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ON LINEAR FUNCTIONS ON THE SPHERE S^2

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1. Let S^2 be a unit sphere in E^3 . Let D be a domain in S^2 . A function $f: D \rightarrow R$ is called *linear* if

$$(1) \quad f(M) = \langle \mathbf{m}, \mathbf{a} \rangle + k,$$

where \mathbf{a} is a constant vector, \mathbf{m} is the position vector of the point $M \in S^2$ with respect to the centre of S^2 , $k \in R$, and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in E^3 . The linear function f is called *homogeneous* or *non-homogeneous* if $k = 0$ or $k \neq 0$, respectively.

In [1], A. Švec found certain conditions for a function f to be linear and homogeneous. These conditions are expressed in terms of partial differential equations on D or on the boundary ∂D of D .

The aim of this paper is to extend the results obtained by A. Švec to the wider class of non-homogeneous linear functions on the domains in S^2 .

2. Let us introduce some notations (see [1]). Consider the unit sphere $S^2 \subset E^3$. With each point M of S^2 , let us associate a tangent orthonormal frame $\{\mathbf{m}, v_1, v_2, v_3\}$ such that \mathbf{m} is the position vector of the point $M \in S^2$, v_1, v_2 are tangent vectors to S^2 at M , and v_3 is a normal vector to S^2 at M . Then we have

$$\begin{aligned} d\mathbf{m} &= \omega^1 v_1 + \omega^2 v_2, & dv_1 &= \omega_1^2 v_2 + \omega^1 v_3, \\ dv_2 &= -\omega_1^2 v_1 + \omega^2 v_3, & dv_3 &= -\omega^1 v_1 - \omega^2 v_2. \end{aligned}$$

Let $f: S^2 \rightarrow R$ be a function. Recall that the covariant derivatives $\mathbf{f}_i, \mathbf{f}_{ij}, P, \dots, S, T_1, \dots, T_5$ ($i, j = 1, 2$) of f with respect to a field of tangent orthonormal frames $\{\mathbf{m}, v_k\}$ ($k = 1, 2, 3$) are defined by the following formulas:

$$(2) \quad df = \mathbf{f}_1 \omega^1 + \mathbf{f}_2 \omega^2;$$

$$(3) \quad \begin{aligned} d\mathbf{f}_1 - \mathbf{f}_2 \omega_1^2 &= \mathbf{f}_{11} \omega^1 + \mathbf{f}_{12} \omega^2, \\ d\mathbf{f}_2 + \mathbf{f}_1 \omega_1^2 &= \mathbf{f}_{12} \omega^1 + \mathbf{f}_{22} \omega^2; \end{aligned}$$

$$\begin{aligned}
(4) \quad & d\mathbf{f}_{11} - 2\mathbf{f}_{12}\omega_1^2 = P\omega^1 + Q\omega^2, \\
& d\mathbf{f}_{12} + (\mathbf{f}_{11} - \mathbf{f}_{22})\omega_1^2 = (Q + \mathbf{f}_2)\omega^1 + (R + \mathbf{f}_1)\omega^2, \\
& d\mathbf{f}_{22} + 2\mathbf{f}_{12}\omega_1^2 = R\omega^1 + S\omega^2; \\
(5) \quad & dP - (3Q + 2\mathbf{f}_2)\omega_1^2 = T_1\omega^1 + T_2\omega^2, \\
& dQ + (P - 2R - 2\mathbf{f}_1)\omega_1^2 = (T_2 + 2\mathbf{f}_{12})\omega^1 + (T_3 + 2\mathbf{f}_{11})\omega^2, \\
& dR + (2Q - S + 2\mathbf{f}_2)\omega_1^2 = (T_3 + 2\mathbf{f}_{22})\omega^1 + (T_4 + 2\mathbf{f}_{12})\omega^2, \\
& dS + (3R + 2\mathbf{f}_1)\omega_1^2 = T_4\omega^1 + T_5\omega^2.
\end{aligned}$$

