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ISOMETRIES OF LATTICE ORDERED GROUPS

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Let $\mathbf{G} = (G; +, \wedge, \vee)$ be a lattice ordered group. A one-to-one mapping f of G onto G is called *an isometry of* $\mathbf{G}$ if the following conditions are valid for each pair of elements $x, y \in G$:

(i) $|f(x) - f(y)| = |x - y|$;

(ii) $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

SWAMY [6] defined the notion of isometry of an abelian lattice ordered group $\mathbf{G}$ as a one-to-one mapping f of G onto G fulfilling (i) identically. It is easy to verify that for abelian lattice ordered groups the condition (ii) is a consequence of (i) (cf. Lemma 1.2 below).

In this paper we shall investigate the relations between isometries of a lattice ordered group $\mathbf{G}$ and direct product decompositions of $\mathbf{G}$.

If f is an isometry of $\mathbf{G}$ and $f(0) = 0$, then f will be called a 0-*isometry*. Let $g \in G$; the translation f_g is defined by $f_g(x) = x + g$ for each $x \in G$. Every translation is an isometry of $\mathbf{G}$. Each isometry can be uniquely represented as a composition of a 0-isometry and a translation. Thus for finding all isometries of $\mathbf{G}$ it suffices to determine all 0-isometries.

It will be shown that for every 0-isometry f of $\mathbf{G}$ there exists a uniquely determined direct product decomposition $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ of $\mathbf{G}$ such that

$$f(x) = x(\mathbf{A}) - x(\mathbf{B})$$

is valid for each $x \in G$, where $x(\mathbf{A})$ and $x(\mathbf{B})$ are components of x in the direct factors $\mathbf{A}$ and $\mathbf{B}$, respectively.

For any lattice ordered group $\mathbf{G}$ we denote by $G^*(\mathbf{G})$, $G_0^*(\mathbf{G})$ and $T(\mathbf{G})$ the set of all isometries, the set of all 0-isometries and the set of all translations of $\mathbf{G}$, respectively. Each of these sets is a group with respect to the composition of mappings. For $f_1, f_2 \in G^*(\mathbf{G})$ we put $f_1 \leqq f_2$ if $f_1(x) \leqq f_2(x)$ is valid for each $x \in G$.

Let $\mathbf{G}$ and $\mathbf{G}'$ be lattice ordered groups. It will be proved that if there exists a one-to-one mapping φ of $G^*(\mathbf{G})$ onto $G^*(\mathbf{G}')$ such that φ is a group isomorphism and an order isomorphism, then $\mathbf{G}$ is isomorphic with $\mathbf{G}'$. This sharpens a result of SWAMY [7].

Let $\mathbf{G}$ be archimedean and let $d(\mathbf{G})$ be the Dedekind completion of $\mathbf{G}$. It will be shown that $G_0^*(\mathbf{G})$ is isomorphic with a subgroup of $G_0^*(d(\mathbf{G}))$. If, moreover, $\mathbf{G}$ is strongly projectable, then $G_0^*(\mathbf{G})$ is isomorphic with $G_0^*(d(\mathbf{G}))$.

We shall use the standard terminology and denotations for lattice ordered groups (cf. FUCHS [2] and CONRAD [1]).

1. THE SYSTEMS M_1 AND M_2

Let $\mathbf{G} = (G; +, \wedge, \vee)$ be a lattice ordered group and let $a, b, x \in G$. It is well-known that

$$|a - b| = (a \vee b) - (a \wedge b) .$$

Since $(a \vee b) - a = b - (a \wedge b)$, we have also

$$|a - b| = a - (a \wedge b) + b - (a \wedge b) = (a \vee b) - a + (a \vee b) - b .$$

1.1. Lemma. *Assume that $\mathbf{G}$ is abelian. The following conditions are equivalent*:

(α) $$|a - b| = |a - x| + |x - b| ;$$

(β) $$x \in [a \wedge b, a \vee b] .$$

Proof. Suppose that (α) is valid. Denote $a \wedge x = p$, $b \wedge x = q$, $p \wedge q = u$, $p \vee q = z$. Clearly $u \leqq a \wedge b$ and $z \leqq x$. Assume that $u < a \wedge b$. Then we have

$$\begin{aligned}|a - x| + |x - b| &= (a - p) + (x - p) + (x - q) + (b - q) \geqq \\ &\geqq (a - p) + (z - p) + (z - q) + (b - q) = \\ &= (a - p) + (q - u) + (p - u) + (b - q) = (a - u) + (b - u) = \\ &= a - (a \wedge b) + b - (a \wedge b) + 2((a \wedge b) - u) > |a - b| ,\end{aligned}$$

which is a contradiction. Thus $a \wedge b = u$ and hence $a \wedge b \leqq x$. The relation $x \leqq \leqq a \vee b$ can be verified dually. Therefore (β) holds.

