0$, then assumption (12) is empty, while assumption (13) is closely related to a condition used in a fixed point theorem due to Altman. For $\dim X < +\infty$ (only then (1) is fulfilled) we get a result about existence of a zero of N. The other extreme case is the case that L is regular. Then assumption (13) is empty, while assumption (12) implies $\|L^{-1}Nx\| \leqq r$ for $\|x\| = r$, which is precisely Rothe's condition for existence of a fixed point for $L^{-1}N$. Similar remarks apply to Theorem 1. For Altman's and Rothe's fixed point theorems see e.g. [5].

The assumptions of Corollary 3 are invariant under a change from L to $-L$. Thus we get also an existence result for $Lx = -Nx$, or, if the assumptions are fulfilled for $-N$ instead of N, another existence result for $Lx = Nx$. More precisely, the conclusion of Corollary 3 also holds if the inequality in (13) is replaced by

$$\|PN(x^* + x_1)\|^2 \geqq \|x^* + PN(x^* + x_1)\|^2 - R^2 , \tag{15}$$

or in the Hilbert space case by

$$\langle PN(x^* + x_1), x^* \rangle \leqq 0 . \tag{16}$$

Again a similar modification can be made in Theorem 1. It is worth noting that the following conditions, Hilbert space versions of which have been used in the literature before, imply the assumptions of Corollary 3. In the following propositions we use the notation of Corollary 3 and assume $L \neq 0$.

Proposition 5. *If there are $J_0 > 0$, $R_0 \geqq 0$ such that $\|Nx\| \leqq J_0$ for all $x \in X$ and $\|PN(x^* + x_1)\|^2 \geqq \|x^* - PN(x^* + x_1)\|^2 - R_0^2$ for all $x^* \in X_0$, $x_1 \in X_1$ with $\|x^*\| \geqq R_0$, $\|x_1\| \leqq \|H(I - P)\| J_0$, then the assumptions* (12), (13) *hold with $R \geqq R_0$, $r = \|H(I - P)\| J_0$.*

Proposition 5 yields a Banach space version of the Cesari-Kannan-McKenna theorem as originally formulated in [2] and [7].

Proposition 6. *If there are $J_0 \geqq 0$, $J_1 > 0$, $0 < \alpha < 1$, $R_0 > 0$, $K_0 \geqq \|H(I - P)\| J_0$, $K_1 > \|H(I - P)\| J_1$ such that $\|Nx\| \leqq J_0 + J_1 \|x\|^\alpha$ for all $x \in X$ and $\|PN(x^* + x_1)\|^2 \geqq \|x^* - PN(x^* + x_1)\|^2 - R_0^2$ for all $x^* \in X_0$, $x_1 \in X_1$ with $\|x^*\| \geqq R_0$, $\|x_1\| \leqq K_0 + K_1 \|x^*\|^\alpha$, then the assumptions* (12), (13) *hold with suitable R, $r > 0$.*

Proof. Let $g : [0, +\infty)^2 \to \mathbb{R}$ be defined by $g(s, t) := J_0 + J_1(1 + s)^\alpha t^\alpha$. It is possible to choose $s_0 > 0$ such that $\|H(I - P)\| J_1(1 + s_0)^\alpha \leqq K_1$ and $g(s_0, R_0) \, . \, \|H(I - P)\| \geqq s_0 R_0$. Since $s_0 t$ increases faster than $g(s_0, t) \, . \, \|H(I - P)\|$ as $t \to \infty$, we can find an $R \geqq R_0$ such that $g(s_0, R) \, . \, \|H(I - P)\| = s_0 R =: r$. We claim that for those $R, r > 0$ the assumptions (12), (13) hold. Let $x^* \in X_0$, $x_1 \in X_1$ with $\|x^*\| \leqq R$, $\|x_1\| = r$. Then we have:

$$\|H(I - P)\| \, . \, \|Nx\| \leqq \|H(I - P)\| \, (J_0 + J_1\|x\|^\alpha) \leqq \|H(I - P)\| \, . \, g(s_0, R) = r \, ,$$

so that (12) is fulfilled.

For $x^* \in X_0$, $x_1 \in X_1$ with $\|x^*\| = R$, $\|x_1\| \leqq r$ we have $\|x_1\| \leqq r = s_0 R = \|H(I - P)\| \, . \, g(s_0, R) \leqq K_0 + \|H(I - P)\| \, J_1(1 + s_0)^\alpha R^\alpha \leqq K_0 + K_1 R^\alpha = K_0 + K_1\|x^*\|^\alpha$, so that by assumption, $\|PN(x^* + x_1)\|^2 \geqq \|x^* - PN(x^* + x_1)\|^2 - R_0^2 \geqq \|x^* - PN(x^* + x_1)\|^2 - R^2$. Thus assumption (13) holds.