By means of these covariant derivatives, one can introduce the following differential operators \mathcal{L} and \mathcal{M} , which play an important role in investigations of the linearity conditions (1):

$$\begin{aligned}
\mathcal{L}\mathbf{f} &= \mathbf{f}_{11} + \mathbf{f}_{22} + 2\mathbf{f}, \\
\mathcal{M}\mathbf{f} &= \mathbf{f}_{11}\mathbf{f}_{22} - \mathbf{f}_{12}^2 + \mathbf{f}(\mathbf{f}_{11} + \mathbf{f}_{22} + \mathbf{f}).
\end{aligned}$$

3. Let $D \subset S^2$ be a domain, ∂D the boundary of D , $\bar{D} = D \cup \partial D$, and let $\mathbf{f}: \bar{D} \rightarrow R$ be a function. In all proofs, we shall use the following

Lemma. *If $L = (\mathcal{L}\mathbf{f})^2 - 4\mathcal{M}\mathbf{f} = 0$ on \bar{D} , then \mathbf{f} is linear on \bar{D} .*

Proof. Supposition $L = (\mathcal{L}\mathbf{f})^2 - 4\mathcal{M}\mathbf{f} = (\mathbf{f}_{11} - \mathbf{f}_{22})^2 + 4\mathbf{f}_{12}^2 = 0$ yields $\mathbf{f}_{11} - \mathbf{f}_{22} = \mathbf{f}_{12} = 0$, and from (4) we get $P = R$, $Q = S$, $Q = -\mathbf{f}_2$, $R = -\mathbf{f}_1$. Then $d\mathbf{f}_{11} = d\mathbf{f}_{22} = P\omega^1 + Q\omega^2 = -\mathbf{f}_1\omega^1 - \mathbf{f}_2\omega^2 = -d\mathbf{f}$. This implies that $\mathbf{f}_{11} = \mathbf{f}_{22} = -\mathbf{f} + c$, where c is an integral constant. Now, let us consider the vector field

$$(6) \quad \mathbf{a} = -\mathbf{f}_1v_1 - \mathbf{f}_2v_2 + (\mathbf{f} - c)v_3$$

on \bar{D} . Then $d\mathbf{a} = 0$ and hence \mathbf{a} is a constant vector. From (6) we get $\mathbf{f} - c = \langle v_3, \mathbf{a} \rangle$. QED.

4. Let \mathcal{L} , \mathcal{M} , L be given as in Sections 2 and 3. In the proofs of the following Theorems 1–8, we shall use the maximum principle in the form described in [2].

Theorem 1. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f}: \bar{D} \rightarrow R$ be a function. If*

1. $L = 0$ on ∂D ,
 2. $(\mathbf{f}_{11} - \mathbf{f}_{22})[(\mathcal{L}\mathbf{f})_{11} - (\mathcal{L}\mathbf{f})_{22}] + 4\mathbf{f}_{12}(\mathcal{L}\mathbf{f})_{12} \geq 0$ on D ,
- then \mathbf{f} is linear on \bar{D} .

Proof. Consider the covariant derivatives of the functions $\mathcal{L}\mathbf{f}$ and L . The formulas (2)–(5) immediately lead to the expressions

$$(7) \quad (\mathcal{L}\mathbf{f})_1 = P + R + 2\mathbf{f}_1, \quad (\mathcal{L}\mathbf{f})_2 = Q + S + 2\mathbf{f}_2;$$

