Czechoslovak Mathematical Journal

Danica Jakubíková-Studenovská

Systems of unary algebras with common endomorphisms. I

Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 3, 406–420

Persistent URL: http://dml.cz/dmlcz/101624

Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

SYSTEMS OF UNARY ALGEBRAS WITH COMMON ENDOMORPHISMS I

DANICA JAKUBÍKOVÁ, Košice

(Received August 30, 1977)

Let A be a nonempty set and let $F = F(A)$ be the system of all mappings of the set A into A. For each $f \in F$ we denote by $H(f)$ the set of all mappings $g \in F$ that are permutable with f, i.e.,

$$H(f) = \{g \in F : f(g(x)) = g(f(x)) \quad \text{for each} \quad x \in A\}\,.$$

Further, we put

$$\mathrm{Eq}(f) = \{f_1 \in F : H(f) = H(f_1)\}\,.$$

In this paper the following question will be investigated: how large can the set $\mathrm{Eq}(f)$ be? It will be shown that the relation

(1) $$\operatorname{card} \mathrm{Eq}(f) \leqq c$$

is always valid (independently of the cardinality of the set A) and that the estimate (1) is the best possible.

A mapping $f \in F$ is called connected if for each pair $x, y \in A$ there are positive integers m, n with $f^m(x) = f^n(y)$. In the present Part I the case when f is connected is dealt with. The results of Part I will be applied in Part II to the general case in order to prove the relation (1).

1. PRELIMINARIES

It will be convenient to use the algebraic terminology concerning unary algebras and their homomorphisms.

Let A and $F = F(A)$ be as above and let $f \in F$. Then we can consider f as a unary operation on A. The algebraic system (A, f) is a monounary algebra (for the terminology, cf. B. JÓNSSON [4]) and $H(f)$ is the set of all endomorphisms of the algebra (A, f). Endomorphisms of monounary algebras have been studied by several authors (for references, cf. M. NOVOTNÝ [7] and L. A. SKORNJAKOV [9]).

The set $\mathrm{Eq}(f)$ defined above is the set of all unary operations $f_1 \in F$ with the property that the system of all endomorphisms of the algebra (A, f) coincides with the system of all endomorphisms of (A, f_1). If $f_1 \in \mathrm{Eq}(f)$, then f and f_1 will be said to be *equivalent with respect to endomorphisms* and we shall write $f_1 \,\mathrm{eq}\, f$.

In terms of the algebraic terminology, the relation (1) can be expressed as follows: Let $\mathscr{S} = \{(A, f_i) : i \in I\}$ be a system of monounary algebras having the same set of endomorphisms. Then card $\mathscr{S} \leqq c$.

We denote by F_c the set of all connected mappings of the set A into A. If $f \in F_c$, then the algebra (A, f) is called *connected.*

A nonempty subset C is called a *cycle of a connected unary algebra* (A, f), if (C, f) is a subalgebra of (A, f) and if there is a positive integer k such that $f^k(y) = y$ for each $y \in C$. If a monounary algebra possesses a cycle, then the cycle is uniquely determined and finite. The cardinality of a cycle C will be called also the *period* of C.

It will be proved that if $f \in F$ is connected and $g \in \mathrm{Eq}(f)$, then g is connected as well. For each $f \in F_c$ we give a constructive description of all mappings $g \in F$ that are equivalent to f with respect to endomorphisms (Thms. 1, 2 and 3). For each $f \in F_c$ the following assertions are valid:

(i) *Suppose that* (A, f) *has a cycle* C *with a period* $p > 1$ *such that* $f(x) \in C$ *for each* $x \in A$. *Then* card $\mathrm{Eq}(f) = \varphi(p)$, *where* φ *is the Euler function.*

(ii) *Let* Z *be the set of all integers. Suppose that there exists a subset* $C_0 = \{x_i \in A : i \in Z\}$ *such that* $x_i \neq x_j$ *whenever* i, j *are distinct elements of* Z, $f(x_i) = x_{i+1}$ *for each* $i \in Z$ *and* $f(y) \in C_0$ *for each* $y \in A$. *Then* card $\mathrm{Eq}(f) = 2$.

(iii) *Suppose that neither the assumptions from* (i) *nor those from* (ii) *are fulfilled. Then* $\mathrm{Eq}(f) = \{f\}$.

From (i)–(iii) it follows that the relation

$$(1') \qquad \mathrm{card}\; \mathrm{Eq}(f) < \aleph_0$$

is valid for each connected mapping f and that the estimate (1′) is the best possible.

For $f, g \in F$ we put $f \leqq g$ if $H(f) \supseteq H(g)$. Then $(F, \leqq)$ is a quasiordered set and $\mathrm{Eq}(f) = \{h \in F : h \leqq f \text{ and } f \leqq h\}$. Let us remark that a similar quasiorder on systems of unary operations (defined by means of homomorphisms) has been investigated in the papers [10], [3], [2]; an analogous quasiorder for systems of abstract machines (i.e., partial monounary algebras) has been studied by W. BARTOL [1].

2. SOME AUXILIARY RESULTS

Let us denote by N the set of all positive integers, $N_0 = N \cup \{0\}$, and let Z be the set of all integers. Further, let Z_p be the set of all integers modulo p.

Let (A, f) be a monounary algebra. The notion of the degree $s_f(x)$ of an element $x \in A$ with respect to the operation f was introduced in the paper [7] (cf. also [5] and [8]) as follows:

Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in N_0}$ of elements belonging to A with the property $x_0 = x$, $f(x_n) = x_{n-1}$ for each $n \in N$. Further, we put $A^{(0)} = \{x \in A : f^{-1}(x) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined $A^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \{x \in A - \bigcup_{\alpha<\lambda} A^{(\alpha)} : f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)}\} .$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. In the former case we put $s_f(x) = \infty$, in the latter we set $s_f(x) = \lambda$.

The following assertions are consequences of the definition of $s_f(x)$ (cf. also [8], Lemma 3.14 and Lemma 3.15):

(a) *Assume that either α is an ordinal or $\alpha = \infty$. If $s_f(x) < \alpha$, then $s_f(f(x)) \leqq \alpha$.*

(b) *If $s_f(x) \neq \infty$, then $s_f(f(x)) > s_f(x)$.*

The construction of the mapping H in the following lemma is analogous to that used in [8], Definition 5.10.