Conversely, assume that (β) is valid. Let p q u be as above. Then $p \vee q = x$, $u = a \wedge b$ hence

$$\begin{aligned}|a - b| &= (a - u) + (b - u) = (a - p) + (p - u) + (b - q) + (q - u) = \\ &= (a - p) + (x - q) + (b - q) + (x - p) = |a - x| + |x - b| .\end{aligned}$$

1.2. Lemma. *Let $\mathbf{G}$ be abelian. Let f be a one-to-one mapping of the set G onto G fulfilling the condition* (i) *for each $x, y \in G$. Then* (ii) *is satisfied for each $x, y \in G$.*

Proof. From (i) it follows that the condition (α) from 1.1 is equivalent to the condition

$$(\alpha_1) \qquad |f(a) - f(b)| = |f(a) - f(x)| + |f(x) - f(b)| .$$

Hence according to 1.1, (β) is equivalent to

$$(\beta_1) \qquad f(x) \in [f(a) \wedge f(b), f(a) \vee f(b)] .$$

Therefore

$$f([a \wedge b, a \vee b]) = [f(a) \wedge f(b), f(a) \vee f(b)]$$

is valid for each $a, b \in G$.

From 1.2 it follows that in the case of an abelian lattice ordered group, the definition of isometry given above coincides with the definition of isometry given in Swamy's paper [6].

It can be easily verified by examples that (i) is not, in general implied by (ii). The question whether (ii) is a consequence of (i) for each lattice ordered group $\mathbf{G}$ remains open.

In what follows, $\mathbf{G}$ is a lattice ordered group; the commutativity of $\mathbf{G}$ will not be assumed.

In 1.3 – 1.7′ we suppose that f is an isometry of $\mathbf{G}$. (Each of the lemmas 1.3 – 1.7′ can be applied also for f^{-1} under the corresponding change of denstations.) We denote by M_1 and M_2 the sets of all intervals $[r, s]$ of $\mathbf{G}$ such that $f(r) \leqq f(s)$ or $f(r) \geqq f(s)$, respectively.

1.3. Lemma. *Let $a, b, c \in G$, $a \leqq b \leqq c$. If $i \in \{1, 2\}$ and $[a, c] \in M_i$, then both the intervals $[a, b]$ and $[b, c]$ belong to M_i.*

This follows immediately from (ii).

Since $M_1 \cap M_2$ contains only one-element intervals, we obtain from 1.3:

1.3′. Corollary. *Let $a, b, c \in M$, $a \leqq b$, $a \leqq c$, $[a, b] \in M_1$, $[a, c] \in M_2$. Then $a = b \wedge c$ (and dually).*

1.4. Lemma. *Let $a, b \in G$, $a \leqq b$. There exist elements $c, d \in [a, b]$ such that $[a, c], [d, b] \in M_1$, $[a, d], [c, b] \in M_2$, $c \wedge d = a$, $c \vee d = b$.*

Proof. Put $c = f^{-1}(f(a) \vee f(b))$, $d = f^{-1}(f(a) \wedge f(b))$. Then (ii) (with f replaced by f^{-1}) yields $c, d \in [a, b]$ and hence $[a, c], [d, b] \in M_1$, $[a, d], [c, b] \in M_2$. Thus according to 1.3, $c \wedge d = a$, $c \vee d = b$.

Suppose that $x, y, u, v \in G$, $x \wedge y = u$, $x \vee y = v$.

1.5. Lemma. *Let $[u, x], [u, y] \in M_1$. Then $f(x) \wedge f(y) = f(u)$ and $f(x) \vee f(y) = f(v)$ (hence $[x, v], [y, v] \in M_1$).*

Proof. We have $f(u) \leqq f(x)$, $f(u) \leqq f(y)$, whence $f(u) \leqq f(x) \wedge f(y)$. On the other hand, from (ii) we infer $f(x) \wedge f(y) \leqq f(u)$, thus $f(u) = f(x) \wedge f(y)$.

Because of $|x - y| = v - u = |v - u|$, the relation

$$|f(x) - f(y)| = |f(v) - f(u)| \tag{1}$$

is valid. From (ii) it follows

$$f(v) \in [f(x) \wedge f(y), f(x) \vee f(y)],$$

hence

$$|f(x) - f(y)| = (f(x) \vee f(y)) - (f(x) \wedge f(y)) =$$
$$= f(x) \vee f(y) - f(u) = f(x) \vee f(y) - f(v) + f(v) - f(u).$$

From this and from (1) we get $f(x) \vee f(y) - f(v) = 0$.