$$\begin{aligned}
(8) \quad & (\mathcal{L}\mathbf{f})_{11} = 2(\mathbf{f}_{11} + \mathbf{f}_{22}) + T_1 + T_3, \\
& (\mathcal{L}\mathbf{f})_{12} = 4\mathbf{f}_{12} + T_2 + T_4, \\
& (\mathcal{L}\mathbf{f})_{22} = 2(\mathbf{f}_{11} + \mathbf{f}_{22}) + T_3 + T_5; \\
(9) \quad & L_1 = 2(\mathbf{f}_{11} - \mathbf{f}_{22})(P - R) + 8\mathbf{f}_{12}(Q + \mathbf{f}_2), \\
& L_2 = 2(\mathbf{f}_{11} - \mathbf{f}_{22})(Q - S) + 8\mathbf{f}_{12}(R + \mathbf{f}_1); \\
(10) \quad & L_{11} = 2(P - R)^2 + 8(Q + \mathbf{f}_2)^2 - 4\mathbf{f}_{22}(\mathbf{f}_{11} - \mathbf{f}_{22}) + 24\mathbf{f}_{12}^2 + \\
& \quad + 2(\mathbf{f}_{11} - \mathbf{f}_{22})T_1 + 8\mathbf{f}_{12}T_2 - 2(\mathbf{f}_{11} - \mathbf{f}_{22})T_3, \\
& L_{12} = 2(P - R)(Q - S) + 8(Q + \mathbf{f}_2)(R + \mathbf{f}_1) + 12\mathbf{f}_{12}(\mathbf{f}_{11} + \mathbf{f}_{22}) + \\
& \quad + 2(\mathbf{f}_{11} - \mathbf{f}_{22})T_2 + 8\mathbf{f}_{12}T_3 - 2(\mathbf{f}_{11} - \mathbf{f}_{22})T_4, \\
& L_{22} = 2(Q - S)^2 + 8(R + \mathbf{f}_1)^2 + 4\mathbf{f}_{11}(\mathbf{f}_{11} - \mathbf{f}_{22}) + 24\mathbf{f}_{12}^2 + \\
& \quad + 2(\mathbf{f}_{11} - \mathbf{f}_{22})T_3 + 8\mathbf{f}_{12}T_4 - 2(\mathbf{f}_{11} - \mathbf{f}_{22})T_5.
\end{aligned}$$

Eliminating T_1, \dots, T_5 from (8) nad (10), one obtains

$$\begin{aligned}
L_{11} + L_{22} - 4L &= 2(\mathbf{f}_{11} - \mathbf{f}_{22})[(\mathcal{L}\mathbf{f})_{11} - (\mathcal{L}\mathbf{f})_{22}] + 8\mathbf{f}_{12}(\mathcal{L}\mathbf{f})_{12} + \\
& \quad + 8(Q + \mathbf{f}_2)^2 + 8(R + \mathbf{f}_1)^2 + 2(P - R)^2 + 2(Q - S)^2.
\end{aligned}$$

Now we can conclude from 2. that this expression satisfies the conditions of the maximum principle for the function L . Thus 1. implies $L = 0$ on \bar{D} and the theorem follows from Lemma. QED.

Theorem 2. Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f}: \bar{D} \rightarrow R$ be a function. If

1. $L = 0$ on ∂D ,
2. $(\mathbf{f}_{11} - \mathbf{f}_{22})[(\mathcal{M}\mathbf{f})_{11} - (\mathcal{M}\mathbf{f})_{22}] + 4\mathbf{f}_{12}(\mathcal{M}\mathbf{f})_{12} \geq 0$ on D ,
3. $\mathcal{M}\mathbf{f} > 0$, $\mathcal{L}\mathbf{f} \geq 0$ on D ,

then \mathbf{f} is linear on \bar{D} .

Proof. Consider the covariant derivatives of the function $\mathcal{M}\mathbf{f}$. We directly obtain from (2)–(5) the identities

$$\begin{aligned}
(11) \quad & (\mathcal{M}\mathbf{f})_1 = (\mathbf{f}_{22} + \mathbf{f})(P + \mathbf{f}_1) + (\mathbf{f}_{11} + \mathbf{f})(R + \mathbf{f}_1) - 2\mathbf{f}_{12}(Q + \mathbf{f}_2), \\
& (\mathcal{M}\mathbf{f})_2 = (\mathbf{f}_{22} + \mathbf{f})(Q + \mathbf{f}_2) + (\mathbf{f}_{11} + \mathbf{f})(S + \mathbf{f}_2) - 2\mathbf{f}_{12}(R + \mathbf{f}_1);
\end{aligned}$$