Lemma 1. *Let (A, f) be a connected monounary algebra, $x, y \in A$, $x \neq y$, $f(x) = f(y)$, $s_f(x) \leqq s_f(y)$. Assume that $f^n(x) \neq x$ and $f^n(y) \neq y$ for each $n \in N$. Then there exists a mapping $H : A \to A$ such that H is a homomorphism with respect to f, $H(x) = y$ and $H(z) = z$ for each $z \in \bigcup_{n \in N_0} f^{-n}(y)$.*

Proof. This is an immediate consequence of Definition 12 and Theorem 1 of [6].

Lemma 2. *Let (A, f) be a connected monounary algebra, $x, y, z \in A$, $x \neq z$, $j \in N_0$, $f^j(y) = x$, $f(x) = f(z)$ and let $f^n(x) \neq x$ for each $n \in N$. Further, let $H : A \to A$ be a homomorphism with respect to f such that $H(x) = H(z) = x$. Then there exists a mapping $H' : A \to A$ such that H' is a homomorphism with respect to f and $H'(x) = H'(z) = x$, $H'(y) = y$.*

Proof. Let the assumption of the lemma hold. The element z does not belong to the cycle of the algebra (A, f), since H is a homomorphism with respect to f, $H(z) = x$ and x does not belong to the cycle. Thus we have also $s_f(z) \leqq s_f(x)$ and according to Lemma 1 we obtain that there is a mapping $H' : A \to A$ such that H' is a homomorphism with respect to f and $H'(x) = H'(z) = x$, $H'(y) = y$.

Lemma 3. *Let $f, g \in F$, f eq g. Then we have:*

(a) *If $x \in A$, $f(x) = g(x)$, then $g^n(x) = f^n(x)$ for each $n \in N$.*

(b) *If $x \in A$, $f(x) = x$, then $g(x) = x$.*

Proof. (a) The mapping f is obviously a homomorphism with respect to f. Therefore f is a homomorphism with respect to g, thus f and g are permutable. Then we get for $x \in A$

$$f^2(x) = f(f(x)) = f(g(x)) = g(f(x)) = g(g(x)) = g^2(x)$$

and by induction we obtain $f^n(x) = g^n(x)$ for each $n \in N$.

(b) Let $x \in A, f(x) = x$. Put $H(z) = x$ for each $z \in A$. The mapping H is a homomorphism with respect to f because, for each $z \in A$,

$$H(f(z)) = x = f(x) = f(H(z))$$

is valid. Thus H is a homomorphism with respect to g as well which yields

$$g(x) = g(H(x)) = H(g(x)) = x .$$

Let $x \in A$, $h \in F$. We denote

$$K_h(x) = \bigcup_{m \in N_0} \bigcup_{n \in N_0} h^{-m}(h^n(x)) .$$

If $y \in h^{-j}(h^i(x))$ for some $i, j \in N_0$, then obviously $K_h(x) = K_h(y)$. The monounary algebra $(K_h(x), h)$ will be called a *connected component* of (A, h) generated by the element x. Let (A', h) be a subalgebra of (A, h); (A', h) will be called a *connected component* of (A, h), if there exists $x \in A$ such that $A' = K_h(x)$.

Lemma 4. *Let $f, g \in F$, f* eq *g and let (A', f) be a connected component of (A, f). Then (A', g) is a connected component of (A, g).*

Proof. (a) First assume that there exists an element $x_0 \in A'$ such that $g(x_0) \notin A'$. Put $H(x) = f(x)$ for each $x \in A'$, $H(x) = x$ for each $x \in A - A'$. The mapping H is a homomorphism with respect to f, hence H is a homomorphism with respect to g and we get

$$g(f(x_0)) = g(H(x_0)) = H(g(x_0)) = g(x_0) .$$

Let $n \in N$, $n > 1$ and suppose that the relation

$$g(f^{n_1}(x_0)) = g(x_0)$$

holds for each $n_1 \in N$, $n_1 < n$. Then

$$g(f^n(x_0)) = f(g(f^{n-1}(x_0))) = f(g(x_0)) = g(f(x_0)) = g(x_0) .$$

Let $x \in f^{-m}(y)$, $y = f^n(x_0)$, $m, n \in N_0$. Then

$$g(x_0) = g(f^n(x_0)) = g(y) = g(f^m(x)) = g(H^m(x)) = H^m(g(x)) .$$

If we assume that $g(x) \in A'$, then $H^m(g(x)) = f^m(g(x)) \in A'$, which is a contradiction with $g(x_0) \notin A'$. Thus we get

$$H^m(g(x)) = g(x) .$$

Hence, for each $x \in \bigcup_{m\in N_0} \bigcup_{n\in N_0} f^{-m}(f^n(x_0)) = K_f(x_0) = A'$, the relation $g(x) = g(x_0)$ holds. This implies

$$f(g(x_0)) = g(f(x_0)) = g(x_0)$$

and, according to Lemma 3(b), $g(g(x_0)) = g(x_0)$. By induction we obtain $g^n(x_0) = g(x_0)$ for every $n \in N$. Now we set $G(x) = g(x_0)$ for each $x \in \bigcup_{n\in N_0} g^{-n}(x_0)$ and $G(x) = x$ for each $x \in A - \bigcup_{n\in N_0} g^{-n}(x_0)$. The mapping G is a homomorphism with respect to g. According to Lemma 3(b), $f(x_0) \neq x_0$. Further we have $g^n(f(x_0)) = g(x_0)$ for each $n \in N$, hence $G(f(x_0)) = f(x_0)$. Thus G fails to be a homomorphism with respect to f, since

$$G(f(x_0)) = f(x_0) \neq g(x_0) = f(g(x_0)) = f(G(x_0)).$$

We arrived at a contradiction, therefore $g(A') \subseteq A'$, i.e., the set A' is closed with respect to the operation g.

