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ON A NONLINEAR INTEGRAL EQUATION

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This paper is a continuation and generalisation of the former author's work [2]. Let B be a Banach space, $I = [0, \infty)$, $\Omega = \{(t, s) \in I^2 : s \leqq t\}$. Let us consider functions φ, L, p, W satisfying the following conditions:

C1. $\varphi \in C^1(I, I)$; $\varphi(0) = 1$, $\lim_{t \to \infty} \varphi(t) = 0$; $(\forall t \in I)\ \varphi'(t) < 0$,

C2. $L \in C(I, I)$; $(\exists q > 0)(\forall t \in I)\ q\, L(t) + (\varphi'(t)/\varphi(t)) \geqq 0$,

C3. $p \in C(I \times B, B)$; $(\forall u \in B)\ p(0, u) = u$; $(\exists N \geqq 0)(\forall t \in I)\ \|p(t, 0)\| \leqq N$;
$(\exists M \geqq 1)(\forall t \in I)(\forall u, v \in B)\ \|p(t, u) - p(t, v)\| \leqq M\, \varphi(t)\, \|u - v\|$,

C4. $W \in C(\Omega \times B, B)$; $(\forall (t, s) \in \Omega)\ W(t, s, 0) = 0$;
$(\exists k \geqq 0)(\forall (t, s) \in \Omega)(\forall u, v \in B)\ \|W(t, s, u) - W(t, s, v)\| \leqq$
$\leqq L(s)\,(\varphi(t)/\varphi(s))\,[\max(\|u\|, \|v\|)]^k\, \|u - v\|$.

We shall examine an integral equation

$$u = p(t, u_0) + \int_0^t W(t, s, u)\, \mathrm{d}s \tag{1}$$

with $u_0 \in B$. We are interested in finding some regions in B, such that for every u_0 belonging to them the equation (1) has a solution on I, this solution is bounded and the solutions "starting" from B are convergent.

As examples of functions φ which satisfy the condition C1 let us mention

$$\varphi_1(t) = \mathrm{e}^{-\alpha t}, \quad \varphi_2(t) = \mathrm{e}^{-\alpha t^\beta}, \quad \varphi_3(t) = \frac{1}{(1 + \gamma t)^\alpha}$$

for $\alpha, \gamma > 0$, $\beta \in (0, 1]$. The inequality from C2 has for these functions the following forms:

$$q\, L(t) \leqq \alpha, \quad q\, L(t) \leqq \frac{\alpha\beta}{t^{1-\beta}}, \quad q\, L(t) \leqq \frac{\alpha\gamma}{1 + \gamma t}.$$

In the paper [2] the case $k = 0$ and $\varphi = \varphi_1$ was considered (the condition $q\, L(t) \leqq \alpha$ had the form $L(t) \leqq \alpha - \varepsilon$ for some $\varepsilon > 0$).

The equation of the type (1) can be obtained from the Cauchy problem $\dot{u} = A(t)\, u + f(t, u)$, $u(0) = u_0$ in the Banach space B, where A is a linear, not necessarily bounded operator in B. Then under some assumptions on A and f there exists a family of linear bounded operators $\{U(t, s) : (t, s) \in I^2\}$ such that every solution of the given problem satisfies the integral equation $u = U(t, 0)\, u_0 + \int_0^t U(t, s) \,.\, f(s, u)\, \mathrm{d}s$. Operator U is connected with the solutions of the equation $\dot{u} = A(t)\, u$ and in many cases it can be estimated by the inequality $\|U(t, s)\| \leqq M\, \mathrm{e}^{\alpha(t-s)}$ for some $\alpha, M \in R$ (see for example [1]). In some cases we have only a weaker estimate for the operator U: $\|U(t, s)\| \leqq M\, \varphi(t)/\varphi(s)$ with a decreasing function φ, for example $\varphi = \varphi_3$. In this paper we are interested in general in this case.

Example 1. Let functions φ, L satisfy the conditions C1, C2 with some constants $k \geqq 0$, $M > 0$, $q > 0$, $P \geqq 0$. Assume that U, f satisfy the following inequalities:

$$\|U(t, s)\| \leqq M \frac{\varphi(t)}{\varphi(s)}, \quad \|f(t, 0)\| \leqq P\, L(t),$$

$$\|f(s, u) - f(s, v)\| \leqq \frac{1}{M} L(s) \left[\max\left(\|u\|, \|v\|\right)\right]^k \|u - v\|,$$

where as usual $U(s, s) = I$ (the identity operator). Then the integral equation

$$u = U(t, 0)\, u_0 + \int_0^t U(t, s) f(s, u)\, \mathrm{d}s$$

has the form (1) and for suitable p and W the conditions C3, C4 are satisfied.