Analogously we obtain:

1.5′. Lemma. *Let* $[u, x], [u, y] \in M_2$. *Then* $f(x) \vee f(y) = f(u)$, $f(x) \wedge f(y) = f(v)$ (*hence* $[x, v], [y, v] \in M_2$).

1.6. Lemma. *Let* $[u, x] \in M_1$, $[u, y] \in M_2$. *Then* $f(u) \wedge f(v) = f(y)$, $f(u) \vee f(v) = f(x)$ (*thus* $[x, v] \in M_2$, $[y, v] \in M_1$).

Proof. According to the assumption we have $f(y) \leqq f(u) \leqq f(x)$. From (ii) it follows that $f(v) \in [f(y), f(x)]$. Hence $[x, v] \in M_2$, $[y, v] \in M_1$. From this and from 1.3′ (applied to f^{-1}) we obtain $f(u) \wedge f(v) = f(y)$, $f(u) \vee f(v) = f(x)$.

1.7. Lemma. *Let* $[u, x] \in M_1$. *Then* $[y, v] \in M_1$.

Proof. According to 1.4 there is $c \in [u, y]$ such that $[u, c] \in M_1$, $[c, y] \in M_2$. Denote $e = c \vee x$. From 1.5 it follows that $[c, e] \in M_1$ and hence according to 1.6, $[y, v] \in M_1$.

Analogously we obtain (by using 1.5′ instead of 1.5)

1.7′. Lemma. *Let* $[u, x] \in M_2$. *Then* $[y, v] \in M_2$.

Now let us suppose that f is a 0-isometry.

1.8. Lemma. *Let* $x \in G$. *Then*

(a) $x \wedge f(x) \geqq 0 \Rightarrow f(x) = x$;

(b) $x \wedge (-f(x)) \geqq 0 \Rightarrow f(x) = -x$;

(c) $x \vee f(x) \leqq 0 \Rightarrow f(x) = x$;

(d) $x \vee (-f(x)) \leqq 0 \Rightarrow f(x) = -x$.

Proof. From $x \wedge (f(x)) \geqq 0$ we obtain $x = |x| = |x - 0| = |f(x) - f(0)| = |f(x)| = f(x)$. The relations (b) – ($d$) can be verified analogously.

1.9. Lemma. *Let* $0 \leqq x \in G$. *Then*

(a) $f(x) = x \Leftrightarrow f(-x) = -x$;

(b) $f(x) = -x \Leftrightarrow f(-x) = x$.

Proof. Suppose that $f(x) = x$. According to 1.4 there exist elements $c, d \in [-x, x]$ such that $[-x, c], [d, x] \in M_1$, $[-x, d], [c, x] \in M_2$. Since $[0, x] \in M_1$, we obtain from 1.3′ that $0 \vee c = x$, whence $0 \in [d, x]$. Then according to (ii), $0 \in [f(d), f(x)]$. Denote $c \wedge 0 = z$. We have $(c - z) \wedge (-z) = 0$, $x = x - 0 = c - z$, hence $x \wedge (-z) = 0$. Thus $2x \wedge (-z) = 0$. On the other hand, $-z = 0 - z = d - (-x)$, thus $2x = x - (-x) = (x - d) + (d - (-x)) = (x - d) + (-z)$ and $x - d \geqq 0$, whence $0 \leqq -z \leqq 2x$. Therefore $z = 0$ and this implies $d = -x$. Further we obtain $f(-x) = f(d) \leqq 0$, hence $f(-x) \vee (-x) \leqq 0$. By 1.8, $f(-x) = -x$. The other implications of the lemma can be proved analogously.

2. THE DIRECT DECOMPOSITION CORRESPONDING TO f

Let $\boldsymbol{G}$ be as above and let f be a 0-isometry of $\boldsymbol{G}$. Denote $A_1 = \{0 \leqq x \in G : f(x) \geqq 0\}$, $B_1 = \{0 \leqq x \in G : f(x) \leqq 0\}$.

2.1. Lemma. *Let* $0 \leqq x \in G$. *There are elements* $p \in A_1$, $q \in B_1$ *such that the relations*

$$x = p + q = p \vee q\,, \quad f(x) = p - q$$

hold. Moreover, $p = \sup (A_1 \cap [0, x])$, $q = \sup (B_1 \cap [0, x])$.