(b) Let $A' = K_f(x_0)$ and let $x \in K_g(x_0)$. Then there are $m, n \in N_0$ such that $g^m(x) = g^n(x_0)$. From part (a) of the proof it follows that $g(x) \in K_f(x)$ and by induction we obtain $g^k(x) \in K_f(x)$ for each $k \in N_0$. Then $g^m(x) \in K_f(x)$. Since $g^m(x) = g^n(x_0) \in K_f(x_0)$, we have $K_f(x) \cap K_f(x_0) \neq \emptyset$ and thus $x \in K_f(x_0)$. Hence $K_g(x_0) \subseteq K_f(x_0)$. Analogously, $K_f(x_0) \subseteq K_g(x_0)$, and therefore $K_f(x_0) = K_g(x_0)$.

Corollary 1. *Let $f \in F_c$, $g \in \mathrm{Eq}(f)$. Then $g \in F_c$.*

3. CONSTRUCTION OF THE OPERATIONS $g \in \mathrm{Eq}(f)$

Let $\mathscr{U}$ be the class of all connected monounary algebras. We introduce the following denotations for classes of monounary algebras:

$\mathscr{O}_1$ – the class of all algebras that belong to $\mathscr{U}$ and contain a one-element cycle;

$\mathscr{O}_2$ – the class of all algebras that belong to $\mathscr{U}$ and possess a cycle with more than one element;

$\mathscr{O}_{20}$ – the class of all algebras (B, g) belonging to $\mathscr{O}_2$ such that $g(x)$ belongs to the cycle of (B, g) for each $x \in B$;

$\mathscr{O}_{21} = \mathscr{O}_2 - \mathscr{O}_{20}$;

$\mathscr{K}$ – the class of all algebras of the class $\mathscr{U}$ with the property that the degree of each element is an ordinal;

$\mathscr{N}_1$ – the class of all algebras belonging to $\mathscr{U} - (\mathscr{O}_1 \cup \mathscr{O}_2)$ and having distinct elements u, u', v, v' such that $f(u) = u'$, $f(v) = v'$, $f(u') = f(v')$ (where f is the corresponding unary operation);

$$\mathscr{N}_2 = \mathscr{U} - (\mathscr{O}_1 \cup \mathscr{O}_2 \cup \mathscr{K} \cup \mathscr{N}_1).$$

Lemma 5. *Let $(A,f) \in \mathcal{N}_2$. Then there is a set $\{x_i \in A : i \in Z\}$ such that $x_i \neq x_j$ for each $i, j \in Z$, $i \neq j$, $f(x_i) = x_{i+1}$ for each $i \in Z$, and $f(y) \in \{x_i : i \in Z\}$ for each $y \in A$.*

Proof. Since $(A,f) \notin \mathcal{K}$, there exists $x \in A$ such that $s_f(x) = \infty$, i.e., there exists a sequence $\{y_i\}_{i \in N_0}$ of elements of A such that $y_0 = x$, $f(y_i) = y_{i-1}$ for each $i \in N$. Put $x_i = f^i(x)$ for each $i \in N_0$, $x_i = y_{-i}$ for each $i \in Z$, $i < 0$. Because of $(A,f) \notin \notin \mathcal{O}_1 \cup \mathcal{O}_2$ the elements $\{x_i\}_{i \in Z}$ are distinct. We have $f(x_i) = x_{i+1}$. Suppose that there exists $y \in A$ with $f(y) \notin \{x_i : i \in Z\}$. Then $y \notin \{x_i : i \in Z\}$. The algebra (A,f) is connected, thus there exists the least $i \in Z$ such that $f^n(y) = x_i$ for some $n \in N$ (obviously, $n \geqq 2$). Let us denote $u = f^{n-2}(y)$, $u' = f^{n-1}(y)$, $v = x_{i-2}$, $v' = x_{i-1}$. The elements u, u', v, v' are distinct and $f(u) = u'$, $f(v) = v'$, $f(u') = f(v')$, which is a contradiction with $(A,f) \notin \mathcal{N}_1$.

Theorem 1. *Let $A = \bigcup_{i \in Z}(\{x_i\} \cup B_i)$, $x_i \neq x_j$ for each $i, j \in Z$, $i \neq j$, $x_i \notin B_i$ for each $i \in Z$. Further, let $f(b_j) = x_{j+1}$ for each $b_j \in \{x_j\} \cup B_j$, $j \in Z$. Then* $\mathrm{Eq}(f) = \{f, g\}$, *where $g(b_j) = x_{j-1}$ for each $b_j \in \{x_j\} \cup B_j$, $j \in Z$.*

Proof. If H is a homomorphism with respect to f, then there exists $i \in Z$ such that

$$H(x_j) = x_{j+i}\,, \quad H(B_j) \subseteq B_{j+i} \cup \{x_{j+i}\} \tag{2}$$

holds for each $j \in Z$. And conversely, if H is a mapping satisfying (2) for some $i \in Z$, then H is a homomorphism with respect to f. Similarly, H is a homomorphism with respect to g if and only if there exists $i \in Z$ such that the condition (2) is fulfilled. Therefore $g \;\mathrm{eq}\; f$.

Now suppose that $h \in \mathrm{Eq}(f)$, $h \neq f$, $h \neq g$. Lemma 4 implies $h \in F_c$. The operation h is a homomorphism with respect to f, thus there exists $i \in Z$ such that $h(x_j) = x_{j+i}$, $h(B_j) \subseteq B_{j+i} \cup \{x_{j+i}\}$ for each $j \in Z$. The component generated by x_j contains x_{j+mi}, $m \in Z$, and it contains no x_k with $k \neq j + mi$ for all $m \in Z$. Since $h \in F_c$, we have $i = 1$ or $i = -1$. If $i = 1$, then $h(x_j) = x_{j+1} = f(x_j)$ for each $j \in Z$, thus there is $j \in Z$, $b_j \in B_j$, $b_{j+1} \in B_{j+1}$ with $h(b_j) = b_{j+1}$. Put $H(z) = z$ for each $z \in A$, $z \neq b_j$, $H(b_j) = x_j$. Then H is a homomorphism with respect to f, but it is not a homomorphism with respect to h, since

$$h(H(b_j)) = h(x_j) = x_{j+1} \neq b_{j+1} = H(h(b_j))\,,$$

which is a contradiction. In the case $i = -1$ we obtain a contradiction analogously as in the case $i = 1$, if we replace f by g.