Now we give two concrete examples (in R and R^2).

Example 2. Consider the differential equation

$$\dot{u} = -\frac{u}{1 + t} + \frac{u^{k+1}}{1 + t} + \gamma(t)$$

with the initial condition $u(0) = u_0$. It is easy to transform this problem to the integral equation

$$u = \frac{u_0}{1 + t} + \int_0^t \frac{1 + s}{1 + t} \gamma(s)\, \mathrm{d}s + \int_0^t \frac{1}{1 + t} u^{k+1}\, \mathrm{d}s.$$

Take

$$p(t, u) = \frac{u}{1 + t} + \int_0^t \frac{1 + s}{1 + t} \gamma(s)\, \mathrm{d}s, \quad W(t, s, u) = \frac{1}{1 + t} u^{k+1}.$$

Suppose that for a constant $N \geqq 0$ we have $|\gamma(t)| \leqq N$. Take

$$\varphi(t) = \frac{1}{1 + t}, \quad M = 1, \quad L(t) = \frac{1 + k}{1 + t}, \quad q = \frac{1}{1 + k}.$$

Then it is easy to see that the conditions C1 – C4 are satisfied. We give here only the proof of the inequality from C4. We have

$$\left|W(t, s, u) - W(t, s, v)\right| = \frac{1}{1+t}\left|u^{k+1} - v^{k+1}\right| = \frac{1}{1+t}\left|\sum_{j=0}^{k} u^j v^{k-j}\right|\left|u - v\right| .$$

But

$$\left|\sum_{j=0}^{k} u^j v^{k-j}\right| \leqq \sum_{j=0}^{k} |u|^j\,|v|^{k-j} \leqq \sum_{j=0}^{k} [\max(|u|, |v|)]^k = (k+1)\,[\max(|u|, |v|)]^k$$

and

$$L(s)\frac{\varphi(t)}{\varphi(s)} = \frac{1+k}{1+s}\,\frac{1+s}{1+t} = \frac{1+k}{1+t},$$

hence

$$\left|W(t, s, u) - W(t, s, v)\right| \leqq L(s)\frac{\varphi(t)}{\varphi(s)}[\max(|u|, |v|)]^k\,|u - v|$$

and C4 is satisfied.

Example 3. Consider the second order equation

$$\ddot{x} + \frac{4}{(t+1)^2}\dot{x} + \frac{2}{(t+1)^2}x = f(t, x, \dot{x}) .$$

Taking

$$u = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

we can rewrite this equation in the form of a system $\dot{u} = A(t)\,u + g(t, u)$ in R^2. The functions

$$x_1 = \frac{1}{1+t}, \quad x_2 = \frac{1}{(1+t)^2}$$

are linearly independent solutions of the corresponding homogeneous equation and thus the fundamental matrix W, $W(0) = I$, of the system $\dot{u} = A(t)\,u$ has the form

$$W(t) = \begin{bmatrix} \dfrac{2t+1}{(t+1)^2} & \dfrac{t}{(t+1)^2} \\ \dfrac{-2t}{(t+1)^3} & \dfrac{-t+1}{(t+1)^3} \end{bmatrix}.$$

Then

$$W^{-1}(t) = \begin{bmatrix} -t^2 + 1 & -t(t+1)^2 \\ 2t(t+1) & (2t+1)(t+1)^2 \end{bmatrix}$$

and for a suitable matrix norm we obtain

$$\|U(t,s)\| = \|W(t)\,W^{-1}(s)\| \leqq A\,\frac{(s+1)^3}{t+1}$$

with a constant A. We can see that this example is different from the first because there exists no function φ from C1 with $\|U(t, s)\| \leqq \varphi(t)/\varphi(s)$. However, this example can be treated as an example to our considerations. Suppose for simplicity that