$$\begin{aligned}
(12) \quad (\mathcal{M}\mathbf{f})_{11} &= 2(P + \mathbf{f}_1)(R + \mathbf{f}_1) - 2(Q + \mathbf{f}_2)^2 + \mathcal{L}\mathbf{f} \cdot \mathbf{f}_{11} - 6\mathbf{f}_{12}^2 + \\
&\quad + 2\mathbf{f}_{22}(\mathbf{f}_{11} + \mathbf{f}) + (\mathbf{f}_{22} + \mathbf{f})T_1 - 2\mathbf{f}_{12}T_2 + (\mathbf{f}_{11} + \mathbf{f})T_3, \\
(\mathcal{M}\mathbf{f})_{12} &= -(R + \mathbf{f}_1)(Q + \mathbf{f}_2) + (S + \mathbf{f}_2)(P + \mathbf{f}_1) + \mathcal{L}\mathbf{f} \cdot \mathbf{f}_{12} + \\
&\quad + 2\mathbf{f}_{12}(\mathbf{f} - \mathbf{f}_{11} - \mathbf{f}_{22}) + (\mathbf{f}_{22} + \mathbf{f})T_2 - 2\mathbf{f}_{12}T_3 + \\
&\quad + (\mathbf{f}_{11} + \mathbf{f})T_4, \\
(\mathcal{M}\mathbf{f})_{22} &= 2(Q + \mathbf{f}_2)(S + \mathbf{f}_2) - 2(R + \mathbf{f}_1)^2 + \mathcal{L}\mathbf{f} \cdot \mathbf{f}_{22} - 6\mathbf{f}_{12}^2 + \\
&\quad + 2\mathbf{f}_{11}(\mathbf{f}_{22} + \mathbf{f}) + (\mathbf{f}_{22} + \mathbf{f})T_3 - 2\mathbf{f}_{12}T_4 + (\mathbf{f}_{11} + \mathbf{f})T_5.
\end{aligned}$$

(10) and (12) yield

$$\begin{aligned}
&(\mathbf{f}_{22} + \mathbf{f})L_{11} - 2\mathbf{f}_{12}L_{12} + (\mathbf{f}_{11} + \mathbf{f})L_{22} - 2\mathcal{L}\mathbf{f}L = \\
&= 2(\mathbf{f}_{11} - \mathbf{f}_{22})[(\mathcal{M}\mathbf{f})_{11} - (\mathcal{M}\mathbf{f})_{22}] + 8\mathbf{f}_{12}(\mathcal{M}\mathbf{f})_{12} + \\
&\quad + 2(\mathbf{f}_{11} + \mathbf{f})(Q + S + 2\mathbf{f}_2)^2 - 4(P + R + 2\mathbf{f}_1)(Q + S + 2\mathbf{f}_2)\mathbf{f}_{12} + \\
&\quad + 2(\mathbf{f}_{22} + \mathbf{f})(P + R + 2\mathbf{f}_1)^2 + 4(\mathcal{L}\mathbf{f})[(R + \mathbf{f}_1)(R - P) + (Q + \mathbf{f}_2)(Q - S)].
\end{aligned}$$

From (9) we get

$$\begin{aligned}
&(R + \mathbf{f}_1)L_1 - (Q + \mathbf{f}_2)L_2 = \\
&= -2(\mathbf{f}_{11} - \mathbf{f}_{22})[(R + \mathbf{f}_1)(R - P) + (Q + \mathbf{f}_2)(Q - S)]
\end{aligned}$$

and hence $[(R + \mathbf{f}_1)(R - P) + (Q + \mathbf{f}_2)(Q - S)] = -\varrho L_1 + \sigma L_2$, where ϱ, σ satisfy $2(\mathbf{f}_{11} - \mathbf{f}_{22})\varrho = R + \mathbf{f}_1$, $2(\mathbf{f}_{11} - \mathbf{f}_{22})\sigma = Q + \mathbf{f}_2$. Thus we have