Hence $\mathrm{Eq}(f) = \{f, g\}$.

Theorem 2. *Let $p \in N$, $p > 1$ and let $A = \bigcup_{i \in Z_p}(\{x_i\} \cup B_i)$, $x_i \neq x_j$ for each $i, j \in Z_p$, $i \neq j$, $x_i \notin B_i$ for each $i \in Z_p$. Further, let $f(b_j) = x_{j+1}$ for each $b_j \in \{x_j\} \cup B_j$, $j \in Z_p$. Then* $\mathrm{Eq}(f) = \{f^i : 0 < i < p$, *$i$ and p are relatively prime$\}$.*

Proof. First assume that g eq f. Since g is a homomorphism with respect to f, we have $g(x_0) \in \{x_j : j \in Z_p\}$. Put $g(x_0) = x_i$. According to Lemma 3(b), $i \neq 0$ holds and we have

$$g(x_j) = g(f^j(x_0)) = f^j(g(x_0)) = f^j(x_i) = x_{i+j}.$$

Suppose that there exists $d \in N$, $d > 1$ such that both p and i are divisible by d. From Lemma 4 it follows that there are $m, n \in N$ with $g^n(x_0) = g^m(x_1)$. By induction it can be easily shown that $g^n(x_0) = x_{ni}$ and hence

$$x_{ni} = g^n(x_0) = g^m(x_1) = g^m(f(x_0)) = f(g^m(x_0)) = f(x_{mi}) = x_{mi+1}.$$

Then $ni \equiv mi + 1 \pmod{p}$ and there exists $l \in Z$ with $ni = mi + 1 + lp$. This implies that 1 is divisible by d, a contradiction. Hence i and p are relatively prime.

Let $j \in Z_p$, $b_j \in B_j$. We obtain

$$f(g(b_j)) = g(f(b_j)) = g(x_{j+1}) = x_{j+1+i},$$

thus $g(b_j) \in f^{-1}(x_{j+i+1}) = \{x_{j+i}\} \cup B_{j+i}$. Suppose that there is $b_{j+i} \in B_{j+i}$ such that $g(b_j) = b_{j+i}$. If we set $H(b_j) = x_j$, $H(z) = z$ for each $z \in A$, $z \neq b_j$, then H would be a homomorphism with respect to f, but it would not be a homomorphism with respect to g, which is a contradiction. Therefore $g(b_j) = x_{j+i}$. We have proved that $g = f^i$.

Now let $g = f^i$, $i \in N$, $0 < i < p$, i and p being relatively prime. Then there exists $k \in N$, $0 < k < p$ such that $ki \equiv 1 \pmod{p}$ and k and p are relatively prime. Since

$$f(x_j) = x_{j+1} = x_{j+ki} = g^k(x_j),$$
$$f(b_j) = x_{j+1} = x_{j+ki} = g^k(x_j) = g^k(b_j)$$

for each $b_j \in B_j$ and each $j \in Z_p$, i.e., $f = g^k$, a mapping $H : A \to A$ is a homomorphism with respect to f, if and only if H is a homomorphism with respect to g.

Corollary 2. *If the assumptions of Theorem 2 are fulfilled, then* card $\mathrm{Eq}(f) = \varphi(p)$, *where φ is the Euler function.*

Remark 1. Observe that in the proof of Theorem 2 it has been shown that the following assertion is valid:

(i) *If f fulfils the assumptions of Theorem 2 and if $g \in \mathrm{Eq}(f)$, then there is $i \in N$ such that $0 < i < p$, i and p are relatively prime and $g = f^i$.*

(ii) *If n is as in* (i) *ind if k is a positive integer such that $k < p$, $ki \equiv 1 \pmod{p}$, k and p are relatively prime, then $f = g^k$.*

(The fact that i and p are relatively prime implies the existence of a positive integer k such that $k < p$, $ki \equiv 1 \pmod{p}$ and k and p are relatively prime.)

Lemma 6. *Let* $A = \{x_0\} \cup B_0$, $x_0 \notin B_0$, $f(z) = x_0$ *for each* $z \in A$. *Then* $\mathrm{Eq}(f) = \{f\}$.

Proof. Let $g \in \mathrm{Eq}(f)$ and let $x \in A$. If we set $H(x) = x_0$, $H(z) = z$ for each $z \in A$, $z \neq x$, the mapping H is a homomorphism with respect to f. Then H is a homomorphism with respect to g and, according to Lemma 3(b), we get (because either $x = x_0$, or we have $g(x) \neq x$ by Lemma 4)

$$f(x) = x_0 = g(x_0) = g(H(x)) = H(g(x)) = g(x).$$

Lemma 7. *Let* (A, f) *be a connected monounary algebra and suppose that* $s_f(x) \neq \infty$ *for each* $x \in A$. *Then* $\mathrm{Eq}(f) = \{f\}$.

Proof. Let $g \in \mathrm{Eq}(f)$. From the assumptions of the lemma and from the fact that g is a homomorphism with respect to f it follows that $f^n(g(x)) \neq x$ for each $x \in A$ and for each $n \in N_0$. Namely, we have $f(x) \neq x$ and $s_f(g(x)) \geqq s_f(x)$ for each $x \in A$. If there exist $x \in A$ and $n \in N_0$ with $f^n(g(x)) = x$, then for $n \geqq 1$ we obtain

$$s_f(x) = s_f(f^n(g(x))) > s_f(g(x)) \geqq s_f(x);$$

hence $n = 0$, $g(x) = x$. According to Lemma 3(b) we have $f(x) = x$, which is a contradiction.