Proof. Denote $u = f(x) \wedge 0$, $v = f(x) \vee 0$. Then $f(x) = u + v$. According to (ii) there are elements $p, q \in [0, x]$ such that $v = f(p)$, $u = f(q)$. Hence by 1.6 we have $p \wedge q = 0$ and $p \vee q = x$. Thus $x = p + q$. Since $p \in A_1$, $q \in B_1$, it follows from 1.8 that $f(p) = p$, $f(q) = -q$, thus $f(x) = p - q$. Let $p' \in A_1 \cap [0, x]$. From 1.3′ we get $p' \wedge q = 0$, whence $p' = p' \wedge x = p' \wedge (p \vee q) = p' \wedge p$. Therefore $p = \sup (A_1 \cap [0, x])$. The relation $q = \sup (B_1 \cap [0, x])$ can be verified similarly.

Analogously we obtain (by using 1.9):

2.1′. Lemma. *Let* $0 \geqq x \in G$. *There are elements* $p \in A_1$, $q \in B_1$ *such that the relations*

$$x = p + q = p \wedge q\,, \quad f(x) = p - q$$

are valid. Moreover, $p = \inf ((-A_1) \cap [x, 0])$, $q = \inf ((-B_1) \cap [x, 0])$.

Let $X \subseteq G$. We denote

$$X^\delta = \{g \in G : |g| \wedge |x| = 0 \text{ for each } x \in X\}.$$

The set X^δ is called *a polar of* $\mathbf{G}$. Each polar of $\mathbf{G}$ is a closed convex l-subgroup of $\mathbf{G}$. (Cf. Šik [8].) Further we put

$$X^{\delta+} = \{y \in X^\delta : y \geqq 0\}.$$

Then $X^{\delta+}$ is a convex sublattice of the lattice $(G; \wedge, \vee)$ and a subsemigroup of the group $(G; +)$.

It is well-known that a polar X^δ is a direct factor of $\mathbf{G}$ if and only it the following condition is fulfilled:

$(*)$ *For each* $0 \leqq x \in G$, *there exists* $\sup(X^{\delta+} \cap [0, x])$ *in the lattice* $(G; \leqq)$.

If $(*)$ holds, then also the dual condition is fulfilled and

$$\mathbf{G} = X^\delta \times X^{\delta\delta}$$

(we write here, in fact, X^δ instead of $(X^\delta; +, \wedge, \vee)$, and similarly for $X^{\delta\delta}$). If this is the case and $0 \leqq y \in G$, $0 \geqq z \in G$, then the components $y(X^\delta)$, $z(X^\delta)$ of y and z in X^δ are given by

$$y(X^\delta) = \sup(X^{\delta+} \cap [0, y]), \quad z(X^\delta) = \inf(X^{\delta-} \cap [z, 0]),$$

where $X^{\delta-} = -X^{\delta+}$.

2.2. Lemma. $A_1^{\delta+} = B_1$ *and* $B_1^{\delta+} = A_1$.

Proof. From 1.3′ we infer that $B_1 \subseteq A_1^{\delta+}$ is valid. Let $x \in A_1^{\delta+}$ and let p, q be as in 2.1. Since $p \in A_1$, we have $x \wedge p = 0$ and hence $p = 0$. Therefore $x = q \in B_1$, $A_1^{\delta+} \subseteq B_1$ and thus $A_1^{\delta+} = B_1$. Similarly we obtain $B_1^{\delta+} = A_1$.

Denote $A_1^\delta = B$, $B_1^\delta = A$, $\mathbf{A} = (A; +. \wedge, \vee)$, $\mathbf{B} = (B; +, \wedge, \vee)$.

2.3. Lemma. $\mathbf{G} = \mathbf{A} \times \mathbf{B}$.

This follows from 2.1 and 2.2.

For $x \in G$ with $x \geqq 0$ or $x \leqq 0$ let p, q have the same meaning as in 2.1 and 2.1′, respectively. Then 2.3 implies $x(\mathbf{A}) = p$, $x(\mathbf{B}) = q$. Hence 2.1, 2.1′ and 2.3 yield

2.4. Lemma. *Let* $x \in G$ *such that either* $x \geqq 0$ *or* $x \leqq 0$. *Then* $f(x) = x(\mathbf{A}) - x(\mathbf{B})$.

2.5. Theorem. *Let* $\mathbf{G} = (G; +, \wedge, \vee)$ *be a lattice ordered group and let* f *be a* 0-*isometry of* $\mathbf{G}$. *Then there are direct factors* $\mathbf{A}, \mathbf{B}$ *of* $\mathbf{G}$ *such that* $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ *and* $f(x) = x(\mathbf{A}) - x(\mathbf{B})$ *holds for each* $x \in G$.

