Marshall Saade

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A NOTE ON SOME VARIETIES OF POINT ALGEBRAS

MARSHALL SAADE, Athens

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1. Introduction. In this note our main purpose is to prove the following result.

Theorem 1.1. *Let n be an integer >1 and $(*)$ a generating operation. Then the class $V(n, *)$ of all point algebras of the form $(S^n, *)$ is a variety.*

We also describe the free algebras in these varieties.

The definitions of point algebra and generating operation are given in [6]. However for completeness they are included here. The definition of a shuffling operation will be formulated essentially as given in [4]. Let S be a set such that $|S| \geqq 2$ and n an integer >1. Let $j : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and $x : \{1, \ldots, n\} \to \{0, 1\}$ be functions. A *shuffling operation*, $(\cdot)$, on S^n is defined as follows:

$$(a_1, a_2, \ldots, a_n) \,.\, (b_1, b_2, \ldots, b_n) = (c_1, c_2, \ldots, c_n)$$

where

$$c_i = \begin{cases} a_{j(i)} & \text{if} \quad x(i) = 0 \\ b_{j(i)} & \text{if} \quad x(i) = 1 \quad \text{for each} \quad i \in \{1, \ldots, n\} \,. \end{cases}$$

Thus, for example, if $n = 3, j(1) = 2, j(2) = 1, j(3) = 1, x(1) = 0, x(2) = 0, x(3) = 1$, then $(\cdot)$ is defined by

$$(a_1, a_2, a_3) \,.\, (b_1, b_2, b_3) = (a_2, a_1, b_1) \,.$$

The elements of S^n are called *points* and the groupoid $(S^n, \cdot)$ is called a *point algebra*. There are $(2n)^n$ s-operations on S^n. An s-operation $(*)$ is called a *generating operation* if for each other s-operation $(\cdot)$ on S^n there is a polynomial Θ in $(*)$, $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ such that

$$(a_1, \ldots, a_n) \,.\, (b_1, \ldots, b_n) = \Theta(*, (a_1, \ldots, a_n), (b_1, \ldots, b_n)) \,.$$

In this case we say $(*)$ generates $(\cdot)$ (or that $(\cdot)$ is generated by $(*)$).

It is shown in [2, p. 365] for $n = 2$ and in [5] for each integer $n > 2$ that the class of point algebras defined by the binary operation,

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_n, b_1, \ldots, b_{n-1}),$$

is a variety. The fact that the binary operation $(\cdot)$ defined above is a generating operation, as will be shown below, will play an integral part in the proof of our theorem.

2. Some preliminary results. Let G and H be point algebras. Then G and H are *similar* point algebras if $G = (S^n, \cdot)$ and $H = (T^n, \cdot)$ where n is an integer >1 and G, H are defined by the same binary operation $(\cdot)$.

Lemma 2.1. *Let G and H be similar point algebras. Then $G \cong H$ if and only if $|G| = |H|$.*

Proof. Trivial.

Thus we may conclude that in the class of all similar point algebras of the form $(S^n, \cdot)$ there is exactly one groupoid of each infinite order to within isomorphism and at most one groupoid of each finite order to within isomorphism. The finite ones, of course, are those of order k^n for each positive integer k.

Lemma 2.2. *If G and H are similar point algebras where $|G| \leqq |H|$ then G can be imbedded in H.*

Proof. Let $G = (S^n, \cdot)$ and $H = (T^n, \cdot)$. Then $|S| \leqq |T|$. Let $U \subseteqq T$ such that $|S| = |U|$. Thus clearly $(U^n, \cdot)$ is a subgroupoid of $(T^n, \cdot)$. Hence by Lemma 2.1, $(S^n, \cdot) \cong (U^n, \cdot)$.

Lemma 2.3. *Let $G = (S^n, \cdot)$ and $H = (T^n, \cdot)$ be similar point algebras such that $|S| \neq 1 \neq |T|$. Then G and H satisfy the same identities.*