Let $x \in A$. Since $f \in F_c$, there are $m, n \in N_0$ with $f^n(x) = f^m(g(x))$. Denote $f^n(x) = z$, $g(x) = y$. Suppose that $m \geqq n$. Then

$$f^{m-n}(g(z)) = f^{m-n}(g(f^n(x))) = f^m(g(x)) = z,$$

which is a contradiction. Thus $n > m$. Put $n - m = i$. Further assume that $f(x) \neq y$. We obtain

$$(*) \qquad f^i(z) = f^{n-m}(z) = f^{n-m}(f^m(y)) = f^n(y) = f^n(g(x)) = g(f^n(x)) = g(z).$$

Lemma 4 implies $g \in F_c$, hence there exist $k, j \in N$ such that $g^k(z) = g^j(f(z))$. It can be easily shown by induction that $g^k(z) = f^{ki}(z)$ for each $k \in N$. Then

$$f^{ki}(z) = g^k(z) = g^j(f(z)) = f(g^j(z)) = g(g^j(z)) = f(f^{ji}(z)) = f^{ji+1}(z);$$

this yields $ki = ji + 1$, hence $i = 1$.

Further, assume that n is the least positive integer with $f^n(x) = f^{n-1}(y)$ (since $f(x) \neq y$, $n > 1$ holds). Denote $f^{n-2}(y) = c$, $f^{n-1}(x) = a$, $f^{n-2}(x) = b$. Then $a \neq c$ and we have

$$g(b) = g(f^{n-2}(x)) = f^{n-2}(g(x)) = f^{n-2}(y) = c,$$
$$g(a) = g(f^{n-1}(x)) = f^{n-1}(g(x)) = f^{n-1}(y) = f^n(x) = f(f^{n-1}(x)) = f(a).$$

According to Lemma 3(a) we have $g^l(a) = f^l(a)$ for each $l \in N_0$. Further we obtain

$$f(c) = f(f^{n-2}(y)) = f^{n-1}(y) = z = f(a),$$
$$f(g(c)) = g(f(c)) = g(f(a)) = f^2(a),$$

thus $g(c) \in f^{-1}(f^2(a))$. By induction it can be shown that

$$g^l(c) \in f^{-1}(f^{l+1}(a)) = f^{-1}(g^{l+1}(a))$$

for each $l \in N$. Since $g \in F_c$, there are $\iota, \varkappa \in N$ with $g^\iota(a) = g^\varkappa(c)$. Hence we have

$$g^\varkappa(c) \in f^{-1}(g^{\varkappa+1}(a)),$$

$$g^{\varkappa+1}(a) = f(g^\varkappa(c)) = f(g^\iota(a)) = g^\iota(f(a)) = g^\iota(g(a)) = g^{\iota+1}(a),$$

thus $\varkappa = \iota$. Now let $\varkappa$ be the least positive integer such that $g^\varkappa(a) = g^\varkappa(c)$. Put $a' = g^{\varkappa-1}(a)$, $c' = g^{\varkappa-1}(c)$; obviously $a' \neq c'$. Then

$$f(a') = f(g^{\varkappa-1}(a)) = g^{\varkappa-1}(f(a)) = g^{\varkappa-1}(f(c)) = f(g^{\varkappa-1}(c)) = f(c').$$

First assume that $s_f(a') \geqq s_f(c')$. From Lemma 1 it follows that there is a mapping $G \in F$ such that G is a homomorphism with respect to f and $G(c') = a'$, $G(b) = b$. Then G is a homomorphism with respect to g and

$$a' = G(c') = G(g^{\varkappa-1}(c)) = G(g^{\varkappa-1}(g(b))) =$$
$$= g^{\varkappa-1}(g(G(b))) = g^{\varkappa-1}(g(b)) = g^{\varkappa-1}(c) = c',$$

which is a contradiction.

Suppose that $s_f(a') < s_f(c')$. Then according to Lemma 1 there exists a mapping $H \in F$ such that H is a homomorphism with respect to f and $H(a') = H(c') = c'$. Since H is also a homomorphism with respect to g, Lemma 2 implies that there is a mapping $H' \in F$ such that H' is a homomorphism with respect to g and $H'(a') = H'(c') = c'$, $H'(b) = b$. Hence we obtain

$$c' = H'(a') = H'(g^{\varkappa-1}(a)) = H'(g^{\varkappa-1}(f(b))) =$$
$$= g^{\varkappa-1}(f(H'(b))) = g^{\varkappa-1}(f(b)) = g^{\varkappa-1}(a) = a',$$

which is a contradiction. We have proved that $f(x) \neq g(x)$ is impossible for every $x \in A$. The proof is complete.

Lemma 8. *Let (A, f) be a connected monounary algebra and let there exist $u, v \in A$, $k \in N$ such that the following conditions are fulfilled:*

(a) $f^k(u) = u$;

(b) $v \neq f^j(u) \neq f(v)$ for each $j \in N_0$;

(c) $u = f^2(v)$.

Then $\mathrm{Eq}(f) = \{f\}$.

Proof. Let k_0 be the least positive integer with $f^{k_0}(u) = u$. Put $C_0 = \{f^j(u) : : j \in N\}$. If $z \in A$, $k_1 \in N$ and if $f^{k_1}(z) = z$, then obviously $z \in C_0$. Let $g \in \mathrm{Eq}(f)$.

Then g is a homomorphism with respect to f and $g(u) = g(f^{k_0}(u)) = f^{k_0}(g(u))$, hence $g(u) \in C_0$. If $k_0 > 1$, then there is $i \in N_0$, $0 \leqq i < k_0$ such that $g(u) = f^i(u)$. This implies $g(f^j(u)) = f^{j+i}(u)$ for each $j \in N$. Since $f(u) \neq u$, we have $i \neq 0$ according to Lemma 3(b). If $k_0 = 1$, then $g(u) = u$ by Lemma 3(b).

(α) In the case $k_0 = 1$ or $i = 1$ we obtain $g(f^j(u)) = f(f^j(u))$ for each $j \in N_0$, i.e., $g(x) = f(x)$ for each $x \in C_0$. For each $n \in N$ we define by induction the set C_n by putting $C_n = f^{-1}(C_{n-1}) - C_{n-1}$. We have $C_n = \{x \in A : f^n(x) \in C_0, f^{n-1}(x) \notin C_0\}$ for every $n \in N$. Clearly, $n > 1$ and $x \in C_n \cap f^{-1}(C_n)$ imply $f^n(x) \in C_0$ and $f(x) \in C_n$ whence $f^n(x) \notin C_0$; this is a contradiction. Thus $x \in C_n \cap f^{-1}(C_n)$ implies $n = 0$.