Proof. Let $\mathbf{A}, \mathbf{B}$ be as in 2.3; hence $\mathbf{G} = \mathbf{A} \times \mathbf{B}$. Let $x \in G$. Denote $x \wedge 0 = u$,

$x \vee 0 = v$. According to 1.4 there exists $r \in [u, 0]$ such that $[u, r] \in M_2$, $[r, 0] \in M_1$. Put $z = x \vee r$. From 1.7 and 1.7′ it follows that $[x, z] \in M_2$, $[z, v] \in M_1$. Hence

$$f(x) = (f(x) - f(z)) - (f(v) - f(z)) + f(v) =$$
$$= |f(x) - f(z)| - |f(v) - f(z)| + f(v).$$

We have

$$|f(x) - f(z)| = |x - z| = |u - r| = |f(u) - f(r)| = f(u) - f(r),$$
$$|f(v) - f(z)| = |v - z| = |0 - r| = |f(0) - f(r)| = |-f(r)| = -f(r),$$

thus $f(x) = f(u) + f(v)$. From 2.4 we obtain

$$f(u) = u(\mathbf{A}) - u(\mathbf{B}), \quad f(v) = v(\mathbf{A}) - v(\mathbf{B}).$$

Hence

$$f(x) = u(\mathbf{A}) - u(\mathbf{B}) + v(\mathbf{A}) - v(\mathbf{B}) = u(\mathbf{A}) + v(\mathbf{A}) - u(\mathbf{B}) - v(\mathbf{B}).$$

The definition of u and v yields $|u| \wedge |v| = 0$ and hence $|u(\mathbf{B})| \wedge |v(\mathbf{B})| = 0$, which implies $u(\mathbf{B}) + v(\mathbf{B}) = v(\mathbf{B}) + u(\mathbf{B})$. Thus

$$f(x) = u(\mathbf{A}) + v(\mathbf{A}) - (u(\mathbf{B}) + v(\mathbf{B})).$$

Clearly $x = u + v$. Therefore $f(x) = x(\mathbf{A}) - x(\mathbf{B})$, which completes the proof.

Remark. *If* $\mathbf{A}$, $\mathbf{B}$ *are as in* 2.5, *then*

$$A = \{x \in G : f(x) = x\}, \quad B = \{x \in G : f(x) = -x\};$$

hence $\mathbf{A}$ *and* $\mathbf{B}$ *are uniquely determined by the* 0*-isometry* f.

From 2.5 it follows that whenever $f \in G_0^*(\mathbf{G})$, then f is an automorphism of the group $(G; +)$ and that $f^2 = e$, where e is the identical mapping on G. If $h \in G^*(\mathbf{G})$ then the mapping f defined by $f(x) = h(x) - h(0)$ for each $x \in G$ is a 0-isometry of $\mathbf{G}$. Hence we have

2.5.1. Corollary. (Cf. [6], Theorem 1.) *For each isometry h of $\mathbf{G}$ there exists just one involutory isometric group automorphism f of $\mathbf{G}$ such that $h(x) = f(x) + h(0)$ for every $x \in G$.*

Remark. In Theorem 1, [6] the assertion 2.5.1 has been proved for the case of an abelian lattice ordered group $\mathbf{G}$ (the commutativity of $\mathbf{G}$ has been essentially used in the proof; namely, the representation of $\mathbf{G}$ as a subdirect product of linearly ordered groups has been applied).

Let h, f be as in 2.5.1 and let $\mathbf{B}$ be as in 2.5. Suppose that h fails to be a translation. Then $B \neq \{0\}$ and for each $0 < b \in B$ we have $h(b)(\mathbf{B}) = -b + h(0)(\mathbf{B}) < h(0)(\mathbf{B})$, whence $h(0) \nleq h(b)$. Thus we have

2.5.2. Corollary. (Cf. [6], Theorem 2.) *An isometry of* $\mathbf{G}$ *is order preserving iff it is a translation.*

In view of Theorem 2.5, we can express this also by saying that (under the denotations as above), h is order preserving iff $B = \{0\}$. Analogously we can verify that h is order reversing iff $A = \{0\}$, i.e., $B = G$. Thus we obtain

2.5.3. Corollary. (Cf. [6], Theorem 3.) *An isometry* h *of* $\mathbf{G}$ *is order reversing iff* $h(x) = h(0) - x$ *for each* $x \in G$.

3. THE GROUP OF ALL ISOMETRIES OF $\mathbf{G}$

As above, let $\mathbf{G} = (G; +, \wedge, \vee)$ be a lattice ordered group. Let card $G > 1$. We denote by $B_0 = B_0(\mathbf{G})$ the system of all direct factors of $\mathbf{G}$ (B_0 being partially ordered by inclusion). Then B_0 is a Boolean algebra.