Proposition 7. *If there are $J_0 \geqq 0$, $J_1 > 0$, $\alpha \geqq 1$, $R_0 > 0$, $K_0 \geqq \|H(I - P)\| J_0$, $K_1 > \|H(I - P)\| J_1$ such that $\|Nx\| \leqq J_0 + J_1\|x\|^\alpha$ for all $x \in X$ and $\|PN(x^* + x_1)\|^2 \geqq \|x^* - PN(x^* + x_1)\|^2 - R_0^2$ for all $x^* \in X_0$, $x_1 \in X_1$ with $\|x^*\| \geqq R_0$, $\|x_1\| \leqq K_0 + K_1\|x^*\|^\alpha$ and if J_1 is sufficiently small (for details see the proof), then the assumptions (12), (13) hold with suitable $R, r > 0$.*

Proof. Define g as in the proof of Proposition 6 and choose $J_1 > 0$ so small that $(\|H(I - P)\| \, J_0 + K_1) \, R_0^{\alpha - 1} \leqq (\|H(I - P)\|^{-1} \, J_1^{-1} K_1)^{1/\alpha} - 1 =: s_0 > 0$, which is possible.

Then $\|H(I - P)\| \, J_1(1 + s_0)^\alpha = K_1$. Note that $\|H(I - P)\| \, . \, g(s_0, t)$ increases faster than $s_0 t$ as $t \to \infty$. On the other hand, $\|H(I - P)\| \, . \, g(s_0, R_0) \leqq s_0 R_0$, so that there is an $R \geqq R_0$ with $\|H(I - P)\| \, . \, g(s_0, R) = s_0 R =: r$. Now we proceed as in the proof of Proposition 6.

These proposition generalize to the situation of Theorem 1 in a suitable way.

Using a result due to R. B. Kellogg [8], we give a sufficient condition for existence of a *unique* solution of $Lx = Nx$, for simplicity of notation only for the situation of Corollary 3.

Theorem 8. *Let the assumptions of Corollary 3 be fulfilled and assume in addition that N is continuously Fréchet differentiable. Furthermore we assume that we have for all $x^* \in X_0$ and $x_1 \in X_1$:*

(17) *If $\|x^*\| < R$ and $\|x_1\| < r$, then*

$$\ker [L - N'(x^* + x_1)] = \{0\} \, .$$

(18) *If $\|x^*\| \leqq R$ and $\|x_1\| = r$ or $\|x^*\| = R$ and $\|x_1\| \leqq R$, then*

$$L(x^* + x_1) \neq N(x^* + x_1) \, .$$

Then $Lx = Nx$ has exactly one solution x with $\|Px\| \leqq R$ and $\|(I - P) \, x\| \leqq r$.

Proof. The existence of such an x follows from Corollary 3. To prove the uniqueness we note that (with the notation of the proof of Theorem 1) the map T_0 (with $\bar{x} := 0$) has no fixed point in $\partial B(R, r)$. T_0 is continuously Fréchet differentiable; if we can show that for all $x \in \operatorname{int} B(R, r)$, 1 is not an eigenvalue of $T_0'(x)$, then the result follows from [8]. Assume that for an $x \in \operatorname{int} B(R, r)$, 1 is an eigenvalue of $T_0'(x)$, so that there exists an $h \neq 0$ with $h = Ph + H(I - P) N'(x) h - PN'(x) h$. By applying P, we obtain $PN'(x) h = 0$, by applying L, we get $Lh = LH(I-P) N'(x) h$. Since $LH(I - P) = I - P$, addition of these two equalities gives $(L - N'(x)) h = 0$, which contradicts (17).

We illustrate our abstract results by applying Corollary 3 to the problem of existence of a periodic solution for an ordinary differential equation. We do not claim, that this particular result could not have been obtained by different means.

