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FUNCTIONAL SEPARATION OF INDUCTIVE LIMITS AND REPRESENTATION OF PRESHEAVES BY SECTIONS PART ONE SEPARATION THEOREMS FOR INDUCTIVE LIMITS OF CLOSURED PRESHEAVES

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INTRODUCTION

This paper is devoted to the problem of when a presheaf of topological (uniform, ...) spaces over a topological space can be represented by sections in its covering space The method used here requires the existence of a Hausdorff topology in the stalks which is coarser than the topology of the inductive limit.

The problem of existence of such a topology, which is important by itself, is discussed in the first part, where some theorems on functional separatedness of inductive limits of presheaves are established. A certain class of categories in which these theorems hold and which contains the categories of topological, uniform, proximal, convergence, topilogical linear, ... spaces, is defined in preliminary chapter 0. The separation theorems are proven for that class of categories. The existence of the above mentioned topology follows from these.

The second part includes embeddings of presheaves into presheaves of compact spaces, from which we get again the theorems on functional separation of inductive limits. Moreover, these embeddings are necessary for the representation theorems in the fourth part and, in the third part, they are used to establish a sufficient condition for the cannonical maps of the projective limit of a presheaf into its members to be homeomorphisms.

In the third part we discuss, for completeness, some cases when an inductive limit has a Hausdorff, completely regular, normal or metrisable topology coarser than that of the inductive limit. The metrisable case enables us to prove a representation theorem in the fourth part. Further, we get there a sufficient condition for the canonical maps of the projective limit of a presheaf into its members to be homeomorphisms.

It is shown that in some cases the tools developed in the foregoing sections can be used for the verification of nonemptiness of projective limits of certain presheaves.

In the fourth part, representation theorems for certain presheaves are proven. They state that there is a closure in the covering space of the presheaf in question, such that the set of all continuous sections over any open set, endowed with the topology of pointwise convergence, is precisely the set of those that canonically correspond to the sets of the presheaf, and that the canonical maps of the spaces of the presheaf onto the spaces of the sections are homeomorphisms. In the final section we find when there is a topology with the mentioned properties in the covering space. The method developed there gives us a representation theorem in terms of topological spaces.

The paper is written in terms of closure spaces as our aim was not to lose generality. The theorems proven here are simple if the presheaf is over a countable set. If it is not the case, a question arises of when the limit of a family of maps of a set into the members of a presheaf is a map onto the projective limit of the presheaf. Some sufficient conditions for this are in the second section of the part one.

The reader who is not interested in the subject of the second and third section of the third part, can go through the paper and skip these.

As the subject studied in the paper is in many places not easy for survey, and as we know that a careful reader may lose two days musing over a place where the author has spared two lines, we tried to write the proofs in detail and shun the phrases "it can be easily seen", "likewise as ... we can prove ..." "it easily follows that ..." as much as possible. To make the paper easier for reading we often recall the foregoing notions and circumstances, being aware of the fact that otherwise the paper could be shortened, but made very bad for reading.

The purpose of the paper is twofold: to study the functional separation of inductive limits ant to deal with the representation of presheaves by sections. As a matter of fact, we have written at the beginning that the purpose was to study the representations and that the studying the separation was necessary for it. However, we might have stated as well that the separation was the main purpose and that the representation followed as its application.

Because of its considerable length the article cannot be published in one piece. The four parts it has been divided into shall come out as separate papers in the Czechoslovak Mathematical Journal. Bearing in mind that all these four papers can and should be considered as parts of one article, we numerate their sections and all other items accordingly. Also the notation and the basic definitions are introduced at the beginning of the first part only and are not rapeated in the others being bound to serve for all the four papers. For instance, Theorem 2.1.7 is in the first section of the second part. If we recall this theorem, say, in the third part, we write simply "see Th. 2.1.7". When we recall Definition 0.5 by writing "see 0.5", the reader will find it in the first part, Chapter 0.

0.1. A partially ordered set $\langle A \leqq \rangle$ is called *right* (*left*) *directed* if for every $a, b \in A$ there is $c \in A$ such that $a \leqq c$, $b \leqq c$ ($c \leqq a$, $c \leqq b$). A set $B \subset A$ is called *cofinal* if for every $a \in A$ there is $b \in B$ such that $a \leqq b$.

0.2. Let $\mathfrak{K} = (\mathcal{O}, \mathcal{M})$ be a category where $\mathcal{O}$ are objects and $\mathcal{M}$ morphisms. *An inductive family* form $\mathfrak{K}$ is a family $\mathcal{S} = \{S_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ where

(a) $\langle A \leqq \rangle$ is a partially ordered set,

(b) $S_\alpha \in \mathcal{O}$ for each $\alpha \in A$,

(c) for every $\alpha, \beta \in A$, $\alpha \leqq \beta$ there is a morphism $\varrho_{\alpha\beta} \in \mathcal{M}$ between $\mathcal{S}_\alpha$ and S_β (we write $\varrho_{\alpha\beta} : S_\alpha \to S_\beta$) such that $\varrho_{\alpha\alpha}$ is identical and $\varrho_{\alpha\gamma} = \varrho_{\beta\gamma} \circ \varrho_{\alpha\beta}$ for all $\alpha, \beta, \gamma \in A$, $\alpha \leqq \beta \leqq \gamma$. We shall sometimes write $\mathcal{S} \subset \mathfrak{K}$ if $\mathcal{S}$ is from $\mathfrak{K}$. $\mathcal{S}$ is called *a presheaf* if $\langle A \leqq \rangle$ is right directed. If $B \subset A$ is a subset of A with the induced order, we put $\mathcal{S}_B = \{S_\alpha | \varrho_{\alpha\beta} | \langle B \leqq \rangle\}$.