Let $\mathbf{A}'$ and $\mathbf{B}'$ be complementary direct factors of $\mathbf{G}$, i.e., $\mathbf{G} = \mathbf{A}' \times \mathbf{B}'$. Then $\mathbf{A}'$ is uniquely determined by $\mathbf{B}'$. Put $h(x) = x(\mathbf{A}') - x(\mathbf{B}')$ for each $x \in G$. We can easily verify that h is a 0-isometry of $\mathbf{G}$. This together with 2.5 implies:

(**) *If we put (under the same denotations as in* 2.5) $\varphi(f) = \mathbf{B}$, *then* φ *is a one-to-one mapping of the set* $G_0^*(\mathbf{G})$ *onto the set* $B_0(\mathbf{G})$.

This result can be slightly sharpened as follows. We denote by $S(B_0)$ the Stone space of B_0. There is an order preserving injection ψ of B_0 onto the system S_1 of all clopen subsets of $S(B_0)$. For each $X \in S_1$ let f_X be the characteristic function of X (i.e., $f_X(t) = 1$ for each $t \in X$ and $f_X(t) = 0$ for each $t \in S(B_0) \setminus X$). Further let $\mathbf{F} = (F; +)$, where (i) F is the set of all functions f_X with X running over S_1, and (ii) the operation $+$ on F is performed as addition modulo 2. Hence $\mathbf{F}$ is a group. Consider the mapping $\psi_1 \colon G_0^*(\mathbf{G}) \to F$ defined by $\psi_1(f) = \psi(\varphi(f))$ for each $f \in G_0^*(\mathbf{G})$, where φ is as in (**). Then 2.3, 2.5 and (**) imply

3.1. Proposition. ψ_1 *is an isomorphism of the group* $G_0^*(\mathbf{G})$ *onto* $\mathbf{F}$.

We can ask to what extent the lattice ordered group $\mathbf{G}$ is determined by the set G and by the group $G_0^*(\mathbf{G})$. Some negative results in this direction are implied by the following examples concerning lattice ordered groups $\mathbf{G} = (G; +, \leqq)$ and $\mathbf{G}_1 = (G; +_1, \leqq_1)$.

3.2. Suppose that $G_0^*(\mathbf{G}) = G_0^*(\mathbf{G}_1)$ and that the operations $+$ and $+_1$ coincide on G. Then it can happen that the partial order $\leqq$ coincides neither with $\leqq_1$ nor with the dual of $\leqq_1$.

Example. Let R be the additive group of all reals with the usual linear order and let G be the set of all pairs (x, y) with $x, y \in R$. We define the operation $+$ in G coordinatewise. For $(x_1, y_1), (x_2, y_2) \in G$ we put $(x_1, y_1) \leqq (x_2, y_2)$ if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 \leqq y_2$. Further we put $(x_1, y_1) \leqq_1 (x_2, y_2)$ if either $y_1 < y_2$, or $y_1 = y_2$ and $x_1 \leqq x_2$. Then $\mathbf{G} = (G; +, \leqq)$ and $\mathbf{G}_1 = (G; +, \leqq_1)$ are linearly ordered groups. Since each linearly ordered group is directly indecomposable, we infer from 2.5 that if f is a 0-isometry of $\mathbf{G}$, then either f is the identity on G or $f(t) = -t$ for each $t \in G$; the same holds for $\mathbf{G}_1$. Thus $G_0^*(\mathbf{G}) = G_0^*(\mathbf{G}_1)$. The linear order $\leqq$ coincides neither with $\leqq_1$ nor with the dual of $\leqq_1$.

3.3. Suppose that $G_0^*(\mathbf{G}) = G_0^*(\mathbf{G}_1)$ and that the partial orders $\leqq$ and $\leqq_1$ are equal. Then the operation $+$ need not coincide with $+_1$.

Example. Let R be the set of all reals and let $+$ and $\leqq$ have the usual meaning. Put $\varphi(t) = t^2$ for each $0 \leqq t \in R$ and $\varphi(t) = -t^2$ for each $0 \geqq t \in R$. For each pair $x, y \in R$ we set $x +_1 y = \varphi(\varphi^{-1}(x) + \varphi^{-1}(y))$. Then $\mathbf{G} = (R; +, \leqq)$ and $\mathbf{G}_1 = (R; +_1, \leqq)$ are linearly ordered groups. We have $G_0^*(\mathbf{G}) = G_0^*(\mathbf{G}_1)$ and the operation $+$ does not coincide with $+_1$.