Proposition 9. *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and bounded, $G := \sup_{s \in \mathbb{R}} |g(s)|$, $h : \mathbb{R} \to \mathbb{R}$ continuous and 2π-periodic, $S := \sup_{s \in \mathbb{R}} |h(s)|$, $\tau \in \mathbb{R}$. Assume that there exist $c, d, \tilde{P}, \tilde{Q} \in \mathbb{R}$ with:*

$$s \geqq d \Rightarrow g(s) \leqq \tilde{Q} \tag{19}$$

$$s \leqq c \Rightarrow g(s) \geqq \tilde{P} \tag{20}$$

$$2\pi\tilde{Q} \leqq \int_0^{2\pi} h(s)\, ds \leqq 2\pi\tilde{P}\,. \tag{21}$$

Then the equation

$$x'(s) + g(x(s - \tau)) = h(s)\,, \quad (s \in \mathbb{R}) \tag{22}$$

has a 2π-periodic continuously differentiable solution $\bar{x}$ with

$$\sup_{s \in \mathbb{R}} |\bar{x}(s)| \leqq 8\pi(G + s) + \max\{d, -c\}\,. \tag{23}$$

Proof. Let X be the Banach space of all continuous, 2π-periodic functions from $\mathbb{R}$ into itself with the supremum-norm. Let $D(L) := \{x \in X / x \text{ continuously differentiable}\}$, and define $L : D(L) \to X$ by $Lx := x'$. L is closed and densely defined, and with the notation of Corollary 3 we have:

$$PX = X_0 = \{x : \mathbb{R} \to \mathbb{R} / x \text{ constant}\}\,,$$

$$(Px)(t) := (2\pi)^{-1} \int_0^{2\pi} x(s)\, ds\,,$$

and

$$(Hx_1)(t) := \int_0^t x_1(s)\, ds \quad \text{for all} \quad t \in \mathbb{R}\,, \quad x \in X\,, \quad x_1 \in X_1 = (I - P) X\,.$$

H is completely continuous with $\|H(I - P)\| = 4\pi$. We define $N : X \to X$ by $(Nx)(t) := -g(x(t - \tau)) + h(t)$. N is continuous and bounded. Choose $r := 4\pi(G + S)$ and $R := \max\{r + d,\ r - c\}$. Then $\|Nx\| \leqq \|H(I - P)\|^{-1} r$ for all $x \in X$, so that assumption (12) is fulfilled.

Let $x^* \in X_0$, $x_1 \in X_1$ with $\|x^*\| = R$ and $\|x_1\| \leqq r$. There are only two possibilities for x^*:

a) $x^*(t) = R$ for all $t \in \mathbb{R}$.

Then $PN(x^* + x_1)(t) = (2\pi)^{-1} \int_0^{2\pi} [h(s) - g(R + x_1(s - \tau))]\, ds =: v_+$ for all $t \in \mathbb{R}$, so that $\|PN(x^* + x_1)\|^2 = v_+^2$ and $\|x^* - PN(x^* + x_1)\|^2 = R^2 - 2Rv_+ + v_+^2$.

Thus assumption (13) is fulfilled if

$$v_+ \geqq 0\,. \tag{24}$$

Since for all $s \in \mathbb{R}$ we have $R + x_1(s - \tau) \geqq R - r \geqq d$, it follows from assumption (19) that $g(R + x_1(s - \tau)) \leqq \tilde{Q}$, which together with (21) implies (24).

b) $x^*(t) = -R$ for all $t \in \mathbb{R}$.

One shows analogously that assumption (13) reduces to

$$v_- \leqq 0 \tag{25}$$

with $v_- := (2\pi)^{-1} \int_0^{2\pi} [h(s) - g(-R + x_1(s - \tau))]\, ds$.

The proof of (25) follows as above from (20) and (21).

Thus the assumptions of Corollary 3 are fulfilled, which implies the existence of an $\bar{x} \in X$ with $Lx = Nx$ and $\|\bar{x}\| \leqq r + R$, i.e., (22) and (23).

It should be noted that Proposition 9 is sharp in the sense that for the linear case ($g = 0$, where we can choose $\tilde{P} = \tilde{Q} = 0$), (21) is necessary and sufficient for the existence of a 2π-periodic solution.

Also, at least for selfadjoint L, the Hilbert space version of Corollary 3 is not only sufficient, but also necessary for the existence of a solution in the linear case (i.e.: $Nx \equiv y$ with $y \in X$). For, in this case, condition (14) reads

$$\langle y, x^* \rangle \geqq 0 \quad \text{for all} \quad x^* \in \ker(L) \tag{26}$$

and hence also $\langle y, -x^* \rangle \geqq 0$ for all $x^* \in \ker(L)$. Thus, condition (14) is equivalent to

$$y \in \ker(L)^\perp \tag{27}$$

in this case. But this is the well-known Fredholm alternative, which is necessary and sufficient for solvability of the linear inhomogeneous equation $Lx = y$.