Proof. Assume $|T| \leqq |S|$. The imbed H in G. (Throughout this proof the image of H in G, under the imbedding, is denoted by H). Clearly any identity satisfied by G is satisfied by H. Now suppose the identity,

$$u(x_1, \ldots, x_m) = v(x_1, \ldots, x_m),$$

is satisfied by H but not by G. Then there are elements $a_i = (a_{1i}, \ldots, a_{ni})$ in S^n, $i = 1, \ldots, m$ such that

$$u(a_1, \ldots, a_m) \neq v(a_1, \ldots, a_m).$$

Thus $u(a_1, \ldots, a_m) = (a_{i_1}, \ldots, a_{i_n}) \neq (a_{k_1}, \ldots, a_{k_n}) = v(a_1, \ldots, a_m)$ where $a_{i_1}, \ldots, a_{i_n}$ and $a_{k_1}, \ldots, a_{k_n}$ are from among the a_{st}, $s = 1, \ldots, n$ and $t = 1, \ldots, m$. Since

$$(a_{i_1}, \ldots, a_{i_n}) \neq (a_{k_1}, \ldots, a_{k_n})$$

then there is a t such that $a_{i_t} \neq a_{k_t}$. This particular a_{i_t} originates from, say, the d-th position in a point a_r, for some r (before actual multiplication of the points in the expression $u(a_1, \ldots, a_m)$ was carried out). Similarly this particular a_{k_t} originates from, say, the e-th position in a point a_s, for some s, in the expression $v(a_1, \ldots, a_m)$. Now if $a_r = a_s$ then $d \neq e$. Thus, say, $d < e$. Let $a, b \in T$ where $a \neq b$ and let $x_0 = (\ldots, a, \ldots, b, \ldots)$ where a is in the d-th position and b is in the e-th position of x_0. The other positions of x_0 may be filled in arbitrarily. Then

$$u(x_0, \ldots, x_0) \neq v(x_0, \ldots, x_0)$$

which contradicts the fact that $u = v$ holds in H. If $a_r \neq a_s$, $r < s$, let $x_0 = (\ldots, a, \ldots)$ and $x_1 = (\ldots, b, \ldots)$ where $a, b \in T$ and a is in the d-th position of x_0 and b is in the e-th position of x_1. Then

$$u(\ldots, x_0, \ldots, x_1, \ldots) \neq v(\ldots, x_0, \ldots, x_1, \ldots)$$

where x_0 is in the r-th place of u and v and x_1 is in the s-th place of u and v. This also contradicts the fact that $u = v$ holds in H. Thus the proof is complete.

Lemma 2.4. *Let $(\cdot)$ be the s-operation defined on S^n as follows:*

$$(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = (a_n, b_1, \ldots, b_{n-1})$$

Then $(\cdot)$ is a generating operation.

Proof. It suffices to prove that any point $(d_1, d_2, \ldots, d_n)$, where each d_i is a fixed a_j or b_k, can be written as a product, under $(\cdot)$, of $(a_1, \ldots, a_n)$'s, $(b_1, \ldots, b_n)$'s. It is easy to see that if we denote by $P(a, n)$ any point containing the element a in the n-th position and by P, any point, that

$$(d_1, d_2, \ldots, d_n) = P(d_1, n) \cdot [P(d_2, n) \cdot [\ldots [P(d_n, n) \cdot P] \ldots]] .$$

Thus it now suffices to show there are points, $P(a_i, n)$, $P(b_j, n)$ for $i, j = 1, 2, \ldots, n$ that can be written as a product of $(a_1, \ldots, a_n)$'s, $(b_1, \ldots, b_n)$'s under $(\cdot)$. However it is easily shown that, if we denote by ${}^k x$ the left associated product of k x's, then

$${}^k(a_1, \ldots, a_n) = (\overbrace{a_n, \ldots, a_n}^{k-1}, a_1, \ldots, a_{n-(k-1)}) = P(a_{n-(k-1)}, n)$$

for $k = 1, 2, \ldots, n + 1$. Similarly, ${}^k(b_1, \ldots, b_n) = P(b_{n-(k-1)}, n)$. Hence, the proof is concluded.

3. Main results. We now prove Theorem 1.1.

Proof. Let $V = V(n, *)$. We will show that V is closed under homomorphisms, direct products and subalgebras. Let V_1 be the class of all point algebras of the form $(S^n, \cdot)$ where $(\cdot)$ is defined by:

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_n, b_1, \ldots, b_{n-1}) .$$

In section 1 it was indicated that V_1 is a variety. From Lemma 2.4 we know that $(\cdot)$ is a generating operation. Since, by hypothesis, $(*)$ is also a generating operation, there are polynomials $f(\cdot, x, y)$ and $g(*, x, y)$ such that

$$x * y = f(\cdot, x, y)$$

and

$$x \,.\, y = g(*, x, y)\,.$$

Let $\mathscr{S} = (S^n, *) \in V$ and let $\Theta : \mathscr{S} \to \mathscr{H} = (H, \circ)$ be a homomorphism from $\mathscr{S}$ onto $\mathscr{H}$. Define the binary operation (∇) on H by

$$s \,\nabla\, t = g(\circ, s, t)\,.$$

Let $\mathscr{S}_1 = (S^n, \cdot)$ and $\mathscr{H}_1 = (H, \nabla)$. If $x, y \in S^n$ then

$$(x \,.\, y)\,\Theta = [g(*, x, y)]\,\Theta = g(\circ, x\Theta, y\Theta) = x\Theta \,\nabla\, y\Theta\,.$$