$$\begin{aligned}
(13) \quad (\mathbf{f}_{22} + \mathbf{f})L_{11} - 2\mathbf{f}_{12}L_{12} + (\mathbf{f}_{11} + \mathbf{f})L_{22} + 4\mathcal{L}\mathbf{f}\varrho L_1 - 4\mathcal{L}\mathbf{f}\sigma L_2 - 2\mathcal{L}\mathbf{f}L = \\
= 2(\mathbf{f}_{11} - \mathbf{f}_{22})[(\mathcal{M}\mathbf{f})_{11} - (\mathcal{M}\mathbf{f})_{22}] + 8\mathbf{f}_{12}(\mathcal{M}\mathbf{f})_{12} + \\
+ 2(\mathbf{f}_{11} + \mathbf{f})(Q + S + 2\mathbf{f}_2)^2 - 4\mathbf{f}_{12}(P + R + 2\mathbf{f}_1)(Q + S + 2\mathbf{f}_2) + \\
+ 2(\mathbf{f}_{22} + \mathbf{f})(P + R + 2\mathbf{f}_1)^2.
\end{aligned}$$

The quadratic form on the right hadn side is positive definite because of 3. This implies that the expression (13) satisfies the conditions of the maximum principle for the function L , and we must have $L = 0$ on \bar{D} . The theorem now follows from Lemma. QED.

Theorem 3. Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \rightarrow R$ be a function. If

1. $L = 0$ on ∂D ,
 2. $\mathcal{L}\mathbf{f}[(\mathcal{L}\mathbf{f})_{11} + (\mathcal{L}\mathbf{f})_{22}] - 2[(\mathcal{M}\mathbf{f})_{11} + (\mathcal{M}\mathbf{f})_{22}] \geq 0$ on D ,
- then \mathbf{f} is linear on \bar{D} .

Proof. From (8), (10) and (12) we get

$$L_{11} + L_{22} = 2\mathcal{L}f[(\mathcal{L}f)_{11} + (\mathcal{L}f)_{22}] - 4[(\mathcal{M}f)_{11} + (\mathcal{M}f)_{22}] + \\ + 2(P + R + 2\mathbf{f}_1)^2 + 2(Q + S + 2\mathbf{f}_2)^2.$$

This expression satisfies the conditions of the maximum principle for the function L , and we must have $L = 0$ on \bar{D} . The theorem now follows from Lemma. QED.

Theorem 4. Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \rightarrow R$ be a function. If

1. $L = 0$ on ∂D ,
 2. $\mathcal{L}f[(\mathbf{f}_{22} + \mathbf{f})(\mathcal{L}f)_{11} - 2\mathbf{f}_{12}(\mathcal{L}f)_{12} + (\mathbf{f}_{11} + \mathbf{f})(\mathcal{L}f)_{22}] - \\ - 2[(\mathbf{f}_{22} + \mathbf{f})(\mathcal{M}f)_{11} - 2\mathbf{f}_{12}(\mathcal{M}f)_{12} + (\mathbf{f}_{11} + \mathbf{f})(\mathcal{M}f)_{22}] \geq 0$ on D ,
 3. $\mathcal{M}f > 0$, $\mathcal{L}f \geq 0$ on D ,
- then \mathbf{f} is linear on \bar{D} .

Proof. From (8), (10) and (12) we get

$$(14) \quad (\mathbf{f}_{22} + \mathbf{f})L_{11} - 2\mathbf{f}_{12}L_{12} + (\mathbf{f}_{11} + \mathbf{f})L_{22} = \\ = 2\mathcal{L}f[(\mathbf{f}_{22} + \mathbf{f})(\mathcal{L}f)_{11} - 2\mathbf{f}_{12}(\mathcal{L}f)_{12} + (\mathbf{f}_{11} + \mathbf{f})(\mathcal{L}f)_{22}] - \\ - 4[(\mathbf{f}_{22} + \mathbf{f})(\mathcal{M}f)_{11} - 2\mathbf{f}_{12}(\mathcal{M}f)_{12} + (\mathbf{f}_{11} + \mathbf{f})(\mathcal{M}f)_{22}] + \\ + 2(\mathbf{f}_{22} + \mathbf{f})(P + R + 2\mathbf{f}_1)^2 - 4\mathbf{f}_{12}(P + R + 2\mathbf{f}_1)(Q + S + 2\mathbf{f}_2) + \\ + 2(\mathbf{f}_{11} + \mathbf{f})(Q + S + 2\mathbf{f}_2)^2.$$