$$f(t, x, \dot{x}) = \frac{1}{(t+1)^\gamma}\, x^{k+1}$$

for some $\gamma \geqq 1$ and $k \geqq 0$. Then

$$|f(t, x, \dot{x}) - f(t, y, \dot{y})| = \frac{1}{(t+1)^\gamma}\,|x^{k+1} - y^{k+1}| \leqq \frac{k+1}{(t+1)^\gamma}\,[\max(|x|, |y|)]^k\,|x - y|$$

for every $x, y, \dot{x}, \dot{y} \in R$ and if

$$u = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad v = \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

then

$$\|g(t, u) - g(t, v)\| = |f(t, x, \dot{x}) - f(t, y, \dot{y})| \leqq$$

$$\leqq \frac{k+1}{(t+1)^\gamma}\,[\max(|x|, |y|)]^k\,|x - y| \leqq \frac{k+1}{(t+1)^\gamma}\,[\max(\|u\|, \|v\|)]^k\,\|u - v\|\,.$$

If we put $p(t, u) = U(t, 0)\,u$, $W(t, s, u) = U(t, s)\,g(s, u)$ and

$$\varphi(t) = \frac{1}{1+t}, \quad M = A\,, \quad L(t) = \frac{k+1}{(1+t)^\gamma}, \quad q = \frac{1}{k+1},$$

then the conditions C1 – C4 are fulfilled for the integral equation

$$u = U(t, 0)\,u_0 + \int_0^t U(t, s)\,g(s, u)\,\mathrm{d}s\,,$$

which is equivalent to the differential equation given at the beginning of this example.

Theorem 1 ($k = 0$). *If the functions in the equation* (1) *satisfy the conditions* C1 – C4 *and* $q > 1$, *then for any* $u_0 \in B$ *the equation* (1) *has exactly one solution on* I (*and on any interval* $[0, T]$, $T > 0$). *This solution is bounded.*

Theorem 1′ ($k > 0$). *Let* R *be any fixed number such that* $0 < R < q^{1/k}$ *and let* $N_0 = R - (1/q)\,R^{k+1}$. *Then for any* $N < N_0$ *and for any* $u_0 \in B$ *such that* $\|u_0\| \leqq$

$\leqq R^{k+1}/qM$, *the equation* (1) *has exactly one solution* u *on* I (*and on any interval* $[0, T]$, $T > 0$); *this solution is bounded* (*i.e.* $(\forall t \in I)\ \|u(t)\| \leqq R$).

Proof of Theorems 1 and 1′. For $k = 0$ let X be the set of all continuous and bounded functions from I to B, and for $k > 0$ let X be the set of all continuous functions from I to B bounded by the number R (i.e. if $u \in X$ then $(\forall t \in I)\ \|u(t)\| \leqq R$). Let $|||u||| = \sup_{t \in I} \|u(t)\|$ for $u \in X$, then $(X, |||\cdot|||)$ is a Banach space with a metric $\varrho(u, v) = |||v - u|||$. In the case $k > 0$, if $u \in X$, then $|||u||| \leqq R$. Consider on X an operator K such that for any $u \in X$

$$(2) \qquad (Ku)(t) = p(t, u_0) + \int_0^t W(t, s, u(s))\, \mathrm{d}s\,.$$

It is easy to show that $Ku \in C(I, B)$ for any $u \in X$. We shall prove that $K : K \to X$. The formulae (2) and C3, C4 imply

$$\|(Ku)(t)\| \leqq \|p(t, u_0)\| + \left\| \int_0^t W(t, s, u(s))\, \mathrm{d}s \right\| \leqq$$

$$\leqq \|p(t, 0)\| + \|p(t, u_0) - p(t, 0)\| + \int_0^t \|W(t, s, u(s))\|\, \mathrm{d}s \leqq$$

$$\leqq N + M\, \varphi(t)\, \|u_0\| + \int_0^t L(s) \frac{\varphi(t)}{\varphi(s)} [\max(\|u(s)\|, 0)]^k\, \|u(s)\|\, \mathrm{d}s\,.$$

But $\max(\|u(s)\|, 0) = \|u(s)\| \leqq \sup_{s \in I} \|u(s)\| = |||u|||$, thus

$$\|(Ku)(t)\| \leqq N + M\|u_0\|\, \varphi(t) + |||u|||^{k+1} \int_0^t L(s) \frac{\varphi(t)}{\varphi(s)}\, \mathrm{d}s\,.$$