Thus $\Theta : \mathscr{S}_1 \to \mathscr{H}_1$ is clearly an onto homomorphism. Since V_1 is a variety $\mathscr{H}_1 \cong$ $\cong \mathscr{H}_2 = (T^n, \cdot) \in V_1$. We now show $\mathscr{H} \cong (T^n, *)$. Let $\mathscr{H}_3 = (T^n, *)$. Thus far we have

$$\mathscr{S} \xrightarrow[\text{onto hom.}]{\Theta} \mathscr{H} \quad \text{and} \quad \mathscr{S}_1 \xrightarrow[\text{onto hom.}]{\Theta} \mathscr{H}_1 \xrightarrow[\text{onto iso.}]{\Psi} \mathscr{H}_2\,.$$

Define $\beta : \mathscr{H} \to \mathscr{H}_3$ as follows. Let $x \in H$ and let s be that unique element of T^n such that $x = s\Psi^{-1}$. Then $x\beta = s$. Now let $y \in H$ such that $y\beta = t$. Hence $y = t\Psi^{-1}$. Also let $x = a\Theta$, $y = b\Theta$ with $a, b \in S^n$. Thus

$$\begin{aligned}(x \circ y)\,\beta &= (a\Theta \circ b\Theta)\,\beta = [(a * b)\,\Theta]\,\beta = [(f(\cdot, a, b))\,\Theta]\,\beta = \\ &= [f(\nabla, a\Theta, b\Theta)]\,\beta = [f(\nabla, x, y)]\,\beta = [f(\nabla, s\Psi^{-1}, t\Psi^{-1})]\,\beta = \\ &= [(f(\circ, s, t))\,\Psi^{-1}]\,\beta = [(s * t)\,\Psi^{-1}]\,\beta = s * t = x\beta * y\beta\,.\end{aligned}$$

[We note: $[(s * t)\,\Psi^{-1}]\,\beta = r$ if and only if $(s * t)\,\Psi^{-1} = r\Psi^{-1}$ if and only if $s * t =$ $= r$ since Ψ^{-1} is injective]. So β is a homomorphism. Suppose $x\beta = y\beta$. Then $s = t$ and thus $s\Psi^{-1} = t\Psi^{-1}$. Hence $x = y$ and so β is injective. Let $s \in T^n$. Then there is an x such that $x = s\Psi^{-1}$. So $x\beta = s$ and so β is surjective. Thus $\mathscr{H} \cong \mathscr{H}_3 \in V$ and so V is closed under homomorphisms.]

Now let $\{G_\lambda \mid G_\lambda = (S^n_\lambda, *),\ \lambda \in \Lambda\}$ be a set of members of V. We show $\prod_\lambda G_\lambda \in V$. Consider $\{G'_\lambda \mid G'_\lambda = (S^n_\lambda, \cdot),\ \lambda \in \Lambda\}$. Then $\prod_\lambda G'_\lambda \cong \mathscr{T} = (T^n, \cdot) \in V_1$. We show $(T^n, *) \cong \prod_\lambda G_\lambda$. Let $\Theta : \mathscr{T} \to \prod_\lambda G'_\lambda$ be an isomorphism onto $\prod_\lambda G'_\lambda$. Let $x, y \in T^n$. Thus

$$(x * y)\,\Theta = (f(\cdot, x, y))\,\Theta = f(\cdot, x\Theta, y\Theta) = x\Theta * y\Theta\,.$$

Hence Θ is a homomorphism and is clearly an isomorphism from $(T^n, *)$ onto $\prod_\lambda G_\lambda$. Thus $\prod_\lambda G_\lambda \in V$ and so V is closed under direct products.