We shall prove that $g(x) = f(x)$ for each $x \in C_n$ and for each $n \in N_0$. If there is an element $x \in C_1$ such that $y = g(x) \neq f(x)$, then $f(x) \in C_0$ and we obtain

$$f(g(x) = g(f(x)) = f(f(x)) = f^2(x),$$

i.e., $g(x) \in f^{-1}(f^2(x))$. Then either $g(x) \in C_0$ or $g(x) \in C_1$. If $g(x) \in C_0$, we have $g(x) = f(x)$. Hence $g(x) \in C_1$. Put

$$H(z) = z \quad \text{for each} \quad z \notin \bigcup_{m \in N_0} f^{-m}(x),$$
$$H(z) = f^{jk_0 - m}(x) \quad \text{for each} \quad z \in f^{-m}(x), \quad m \in N_0,$$
where j is an element of N such that $jk_0 - m > 0$.

The mapping H is a homomorphism with respect to f since

$$f(H(z)) = f(z) = H(f(z)) \quad \text{for each} \quad z \notin \bigcup_{m \in N_0} f^{-m}(x),$$
$$f(H(x)) = f(f^{k_0}(x)) = f^{k_0}(f(x)) = H(f(x)),$$
$$f(H(z)) = f(f^{jk_0 - m}(x)) = f^{jk_0 - (m-1)}(x) = H(f(z)) \quad \text{for each} \quad z \in f^{-m}(x), \quad m \in N.$$

Further, we have

$$g(H(x)) = g(f^{k_0}(x)) = f(f^{k_0}(x)) = f^{k_0+1}(x) \neq g(x) = H(g(x))$$

because $f^{k_0+1}(x) \in C_0$. $g(x) \notin C_0$. Thus H fails to be a homomorphism with respect to g. Hence we have proved that $g(x) = f(x)$ for each $x \in C_1$.

Let $n \in N$, $n > 1$ and suppose that for each $m \in N$, $m < n$ and each $x' \in C_m$ the relation $g(x') = f(x')$ holds. If there is $x \in C_n$ such that $g(x) = y \neq f(x)$, then $x' = f(x) \in C_{n-1}$ and we obtain

$$f(g(x)) = g(f(x)) = g(x') = f(x') = f^2(x),$$

i.e., $g(x) \in f^{-1}(f^2(x))$. Since $f^2(x) \in C_{n-2}$, we have $g(x) \in f^{-1}(C_{n-2}) \subseteq C_{n-2} \cup C_{n-1}$. If $y \in C_{n-2}$, then $y \in C_{n-2} \cap f^{-1}(C_{n-2})$ and thus $n = 2$; put

$$H_1(z) = z \quad \text{for each} \quad z \notin \bigcup_{m \in N_0} g^{-m}(x),$$
$$H_1(z) = g^{jk_0 - m}(x) \quad \text{for each} \quad z \in g^{-m}(x), \quad m \in N_0,$$

where j is an element of N such that $jk_0 - m > 0$. The mapping H_1 is a homomorphism with respect to g, but it is not a homomorphism with respect to f since

$$H_1(f(x)) = H_1(x') = x' \neq y = g^{k_0}(y) = g^{k_0-1}(g(y)) =$$
$$= g^{k_0-1}(f(f(x))) = g^{k_0-1}(g(x')) = g^{k_0}(f(x)) = f(g^{k_0}(x)) = f(H_1(x)),$$

which is a contradiction. Thus $y \in C_{n-1}$. Then $f(y) = f(g(x)) = g(f(x)) = g(x') = f(x')$. If $s_f(x') \leqq s_f(y)$, then Lemma 1 implies that there is a mapping $H_2 \in F$ such that H_2 is a homomorphism with respect to f and $H_2(x') = y$, $H_2(y) = y$. Therefore H_2 is a homomorphism with respect to g and according to Lemma 2 there exists a mapping $H'_2 \in F$ such that H'_2 is a homomorphism with respect to g and $H'_2(x') = H'_2(y) = y$, $H'_2(x) = x$. Thus

$$H'_2(f(x)) = H'_2(x') = y \neq x' = f(x) = f(H'_2(x)),$$

and this is a contradiction. If $s_f(x') > s_f(y)$, then it follows from Lemma 1 that there is a mapping $H_3 \in F$ such that H_3 is a homomorphism with respect to f and $H_3(x') = H_3(y) = x'$, $H_3(x) = x$. Therefore

$$H_3(g(x)) = H_3(y) = x' \neq y = g(x) = g(H_3(x)),$$

which is a contradiction. Hence we have proved that $g(x) = f(x)$ for each $x \in \bigcup_{n \in N_0} C_n = A$.

(β) Now let $i > 1$. Lemma 3(a) implies $g(z) \neq f(z)$ for each $z \in A$. Put $x = f(v)$; then $f(g(x)) = g(f(x)) = g(u) = f^i(u)$, i.e., $g(x) \in f^{-1}(f^i(u))$. Since $g \in F_c$, there exists the least positive integer n_0 with $g^{n_0}(x) \in C_0$. If $n_0 = 1$, then $g(x) = f^{i-1}(u)$; we put

$$G(z) = z \quad \text{for each} \quad z \notin \bigcup_{n \in N_0} g^{-n}(x),$$
$$G(z) = f^{jk_0 - ni - 1}(u) \quad \text{for each} \quad z \in g^{-n}(x), \quad n \in N_0,$$
where $j \in N$ is such that $jk_0 - ni - 1 \geqq 0$.

The mapping G is a homomorphism with respect to g since

$$G(g(z)) = g(z) = g(G(z)) \quad \text{for each} \quad z \notin \bigcup_{n \in N_0} g^{-n}(x),$$
$$G(g(x)) = g(x) = f^{i-1}(u) = f^{k_0+i-1}(u) = f^{k_0-1}(f^i(u)) =$$
$$= f^{k_0-1}(g(u)) = g(f^{k_0-1}(u)) = g(G(x)),$$
$$G(g(z)) = f^{jk_0-(n-1)i-1}(u) = f^{jk_0-ni-1}(f^i(u)) =$$
$$= f^{jk_0-ni-1}(g(u)) = g(f^{jk_0-ni-1}(u)) = g(G(z)) \quad \text{for}$$
each $z \in g^{-n}(x)$, $n \in N$.