From C1, C2 we have

$$(3) \qquad \int_0^t L(s) \frac{\varphi(t)}{\varphi(s)}\, \mathrm{d}s \leqq -\frac{1}{q}\, \varphi(t) \int_0^t \frac{\varphi'(s)}{\varphi^2(s)}\, \mathrm{d}s = \frac{1}{q}(1 - \varphi(t))\,,$$

hence

$$(4) \qquad \|(Ku)(t)\| \leqq N + M\|u_0\|\, \varphi(t) + \frac{1}{q}(1 - \varphi(t))\, |||u|||^{k+1}\,.$$

For $k = 0$ we obtain that

$$\|(Ku)(t)\| \leqq N + M\|u_0\| + \frac{1}{q}\, |||u|||$$

(because $0 < \varphi(t) \leqq 1$), hence $K|||u||| \leqq N + M\|u_0\| + (1/q)\, |||u|||$; Ku is a bounded continuous function, $Ku \in X$, and finally for $k = 0$ we have $K : X \to X$.

Let now $k > 0$. We consider then such functions u that $|||u||| \leqq R$, such u_0 that $\|u_0\| \leqq 1/Mq) R^{k+1}$, and such N that $N \leqq N_0 = R - (1/q) R^{k+1}$. Then by the inequality (4)

$$\|(Ku)(t)\| \leqq R - \frac{1}{q} R^{k+1} + \frac{1}{q} R^{k+1} \varphi(t) + \frac{1}{q}(1 - \varphi(t)) R^{k+1} = R,$$

i.e. $|||Ku||| \leqq R$, $K : X \to X$.

Now we want to prove that K is a contractive operator. For $u, v \in X$ we have

$$\|(Ku)(t) - (Kv)(t)\| = \left\| \int_0^t W(t, s, u(s)) - W(t, s, v(s)) \, ds \right\| \leqq$$

$$\leqq \int_0^t \|W(t, s, u(s)) - W(t, s, v(s))\| \, ds \leqq$$

$$\leqq \int_0^t L(s) \frac{\varphi(t)}{\varphi(s)} [\max(\|u(s)\|, \|v(s)\|)]^k \|u(s) - v(s)\| \, ds .$$

But $\max(\|u(s)\|, \|v(s)\|) \leqq \max(|||u|||, |||v|||)$ and furthermore $|||u||| \leqq R$, $|||v||| \leqq R$, hence $\max(\|u(s)\|, \|v(s)\|) \leqq R$; taking into account also that $\|u(s) - v(s)\| \leqq$ $\leqq |||u - v|||$ we obtain

$$\|(Ku)(t) - (Kv)(t)\| \leqq R^k |||u - v||| \int_0^t L(s) \frac{\varphi(t)}{\varphi(s)} \, ds \leqq R^k |||u - v||| \frac{1}{q},$$

i.e.

$$|||Ku - Kv||| \leqq \frac{1}{q} R^k |||u - v|||$$

or

$$\varrho(Ku, Kv) \leqq \alpha \varrho(u, v), \quad \text{where} \quad \alpha = \frac{1}{q} R^k .$$

In the case $k = 0$ we have $\alpha = 1/q$, but $q > 1$ and so $\alpha < 1$; in the case $k > 0$ we have $R < q^{1/k}$, hence $R^k < q$ and $(1/q) R^k < 1$, so $\alpha < 1$. The operator K is contractive.

From Banch's principle it follows that the equation (1) has a unique solution on I, this solution is bounded (for $k = 0$ we have shown it directly, for $k > 0$ it results from the definition of the space X). For the proof that the equation (1) has a unique solution on every interval $[0, T]$, $T > 0$, it is sufficient to repeat the same arguments as above for the space X of the same functions as before but defined only on $[0, T]$.

Theorem 2. *The solutions considered in Theorems* 1 *and* 1' *are convergent (i.e. for any two such solutions* u, v $\|u(t) - v(t)\| \to 0$ *as* $t \to \infty$).