Corollary 10. *For all $\tau \in \mathbb{R}$, the equation*

$$x'(s) = \arctan(x(s - \tau)) + \sin(s)\,, \quad (s \in \mathbb{R}) \tag{28}$$

has a 2π-periodic continuously differentiable solution $\bar{x}$ with

$$\sup_{s\in\mathbb{R}} |\bar{x}(s)| \leqq 4\pi^2 + 33\pi/4 . \tag{29}$$

Remark 11. The following extension of Theorem 1 to problems involving two Banach spaces is of interest. Let X, Y be real Banach spaces, let $L: D(L) \to Y$ be a linear operator with domain $D(L) \subseteq X$ dense in X, $X_0 = \ker(L) \subseteq X$, and $Y_1 = \text{range}(L) \subseteq Y$. Let $N : X \to Y$ be continuous and bounded (not necessarily linear). Let us assume that

(1)′ $X_0 = \ker(L)$ is closed in X ; $\dim X_0 < +\infty$.

(2)′ $Y_1 = \text{range}(L)$ is closed in Y;

(3)′ $X = X_0 \oplus X_1$, $Y = Y_0 \oplus Y_1$, where $X_0 = PX$, $X_1 = (I - P)X$, $Y_0 = QY$, $Y_1 = (I - Q)Y$, and $P : X \to X$, $Q : Y \to Y$ are projectors in X and Y, respectively.

Then, (6) and (7) are replaced by the equations

$$x = Px + H(I - Q)Nx , \tag{6'}$$

$$QNx = 0 . \tag{7'}$$

If $S : Y_0 \to X_0$ is any continuous bounded operator (not necessarily linear) such that $S^{-1}(0) = 0$, then the system (6)′, (7)′ is equivalent to the fixed point problem $x = T_0 x$, where $T_0 x = Px + H(I - Q)Nx - SQNx$.

Under these assumptions, Theorem 1 still holds, where (9) and (10) are replaced by

(9)′ For $x \in X$ with $\|P(x - \bar{x})\| \leqq R$ and $\|(I - P)(x - \bar{x})\| = r$ we have $\|Nx - L\bar{x}\| \leqq \|H(I - Q)\|^{-1} r$.

(10)′ For $x \in X$ with $\|P(x - \bar{x})\| = R$ and $\|(I - P)(x - \bar{x})\| \leqq r$ we have $\|SQNx\|^2 \geqq \|P(x - \bar{x}) - SQNx\|^2 - R^2$.

The proof is the same. Whenever $X = Y$, $P = Q$, $S = I$, the present statement reduces to Theorem 1.

References

[1] *L. Cesari:* Functional analysis, nonlinear differential equations, and the alternative method, in: Nonlinear Functional Analysis and Differential Equations (L. Cesari, R. Kannan, J. Schuur, eds.), Dekker, New York, 1976, 1—197.

[2] *L. Cesari - R. Kannan:* An abstract existence theorem at resonance, Proc. Amer. Math. Soc. *63* (1977), 221—225.

[3] *L. Cesari:* Nonlinear oscillations across a point of resonance for nonselfadjoint systems, Journ. Differential Equations *28* (1978), 43—59.
[4] *L. Cesari:* Functional analysis and periodic solutions of nonlinear differential equations, in: Contributions to Differential Equations 1, Wiley, New York, 1963, 149—187.
[5] *K. Deimling:* Nichtlineare Gleichungen und Abbildungsgrade, Springer, Berlin, 1974.
[6] *S. Fučík:* Further remarks on a theorem by E. Landesman and A. C. Lazer, Comment. Math. Univ. Carol. *15* (1974), 259—271.
[7] *R. Kannan - P. McKenna:* An existence theorem by alternative methods for semilinear abstract equations at resonance, Boll. Un. Mat. Ital. (5) *14*-A (1977), 355—358.
[8] *R. B. Kellogg:* Uniqueness in the Schauder fixed point theorem, Proc. Amer. Math. Soc. *60* (1976), 207—210.
[9] *E. Landesman - A. Lazer:* Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. *19* (1970), 609—623.
[10] *A. Lazer - D. Leach:* Bounded perturbations of forced harmonic oscillations at resonance, Ann. Mat. Pura Appl. *72* (1969), 49—68.
[11] *N. Lloyd:* Degree Theory, Cambridge University Press, Cambridge, 1978.
[12] *S. A. Williams:* A connection between the Cesari and Leray-Schauder methods, Michigan Math. Journ. *15* (1968), 441—448.
[13] *S. A. Williams:* A sharp sufficient condition for solution of a nonlinear elliptic boundary value problem, Journ. Differential Equations *8* (1970), 580—586.

Authors' addresses: L. Cesari, Department of Mathematics, University of Michigan, Ann Arbor, Michigan, U.S.A.; H. W. Engl, Institut für Mathematik, Johannes-Kepler-Universität, Linz, Austria.