Further, we have $f(g(v)) = g(f(v)) = g(x) = f^{i-1}(u)$, i.e., $g(v) \in f^{-1}(f^{i-1}(u))$. If

$g(v) \in C_0$, then $g(v) = f^{k_0+i-2}(u) \in C_0$, and $g^n(v) \in C_0$ for each $n \in N$. Thus $g^n(v) \neq x \notin C_0$. Therefore G fails to be a homomorphism with respect to f since

$$G(f(v)) = G(x) = f^{k_0-1}(u) \neq x = f(v) = f(G(v)).$$

Hence $g(v) \notin C_0$ and we set

$$G_1(z) = z \quad \text{for each} \quad z \notin \bigcup_{n \in N_0} f^{-n}(g(v)),$$

$$G_1(z) = f^{jk_0-n}(g(v)) \quad \text{for each} \quad z \in f^{-n}(g(v)), \quad n \in N_0,$$

where j is an element of N such that $jk_0 - n > 0$.

The mapping G_1 is a homomorphism with respect to f. Because of $v \neq g(v)$, $f(v) \neq g(v)$ and $f^n(v) \in C_0$ for each $n \in N$, $n \geqq 2$, we have $f^n(v) \neq g(v) \notin C_0$ for each $n \in N_0$. Then $G_1(v) = v$ and G_1 is not a homomorphism with respect to g since

$$G_1(g(v)) = f^{k_0}(g(v)) \in C_0, \quad g(G_1(v)) = g(v) \notin C_0,$$

and this is a contradiction.

In the case $n_0 > 1$ the relation $g(x) \in f^{-1}(f^i(u)) - C_0$ is valid. Put

$$G_2(z) = z \quad \text{for each} \quad z \notin \bigcup_{n \in N_0} f^{-n}(g(x)),$$

$$G_2(z) = f^{jk_0-n}(g(x)) \quad \text{for each} \quad z \in f^{-n}(g(x)), \quad n \in N_0,$$

where $j \in N$ is such that $jk_0 - n > 0$.

The mapping G_2 is a homomorphism with respect to f. Since $x \neq g(x) \neq f(x)$ and $f^n(x) \in C_0$ for each $1 < n \in N$, $g(x) \notin C_0$, we have $f^n(x) \neq g(x)$ for each $n \in N_0$. Then $G_2(x) = x$ and G_2 fails to be a homomorphism with respect to g since

$$G_2(g(x)) = f^{k_0}(g(x)) \in C_0, \quad g(G_2(x)) = g(x) \notin C_0,$$

which is a contradiction. Hence we have proved that $\mathrm{Eq}(f) = \{f\}$.

Lemma 9. *Let $(A. f)$ be a connected monounary algebra such that $f^n(z) \neq z$ for each $z \in A$, $n \in N$. Assume that there are distinct elements u, u', v, v' of A with the property*

3) $$f(u) = u', \quad f(v) = v', \quad f(u') = f(v').$$

Then $\mathrm{Eq}(f) = \{f\}$.

Proof. The case when $s_f(z) \neq \infty$ for each $z \in A$ has been investigated in Lemma 7. Hence we shall study the opposite case. Let $g \in \mathrm{Eq}(f)$, $g \neq f$. Suppose that x is an element of A with $g(x) = y \neq f(x)$. Since $f \in F_c$, there are $n, m \in N_0$ with $f^n(x) = f^m(y)$. Put $f^n(x) = z$. If $n = m$, then

$$g(z) = g(f^n(x)) = f^n(g(x)) = f^n(y) = z$$

and, according to Lemma 3(b), $f(z) = z$, which is a contradiction with the assumption of the lemma. Let $i = n - m$ and assume first that $i > 0$. In this case we use the same reasoning as in the proof of Lemma 7 beginning with the relation (*) (let us remark that in the part of the proof of Lemma 7 that follows the relation (*) we did not use the assumption that $s_f(u) \neq \infty$ for each $u \in A$). By the same method as in Lemma 7 we arrive at a contradiction.

Assume now that $i < 0$. We have

$$z = f^m(y) = f^m(g(x)) = g(f^m(x)) = g(f^{m-n}(f^n(x))) =$$
$$= g(f^{m-n}(z)) = f^{m-n}(g(z)) = f^{-i}(g(z)),$$

hence $g(z) \in f^i(z)$. By induction it can be shown that, for each $l \in N$, the relation $g^l(z) \in f^{li}(z)$ holds. Put $x_r = f^r(z)$ for each $r \in N_0$, $x_r = f^{-li+r}(g^l(z))$ for each $r \in Z$, $r < 0$, where l is the least positive integer with $-li + r \geqq 0$. If $r \in Z$, $r < 0$, we obtain

$$f^{-r}(x_r) = f^{-r}(f^{-li+r}(g^l(z))) = f^{-li}(g^l(z)) = z,$$

i.e., $x_r \in f^r(z)$. Since $x_r = f^r(z)$ for each $r \in N_0$, we have $x_r \neq x_p$ for each $r, p \in Z$, $r \neq p$. From $g \in F_c$ it follows that there are $k, j \in N$ with $g^k(z) = g^j(f(z))$ and then

$$f^{-ji}(z) = f^{-ji}(f^{-ki}(g^k(z))) = f^{-ji-ki}(g^j(f(z))) =$$
$$= f^{-ji-ki+1}(g^j(z)) = f^{-ki+1}(f^{-ji}(g^j(z))) = f^{-ki+1}(z);$$

hence $i = -1$. Since $x_r = f^{l+r}(g^l(z))$ for $r \in Z$, $r < 0$ and l is the least positive integer with $l + r \geqq 0$, we get $l = -r$. Thus $x_r = g^{-r}(z)$ for each $r \in Z$, $r < 0$. Obviously $g(x_r) = x_{r-1}$ for each $r \in Z$, $r < 0$. If $r > 0$, we have

$$g(x_r) = g(f^r(z)) = f^r(g(z)) = f^{r-1}(f(g(z))) = f^{r-1}(z) = x_{r-1}.$$

The identity $g(x_0) = g(z) = x_{-1}$ holds as well. Obviously $f(x_r) = x_{r+1}$ for each $r \in N_0$. If $r < -1$, then

$$f(x_r) = f(g^{-r}(z)) = g^{-r}(f(z)) = g^{-r-1}(g(f(z))) = g^{-r-1}(z) = x_{r+1}.$$