The quadratic form on the right hand side is positive definite because of 3. This implies that the expression (14) satisfies the conditions of the maximum principle for the function L , and we must have $L = 0$ on \bar{D} . The theorem now follows from Lemma. QED.

5. Let us consider the orthonormal tangent vector fields V_1, V_2 on S^2 . Let the tangent frames on S^2 be chosen in such a way that $v_i = V_i (i = 1, 2)$. Put $v_i \mathbf{f} = d\mathbf{f}(v_i)$.

Theorem 5. Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \rightarrow R$ be a function satisfying $\mathbf{f}_{12} = 0$ on D . If

1. $L = 0$ on ∂D ,
2. on D , there is a couple of orthonormal tangent vector fields V_1, V_2 such that

$$(\mathbf{f}_{11} - \mathbf{f}_{22})(V_1 V_1 - V_2 V_2) \mathcal{L}f \geq 0,$$

then \mathbf{f} is linear on \bar{D} .

Proof. From $d\mathbf{f}_{12} = 0$ we get $(\mathbf{f}_{11} - \mathbf{f}_{22})\omega_1^2 = (Q + \mathbf{f}_2)\omega^1 + (R + \mathbf{f}_1)\omega^2$. Consequently, there are functions α, β such that

$$(15) \quad \omega_1^2 = \alpha\omega^1 + \beta\omega^2,$$

and α, β satisfy $(\mathbf{f}_{11} - \mathbf{f}_{22})\alpha = Q + \mathbf{f}_2$, $(\mathbf{f}_{11} - \mathbf{f}_{22})\beta = R + \mathbf{f}_1$. We have $v_1\mathcal{L}\mathbf{f} = P + R + 2\mathbf{f}_1$, $v_2\mathcal{L}\mathbf{f} = Q + S + 2\mathbf{f}_2$. This together with (15) implies

$$(16) \quad \begin{aligned} v_1v_1\mathcal{L}\mathbf{f} &= 2(\mathbf{f}_{11} + \mathbf{f}_{22}) + (Q + S + 2\mathbf{f}_2)\alpha + T_1 + T_3, \\ v_2v_2\mathcal{L}\mathbf{f} &= 2(\mathbf{f}_{11} + \mathbf{f}_{22}) - (P + R + 2\mathbf{f}_1)\beta + T_3 + T_5. \end{aligned}$$

Eliminating T_1, \dots, T_5 from (10) and (16) we obtain

$$\begin{aligned} L_{11} + L_{22} - 4L &= 2(\mathbf{f}_{11} - \mathbf{f}_{22})(v_1v_1 - v_2v_2)\mathcal{L}\mathbf{f} + \\ &+ 2[(P + \mathbf{f}_1) - \frac{3}{2}(R + \mathbf{f}_1)]^2 + 2[(S + \mathbf{f}_2) - \frac{3}{2}(Q + \mathbf{f}_2)]^2 + \\ &+ \frac{7}{2}[(Q + \mathbf{f}_2)^2 + (R + \mathbf{f}_1)^2]. \end{aligned}$$

Now we can conclude from 2. that this expression satisfies the conditions of the maximum principle for the function L , and $L = 0$ on \bar{D} . The theorem follows from Lemma. QED.