Let $\mathbf{G} = (G; +, \wedge, \vee)$ and $\mathbf{G}' = (G'; +, \wedge, \vee)$ be lattice ordered groups with $G \cap G' = \emptyset$. Let φ be a one-to-one mapping of the set $G^*(\mathbf{G})$ onto $G^*(\mathbf{G}')$. Both these sets are taken as partially ordered (cf. Introduction). Consider the following conditions for φ:

(a) φ is a group isomorphism.

(b) φ is an order isomorphism.

(c) φ carries translations onto translations.

The following theorem is the main result of the paper [7]:

(S) *Let $\mathbf{G}$ and $\mathbf{G}'$ be abelian lattice ordered groups.* (i) *If there exists a mapping φ of $G^*(\mathbf{G})$ onto $G^*(\mathbf{G}')$ fulfilling the conditions* (a), (b) *and* (c), *then $\mathbf{G}$ is isomorphic with $\mathbf{G}'$.* (ii) *If $\mathbf{G}$ and $\mathbf{G}'$ are divisible and if there exists a mapping φ of $G^*(\mathbf{G})$ onto $G^*(\mathbf{G}')$ fulfilling* (a) *and* (b), *then $\mathbf{G}$ and $\mathbf{G}'$ are isomorphic.*

We shall show that the assertion (ii) remains valid without assuming that $\mathbf{G}$ and $\mathbf{G}'$ are abelian and divisible.

3.4. Lemma. *Let $\mathbf{G}$ and $\mathbf{G}'$ be lattice ordered groups. Suppose that φ is a one-to-one mapping of the set $G^*(\mathbf{G})$ onto $G^*(\mathbf{G}')$ fulfilling the conditions* (a) *and* (b). *Then φ fulfils the condition* (c) *as well.*

Proof. Let e and e' be the neutral elements of $G^*(\mathbf{G})$ and $G^*(\mathbf{G}')$, respectively. For each $c \in G$ we denote by f_c the translation of $\mathbf{G}$ defined by $f_c(t) = t + c$ for each $t \in G$. Let $0 < c \in G$. Then $f_c > e$. The isometry $\varphi(f_c)$ can be written as a composition of a 0-isometry and a translation, hence there are $f \in G_0^*(\mathbf{G}')$ and $c' \in G'$ such that $\varphi(f_c)(t') = f(t') + c'$ is valid for each $t' \in G'$. Since φ fulfils (a) and (b), we have $\varphi(f_c) > e'$, hence

$$(\alpha) \qquad f(t') + c' \geqq t'$$

holds for each $t' \in G'$.

Assume that $f \neq e'$. Then (under analogous denotation as in 2.5, taking $\boldsymbol{G}'$ instead of $\boldsymbol{G}$) we have $B \neq \{0\}$, hence the lattice $(B; \leqq)$ has no greatest element. Thus there is $0 < b \in B$ with $c'(B) \ngeqq b$. Since $f(b) = -b$, we obtain $-b + c' \geqq b$ by putting $t' = b$ into (α), hence $c'(\boldsymbol{B}) \geqq 2b(\boldsymbol{B}) = 2b$, which is a contradiction. Therefore $\varphi(f_c)$ is a translation whenever $c > 0$.

If $0 > c \in G$, then $f_c = (f_{-c})^{-1}$, hence $\varphi(f_c)$ is a translation as well. For each $d \in G$ we have $d = u + v$ with $u = d \wedge 0$, $v = d \vee 0$. Since $\varphi(f_d) = \varphi(f_u f_v) = \varphi(f_u)\,\varphi(f_v)$ we infer that $\varphi(f_d)$ is a translation.

3.4.1. Corollary. *Let $\boldsymbol{G}, \boldsymbol{G}'$ and φ be as in* 3.4. *Then the partial mapping $\varphi_{T(\boldsymbol{G})}$ is an isomorphism of the partially ordered group $T(\boldsymbol{G})$ onto $T(\boldsymbol{G}')$.*

Since $T(\boldsymbol{G})$ is isomorphic with $\boldsymbol{G}$ and $T(\boldsymbol{G}')$ is isomorphic with $\boldsymbol{G}'$, we obtain

3.5. Proposition. *Let $\boldsymbol{G}$ and $\boldsymbol{G}'$ be lattice ordered groups. If there exists a mapping φ of $G^*(\boldsymbol{G})$ onto $G^*(\boldsymbol{G}')$ fulfilling the conditions* (a) *and* (b), *then $\boldsymbol{G}$ is isomorphic with $\boldsymbol{G}'$.*

Let $\boldsymbol{G} = (G; +, \wedge, \vee)$ be an l-subgroup of a lattice ordered group $\boldsymbol{G}' = (G; +, \wedge, \vee)$. For $f' \in G^*(\boldsymbol{G}')$ we denote by f'_G the corresponding partial mapping of the set G into G'. Consider the following conditions:

(a_1) $f'_G \in G_0^*(\boldsymbol{G})$ for each $f' \in G_0^*(\boldsymbol{G}')$.