Since also $f(x_{-1}) = f(g(z)) = z$, we have $f(x_r) = x_{r+1}$, $g(x_r) = x_{r-1}$ for each $r \in Z$. From this and from Lemma 3(a) it follows that $g(w) \neq f(w)$ for each $w \in A$. Put $C_0 = \{x_r : r \in Z\}$. According to the assumption, $f(u') = f(v')$, $u' \neq v'$, thus at least one of the elements u', v' does not belong to C_0; let $v' \notin C_0$. Because of $f \in F_c$, there is the least positive integer l such that $f^l(v) \in C_0$. Since $f(v) = v' \notin C_0$, we get $l > 1$. There exists $r \in Z$ with $f^l(v) = x_r$. Put $w = f^{l-2}(v)$, $w' = f^{l-1}(v)$. Thus $w, w' \notin C_0$ and we have

$$f(g(w')) = g(f(w')) = g(x_r) = x_{r-1},$$

i.e., $g(w') \in f^{-1}(x_{r-1})$. If $g(w') \in C_0$, then $g(w') = x_{r-2}$; we set

$$G(a) = a \quad \text{for each} \quad a \notin \bigcup_{t \in N_0} g^{-t}(w'),$$
$$G(a) = x_{r+t-1} \quad \text{for each} \quad a \in g^{-t}(w'), \quad t \in N_0.$$

The mapping G is a homomorphism with respect to g. Further, we have

$$f(g(w)) = g(f(w)) = g(w') = x_{r-2},$$

i.e., $g(w) \in f^{-1}(x_{r-2})$. If $g(w) \in C_0$, then $g(w) = x_{r-3}$ and $g^t(w) \in C_0$ for each $t \in N$, hence $g^t(w) \neq w' \notin C_0$ for each $t \in N_0$, and $G(w) = w$. Then we get

$$G(f(w)) = G(w') = x_{r-1} \neq w' = f(w) = f(G(w)),$$

which is a contradiction, therefore $g(w) \notin C_0$. We put

$$G_1(a) = a \quad \text{for each} \quad a \notin \bigcup_{t \in N_0} f^{-t}(g(w)),$$
$$G_1(a) = x_{r-t-3} \quad \text{for each} \quad a \in f^{-t}(g(w)), \quad t \in N_0.$$

The mapping G_1 is a homomorphism with respect to f. Because of $w \neq g(w), f(w) \neq g(w)$ and $f^t(w) \in C_0$ for each $t \in N$, $t \geqq 2$, we obtain $f^t(w) \neq g(w) \notin C_0$ for each $t \in N_0$, and $G_1(w) = w$. The mapping G_1 fails to be a homomorphism with respect to g since

$$G_1(g(w)) = x_{r-3} \neq g(w) = g(G_1(w)),$$

and this is a contradiction.

In the case $g(w') \in f^{-1}(x_{r-1}) - C_0$ we set

$$G_2(a) = a \quad \text{for each} \quad a \notin \bigcup_{t \in N_0} f^{-1}(g(w')),$$
$$G_2(a) = x_{r-t-2} \quad \text{for each} \quad a \in f^{-t}(g(w')), \quad t \in N_0.$$

The mapping G_2 is a homomorphism with respect to f. Since $w' \neq g(w'), f^t(w') \in C_0$ for each $t \in N$, $g(w') \notin C_0$, the relation $f^t(w') \neq g(w')$ holds for each $t \in N_0$, and $G_2(w') = w'$. Then we have

$$G_2(g(w')) = x_{r-2} \neq g(w') = g(G_2(w')),$$

hence G_2 is not a homomorphism with respect to g, which is a contradiction. Thus we have proved that if $g \in \mathrm{Eq}(f)$, then $g = f$.

Theorem 3. *Let (A, f) be an algebra belonging to some of the classes $\mathscr{K}, \mathscr{O}_1, \mathscr{O}_{21}$ and $\mathscr{N}_1$. Then* $\mathrm{Eq}(f) = \{f\}$.

Proof. The assertion follows from Lemmas 6, 7, 8 and 9.

The assertions (i), (ii) and (iii) from § 1 are consequences of Theorems 2, 1 and 3.

References

[1] *W. Bartol:* Algebraic complexity of machines, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr., Phys., *22* (1974), 851—856.

[2] *М. Гавалец, Д. Якубикова:* О ширине категории моноунарных алгебр, Math. Slov. *28* (1978), 263—276.

[3] *Д. Якубикова:* Гомоморфизмы унарных алгебр, Math. Slov. *26* (1976), 317—322.

[4] *B. Jónsson:* Topics in universal algebra, Berlin 1972.

[5] *O. Kopeček, M. Novotný:* On some invariants of unary algebras, Czech. Math. J. *24* (*99*), (1974), 219—246.

[6] *M. Novotný:* On a problem of the theory of mappings (Czech), Publ. Fac. Sci. Univ. Masaryk, No 344 (1953), 53—64.

[7] *M. Novotný:* Über Abbildungen von Mengen, Pacif. J. Math., *13* (1963), 1359—1369.

[8] *M. Novotný:* Construction of all homomorphisms of unary algebras (Czech), Katedra matematiky pedagogické fakulty UJEP, Seminář o nových směrech ve vyučování matematice, Brno, 1973.

[9] *L. A. Skornjakov:* Unars, Colloquium on universal algebra, Esztergom, 1977, 133—136.

[10] *И. Жемберы:* Характеризация операций в алгебрах при помощи частично упорядоченных множеств, Matem. časopis 24 (1974), 277—282.

Author's address: 041 54 Košice, Komenského 14, ČSSR (Katedra geometrie a algebry Prírodovedeckej fakulty UPJŠ).