Theorem 6. Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f}: \bar{D} \rightarrow \mathbb{R}$ be a function satisfying $\mathbf{f}_{12} = 0$ on D . If

1. $L = 0$ on ∂D ,
2. on D . there is a couple of orthonormal tangent vector fields V_1, V_2 such that

$$(\mathbf{f}_{11} - \mathbf{f}_{22})(V_1V_1 - V_2V_2)\mathcal{M}\mathbf{f} \geq 0,$$

3. $\mathcal{M}\mathbf{f} > 0$, $\mathcal{L}\mathbf{f} \geq 0$ on D ,
4. $\frac{4}{11} \leq \frac{(\mathbf{f}_{22} + \mathbf{f})^2}{(\mathbf{f}_{11} + \mathbf{f})^2} \leq \frac{11}{4}$ on D ,

then \mathbf{f} is linear on \bar{D} .

Proof. We have

$$\begin{aligned} v_1\mathcal{M}\mathbf{f} &= (\mathbf{f}_{11} + \mathbf{f})(R + \mathbf{f}_1) + (\mathbf{f}_{22} + \mathbf{f})(P + \mathbf{f}_1), \\ v_2\mathcal{M}\mathbf{f} &= (\mathbf{f}_{11} + \mathbf{f})(S + \mathbf{f}_2) + (\mathbf{f}_{22} + \mathbf{f})(Q + \mathbf{f}_2); \end{aligned}$$

from (15), we obtain

$$(17) \quad \begin{aligned} v_1v_1\mathcal{M}\mathbf{f} &= 2(P + \mathbf{f}_1)(R + \mathbf{f}_1) - 2(Q + \mathbf{f}_2)^2 + 2\mathbf{f}_{22}(\mathbf{f}_{11} + \mathbf{f}) + \\ &+ \mathcal{L}\mathbf{f} \cdot \mathbf{f}_{11} + [(\mathbf{f}_{11} + \mathbf{f})(S + \mathbf{f}_2) + (\mathbf{f}_{22} + \mathbf{f})(Q + \mathbf{f}_2)]\alpha + \\ &+ (\mathbf{f}_{22} + \mathbf{f})T_1 + (\mathbf{f}_{11} + \mathbf{f})T_3, \\ v_2v_2\mathcal{M}\mathbf{f} &= 2(Q + \mathbf{f}_2)(S + \mathbf{f}_2) - 2(R + \mathbf{f}_1)^2 + 2\mathbf{f}_{11}(\mathbf{f}_{22} + \mathbf{f}) + \\ &+ \mathcal{L}\mathbf{f} \cdot \mathbf{f}_{22} - [(\mathbf{f}_{11} + \mathbf{f})(R + \mathbf{f}_1) + (\mathbf{f}_{22} + \mathbf{f})(P + \mathbf{f}_1)]\beta + \\ &+ (\mathbf{f}_{22} + \mathbf{f})T_3 + (\mathbf{f}_{11} + \mathbf{f})T_5. \end{aligned}$$

Eliminating T_1, \dots, T_5 from (10) and (17) we obtain

$$\begin{aligned}
& (\mathbf{f}_{22} + \mathbf{f}) L_{11} + (\mathbf{f}_{11} + \mathbf{f}) L_{22} - 2\mathcal{L}\mathbf{f}L = 2(\mathbf{f}_{11} - \mathbf{f}_{22})(v_1v_1 - v_2v_2) \mathcal{M}\mathbf{f} + \\
& + 2(\mathbf{f}_{11} + \mathbf{f}) \left\{ \left[(S + \mathbf{f}_2) - \left(\frac{1}{2} + \frac{\mathbf{f}_{22} + \mathbf{f}}{\mathbf{f}_{11} + \mathbf{f}} \right) (Q + \mathbf{f}_2) \right]^2 + \right. \\
& \quad \left. + \left(\frac{11}{4} - \frac{(\mathbf{f}_{22} + \mathbf{f})^2}{(\mathbf{f}_{11} + \mathbf{f})^2} \right) (Q + \mathbf{f}_2)^2 \right\} + \\
& + 2(\mathbf{f}_{22} + \mathbf{f}) \left\{ \left[(P + \mathbf{f}_1) - \left(\frac{1}{2} + \frac{\mathbf{f}_{11} + \mathbf{f}}{\mathbf{f}_{22} + \mathbf{f}} \right) (R + \mathbf{f}_1) \right]^2 + \right. \\
& \quad \left. + \left(\frac{11}{4} - \frac{(\mathbf{f}_{11} + \mathbf{f})^2}{(\mathbf{f}_{22} + \mathbf{f})^2} \right) (R + \mathbf{f}_1)^2 \right\}.
\end{aligned}$$

Now we can conclude from 2., 3. and 4. that this expression satisfies the conditions of the maximum principle for the function L , and $L = 0$ on \bar{D} . The theorem now follows from Lemma. QED.