(b_1) For every $f \in G_0^*(\boldsymbol{G})$ there exists $f' \in G_0^*(\boldsymbol{G}')$ such that $f = f'_G$.

The Dedekind completion of an archimedean lattice ordered group $\boldsymbol{G}$ will be denoted by $d(\boldsymbol{G})$; under the natural embedding, $\boldsymbol{G}$ is an l-subgroup of $d(\boldsymbol{G})$. A lattice ordered group is called strongly projectable if each its polar is a direct factor.

3.6. Proposition. *Let $\boldsymbol{G}$ be an archimedean lattice ordered group and let $\boldsymbol{G}' = d(\boldsymbol{G})$. Then the condition* ($b_1$) *is valid.*

Proof. Let $f \in G_0^*(\boldsymbol{G})$. Let $\boldsymbol{A}, \boldsymbol{B}$ be as in 2.5. From $\boldsymbol{G} = \boldsymbol{A} \times \boldsymbol{B}$ it follows that $d(\boldsymbol{G}) = d(\boldsymbol{A}) \times d(\boldsymbol{B})$ (cf. [3]). Put $f'(z) = z(d(\boldsymbol{A})) - z(d(\boldsymbol{B}))$ for each $z \in d(\boldsymbol{G})$. Then $f' \in G_0(\boldsymbol{G}')$ and $f'_G = f$; hence (b_1) holds.

From 3.6 we easily obtain the following corollary:

3.7. Corollary. *Let $\boldsymbol{G}$ be an archimedean lattice ordered group. Then $G_0^*(\boldsymbol{G})$ is isomorphic with a subgroup of $G_0^*(d(\boldsymbol{G}))$.*

The notion of generalized Dedekind completion $D(\boldsymbol{G})$ of a lattice ordered group $\boldsymbol{G}$ (where $\boldsymbol{G}$ need not be archimedean) has been introduced in [4]. If $\boldsymbol{G}$ is archimedean, then $D(\boldsymbol{G}) = d(\boldsymbol{G})$. In [5] it was proved that to each direct product decomposition $\boldsymbol{G} = \boldsymbol{A} \times \boldsymbol{B}$ of $\boldsymbol{G}$ the corresponding completion is the direct product decomposition $D(\boldsymbol{G}) = D(\boldsymbol{A}) \times D(\boldsymbol{B})$. This implies that the condition (b_1) holds whenever $\boldsymbol{G}$ is a lattice ordered group and $\boldsymbol{G}' = D(\boldsymbol{G})$.

3.8. Proposition. *Let $\mathbf{G}$ be a lattice ordered group. Suppose that $\mathbf{G}$ is archimedean and strongly projectable. Let $\mathbf{G}' = d(\mathbf{G})$. Then the condition* (a_1) *holds.*

Proof. Let $f' \in G_0^*(\mathbf{G}')$. According to 2.5 there is a direct product decomposition $\mathbf{G}' = \mathbf{A}' \times \mathbf{B}'$ of $\mathbf{G}'$ such that $f'(z) = z(\mathbf{A}') - z(\mathbf{B}')$ is valid for each $z \in \mathbf{G}'$. Put $A = A' \cap G$, $B = B' \cap G$, where A' and B' are the underlying sets of $\mathbf{A}'$ or $\mathbf{B}'$, respectively. Then $A = B^\delta$ and $B = A^\delta$ hold in $\mathbf{G}$. Denote $\mathbf{A} = (A; +, \leqq)$, $\mathbf{B} = (B; +, \leqq)$. Since $\mathbf{G}$ is strongly projectable, we have $\mathbf{G} = \mathbf{A} \times \mathbf{B}$. It can be easily verified that $x(\mathbf{A}) = x(\mathbf{A}')$ and $x(\mathbf{B}) = x(\mathbf{B}')$ for each $x \in G$. This yields $f'_G \in G_0^*(\mathbf{G})$ and hence (a_1) holds.

3.9. Corollary. *Let $\mathbf{G}$ be a lattice ordered group. Suppose that $\mathbf{G}$ is archimedean and strongly projectable. Then the groups $G_0^*(\mathbf{G})$ and $G_0^*(d(\mathbf{G}))$ are isomorphic.*

It can be shown by examples that the assertion of 3.8 need not hold if the strong projectability of $\mathbf{G}$ is not assumed.

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