Now let $\mathscr{S} = (S^n, *) \in V$ and suppose $\mathscr{H} = (H, *)$ is a subgroupoid of $\mathscr{S}$. However $\mathscr{S}_1 = (S^n, \cdot) \in V_1$ and thus $\mathscr{H}_1 = (H, \cdot) \cong (T^n, \cdot) \in V_1$. Let $\Theta : (T^n, \cdot) \to \mathscr{H}_1$ be an isomorphism from $(T^n, \cdot)$ onto $\mathscr{H}_1$. We show $(T^n, *) \cong \mathscr{H}$. For if $x, y \in T^n$, then

$$(x * y)\, \Theta = (f(\cdot, x, y))\, \Theta = f(\cdot, x\Theta, y\Theta) = x\Theta * y\Theta\, .$$

It follows easily that $\mathscr{H} \cong (T^n, *)$. Thus V is closed under subalgebras and hence by Birkhoff's Theorem [1], V is a variety.

We now describe the V-free algebras in the following two theorems.

Theorem 3.1. *Every infinite groupoid in V is V-free.*

Proof. Let $(S^n, *) \in V$ where S is infinite. Now let $F_V(|S|)$ be the V-free groupoid on $|S|$ free generators. Since $|S^n| = |F_V(|S|)|$ then from Lemma 2.1, $(S^n, *) \cong F_V(|S|)$.

Theorem 3.2. *Let $\mathscr{S} = (S^n, *)$ be a finite member of V. Then $\mathscr{S}$ is V-free if and only if $|S| = kn$ for some positive integer k.*

Proof. Suppose $|S| = kn$. We show $(S^n, *)$ is free on k generators. Now let $T = \{(a_{11}, a_{12}, \ldots, a_{1n}), \ldots, (a_{k1}, a_{k2}, \ldots, a_{kn})\}$ where the a_{ij}, $1 \leqq i \leqq k$, $1 \leqq j \leqq n$, are the distinct members of S. We first show that $(S^n, \cdot)$ is generated by T where $(\cdot)$ is the binary operation defined previously by

$$(a_1, \ldots, a_n) \,.\, (b_1, \ldots, b_n) = (a_n, b_1, \ldots, b_{n-1})\, .$$

Let $(a_1, \ldots, a_n) \in S^n$. Then with reference to the proof of Lemma 2.4 one sees that

$$(a_1, \ldots, a_n) = P(a_1, n) \,.\, [P(a_2, n) \,.\, [\ldots [P(a_n, n) \,.\, P(a_n, n)] \ldots]]\, .$$

Thus it suffices to show there is a point, $P(a_t, n) \in [T, \cdot]$ (the groupoid generated by T under $(\cdot)$) for each $a_t \in S$. However, with reference again to the proof of Lemma 2.4,

$${}^k(a_{i1}, a_{i2}, \ldots, a_{in}) = (a_{in}, \ldots, a_{in}, a_{i1}, \ldots, a_{i(n-(k-1))}) = P(a_{i(n-(k-1))}, n)$$

for each i, j. Thus T generates $(S^n, \cdot)$. Recalling that $x \,.\, y = g(*, x, y)$ for all $x, y \in S^n$ we let $(a_1, \ldots, a_n) \in S^n$. Then $(a_1, \ldots, a_n) \in [T, \cdot]$. Thus there is a polynomial Φ such that

$$(a_1, \ldots, a_n) = \Phi(\cdot, (a_{11}, \ldots, a_{1n}), \ldots, (a_{k1}, \ldots, a_{kn}))\, .$$

However each product, $(a_{i1}, \ldots, a_{in}) \,.\, (a_{j1}, \ldots, a_{jn})$ in Φ may be written as $g(*, (a_{i1}, \ldots, a_{in}), (a_{j1}, \ldots, a_{jn}))$. Thus $(a_1, \ldots, a_n) \in [T, *]$. That is, $(S^n, *)$ is generated by T. Let $\Theta : F_V(k) \to (S^n, *)$ be a homomorphism from the V-free groupoid on k

generators onto $(S^n, *)$. Clearly $|F_V(k)| \leqq (kn)^n$ since if $F_V(k)$ is generated by $B = \{b_{11}, \ldots, b_{1n}), \ldots, (b_{k1}, \ldots, b_{kn})\}$ then $[B, *] = F_V(k)$ has at most $(kn)^n$ points from the fact there are at most kn choices for each b_i. However $|(S_n, *)| = (kn)^n$. Thus $|F_V(k)| \geqq (kn)^n$. Hence $|F_V(k)| = (kn)^n$. Then by Lemma 2.1, $(S^n, *) \cong F_V(k)$. Thus $(S^n, *)$ is V-free on k generators. The second part of the proof follows easily.

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Author's address: University of Georgia, Athens, Georgia, U.S.A.