Theorem 7. Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \rightarrow R$ be a function satisfying $\mathbf{f}_{12} = 0$ on D . If

1. $L = 0$ on ∂D ,
2. on D , there is a couple of orthonormal tangent vector fields V_1, V_2 such that

$$\mathcal{L}\mathbf{f}(V_1V_1 + V_2V_2) \mathcal{L}\mathbf{f} - 2(V_1V_1 + V_2V_2) \mathcal{M}\mathbf{f} \geq 0,$$

then \mathbf{f} is linear on \bar{D} .

Proof. From (10), (16) and (17) we obtain

$$\begin{aligned}
& L_{11} + L_{22} + \alpha L_2 - \beta L_1 = 2\mathcal{L}\mathbf{f}(v_1v_1 + v_2v_2) \mathcal{L}\mathbf{f} - \\
& - 4(v_1v_1 + v_2v_2) \mathcal{M}\mathbf{f} + 2(P + R + 2\mathbf{f}_1)^2 + 2(Q + S + 2\mathbf{f}_2)^2.
\end{aligned}$$

This expression satisfies the conditions of the maximum principle for the function L , and we must have $L = 0$ on \bar{D} . The theorem now follows from Lemma. QED.

Theorem 8. Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \rightarrow R$ be a function satisfying $\mathbf{f}_{12} = 0$ on D . If

1. $L = 0$ on ∂D ,
2. on D , there is a couple of orthonormal tangent vector fields V_1, V_2 such that

$$\begin{aligned}
& \mathcal{L}\mathbf{f}[(\mathbf{f}_{22} + \mathbf{f}) V_1V_1 \mathcal{L}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f}) V_2V_2 \mathcal{L}\mathbf{f}] - \\
& - 2[(\mathbf{f}_{22} + \mathbf{f}) V_1V_1 \mathcal{M}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f}) V_2V_2 \mathcal{M}\mathbf{f}] \geq 0,
\end{aligned}$$

3. $\mathcal{M}\mathbf{f} > 0$, $\mathcal{L}\mathbf{f} \geq 0$ on D ,
then \mathbf{f} is linear on \bar{D} .

Proof. From (10), (16) and (17) we obtain

$$\begin{aligned}
 (18) \quad & L(\mathbf{f}_{22} + \mathbf{f})L_{11} + (\mathbf{f}_{11} + \mathbf{f})L_{22} - (\mathbf{f}_{11} + \mathbf{f})\beta L_1 + (\mathbf{f}_{22} + \mathbf{f})\alpha L_2 = \\
 & = 2\mathcal{L}\mathbf{f}[(\mathbf{f}_{22} + \mathbf{f})v_1v_1\mathcal{L}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f})v_2v_2\mathcal{L}\mathbf{f}] - \\
 & \quad - 4[(\mathbf{f}_{22} + \mathbf{f})v_1v_1\mathcal{M}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f})v_2v_2\mathcal{M}\mathbf{f}] + \\
 & \quad + 2(\mathbf{f}_{22} + \mathbf{f})(P + R + 2\mathbf{f}_1)^2 + 2(\mathbf{f}_{11} + \mathbf{f})(Q + S + 2\mathbf{f}_2)^2.
 \end{aligned}$$

The quadratic form on the right hand side is positive definite because of 3. This implies that the expression (18) satisfies the conditions of the maximum principle for the function L , and we must have $L = 0$ on \bar{D} . The theorem now follows from Lemma. QED.

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