Proof. Let u, v be solutions of the equation (1) satisfying the conditions of Theorem 1 or 1', let $u(0) = u_0$, $v(0) = v_0$. Then

$$u(t) - v(t) = p(t, u_0) - p(t, v_0) + \int_0^t [W(t, s, u(s)) - W(t, s, v(s))] \, ds$$

and

$$\|u(t) - v(t)\| \leqq M \varphi(t) \|u_0 - v_0\| + \\ + \int_0^t L(s) \frac{\varphi(t)}{\varphi(s)} [\max(\|u(s)\|, \|v(s)\|)]^k \|u(s) - v(s)\| \, ds .$$

But $[\max(\|u(s)\|, \|v(s)\|)]^k = 1$ for $k = 0$, and $\max(\|u(s)\|, \|v(s)\|) \leqq R$ for $k > 0$, hence

$$\|u(t) - v(t)\| \leqq M \|u_0 - v_0\| \varphi(t) + \int_0^t R^k L(s) \frac{\varphi(t)}{\varphi(s)} \|u(s) - v(s)\| \, ds$$

and

$$\frac{\|u(t) - v(t)\|}{\varphi(t)} \leqq M \|u_0 - v_0\| + \int_0^t R^k L(s) \frac{\|u(s) - v(s)\|}{\varphi(s)} \, ds .$$

This inequality yields

$$\frac{\|u(t) - v(t)\|}{\varphi(t)} \leqq M \|u_0 - v_0\| \exp\left(R^k \int_0^t L(s) \, ds\right)$$

and

$$\|u(t) - v(t)\| \leqq M \|u_0 - v_0\| \varphi(t) \exp\left(R^k \int_0^t L(s) \, ds\right) .$$

The condition C2 implies

$$q L(t) + \frac{\varphi'(t)}{\varphi(t)} \leqq 0 ,$$

hence

$$q \int_0^t L(s) \, ds + \int_0^t \frac{\varphi'(s)}{\varphi(s)} \, ds \leqq 0 .$$

We obtain

$$q \int_0^t L(s) \, ds + \ln \varphi(t) \leqq 0 , \quad R^k \int_0^t L(s) \, ds \leqq - \frac{R^k}{q} \ln \varphi(t)$$

and finally

$$\exp\left(R^k \int_0^t L(s) \, ds\right) \leqq [\varphi(t)]^{-R^k/q} .$$

Thus

$$\|u(t) - v(t)\| \leqq M\|u_0 - v_0\| \, [\varphi(t)]^{1-(R^k/q)} . \tag{5}$$

For $k = 0$ we have $q > 1$, hence $1 - (R^k/q) = (q - 1)/q > 0$; for $k > 0$ we have $R < q^{1/k}$, thus $R^k < q$ and $1 - (R^k/q) = (q - R^k)/q > 0$. Since $\varphi(t) \to 0$ as $t \to \infty$, then in both cases $\|u(t) - v(t)\| \to 0$ as $t \to \infty$.

Remark 1. If $p(t, 0) = 0$, then the equation (1) has the trivial solution $u = 0$; Theorem 2 implies that in this case all solutions considered tend to zero as $t \to \infty$.

Remark 2. The inequality (5) which was shown in the proof of Theorem 2 implies that all solutions described are asymptotically stable.

Remark 3. Consider the equation from Example 2 with $k = 1$ and $\gamma(t) = N/(1 + t)$. The solutions of this equation have the forms:
if $N < \frac{1}{4}$ then

$$u = \frac{1}{2} + \frac{B(u_0 - \frac{1}{2} + B) + B(u_0 - \frac{1}{2} - B)(1 + t)^{2B}}{(u_0 - \frac{1}{2} + B) - (u_0 - \frac{1}{2} - B)(1 + t)^{2B}}$$

where $B = \sqrt{(\frac{1}{4} - N)}$;
if $N = \frac{1}{4}$ then

$$u = \frac{1}{2} + \frac{2u_0 - 1}{2 - (2u_0 - 1)\ln(1 + t)} ;$$

if $N > \frac{1}{4}$ then

$$u = \frac{1}{2} + B\frac{(2u_0 - 1) + 2B \operatorname{tg}(B \ln(1 + t))}{2B - (2u_0 - 1)\operatorname{tg}(B \ln(1 + t))}$$

where $B = \sqrt{(N - \frac{1}{4})}$. These solutions are defined (and bounded) on I only if $N \leqq \frac{1}{4}$ and $u_0 \leqq \frac{1}{2} + \sqrt{(\frac{1}{4} - N)}$. This result shows that the restrictions of R, N and u_0 in the theorems arise from the nature of the problem.

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