0.3. A family $\mathcal{G} = \{g_\alpha \mid \alpha \in A\}$ of $\mathfrak{K}$-morphisms is called *inductive* for $\mathcal{S}$ if each g_α is a morphism between S_α and a common object $R \in \mathcal{O}$. We say that $\mathcal{G}$ is *a family between* $\mathcal{S}$ *and* R; $\mathcal{G}$ is called *compatible* if $g_\alpha = g_\beta \circ \varrho_{\alpha\beta}$ for all $\alpha \leqq \beta$.

0.4. An object $I \in \mathcal{O}$ is called *an inductive limit* of $\mathcal{S}$ if

(a) there is a compatible family $\Phi = \{\xi_\alpha \mid \alpha \in A\}$ between $\mathcal{S}$ and I,

(b) if $\mathcal{F} = \{f_\alpha \mid \alpha \in A\}$ is a compatible family between $\mathcal{S}$ and an object R, then there is a unique $\mathfrak{K}$-morphism $f : I \to R$ with $f_\alpha = f \circ \xi_\alpha$ for all $\alpha \in A$. Then f is denoted by $\varinjlim \mathcal{F}$ and the inductive limit of $\mathcal{S}$ by $\varinjlim \mathcal{S}$. The morphism $\xi_\alpha \in \Phi$ and f are called *natural* or *canonical*. If $\mathfrak{K}$ is a category whose morphisms are maps, $p \in I$, $\alpha \in A$, $a \in S_\alpha$ and $\xi_\alpha(a) = p$ then a is called *a representative of* p.

It can be easily seen from (a), (b) that if an object $J \in \mathcal{O}$ and a family $\Omega = \{\eta_\alpha \mid \alpha \in A\}$ fulfil (a), (b), then there are $\mathfrak{K}$-morphisms $i : I \to J$ and $j : J \to I$ such that $i \circ j$ and $j \circ i$ are identical and $\eta_\alpha = i \circ \xi_\alpha$ for each $\alpha \in A$. Thus I and J are $\mathfrak{K}$-isomorphic.

0.5. A category $\mathfrak{K}$ is called *inductive* if every presheaf $\mathcal{S}$ from $\mathfrak{K}$ has an inductive limit in $\mathfrak{K}$. It is known that each of the following categories is inductive: The category of sets, closure spaces, topological spaces, semiuniform spaces, proximity spaces, topological linear spaces, ... These are denoted by *SET, CLOS, TOP, SEM, PROX, UNIF, LIN*, ...

0.6. A family $\mathcal{G} = \{g_\alpha \mid \alpha \in A\}$ of $\mathfrak{K}$-morphisms is called *projective for* $\mathcal{S}$ if each g_α is between a common object T and S_α. We then say that $\mathcal{G}$ *is between* P *and* $\mathcal{S}$. $\mathcal{G}$ is called *compatible* if $g_\beta = \varrho_{\alpha\beta} \circ g_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$. An object T

of $\mathfrak{K}$ is called *a projective limit of* $\mathscr{S}$ if

(a) there is a compatible family $\Psi = \{p_\alpha \mid \alpha \in A\}$ between P and $\mathscr{S}$,

(b) if $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ is a compatible family between an object T and $\mathscr{S}$, then there is a unique $\mathfrak{K}$-morphism $f : T \to P$ with $f_\alpha = p_\alpha \circ f$ for all $\alpha \in A$. Then f is denoted by $\varprojlim \mathscr{F}$ and the projective limit P of $\mathscr{S}$ by $\varprojlim \mathscr{S}$. As in 0.4 we can prove that any two projective limits of $\mathscr{S}$ are $\mathfrak{K}$-isomorphic. A category $\mathfrak{K}$ is called *projective* if any inductive family from $\mathfrak{K}$ has a projective limit in $\mathfrak{K}$. Each category from 0.5 is projective. If $\mathscr{S} = \{S_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ is a projective family from SET then $\varprojlim \mathscr{S}$ consists of all families $a = \{a_\alpha \in S_\alpha \mid \alpha \in A\}$ with $\varrho_{\alpha\beta}(a_\alpha) = a_\beta$ for every $\alpha, \beta \in A$, $\alpha \leqq \beta$.

0.7. Let $\mathfrak{K}$, $\mathfrak{L}$ be two inductive categories, $\mathfrak{K} \subset \mathfrak{L}$. If a presheaf $\mathscr{S}$ is from $\mathfrak{K}$, then it is also from $\mathfrak{L}$. There thus is $I = \varinjlim \mathscr{S}$ in $\mathfrak{K}$ and also $J = \varinjlim \mathscr{S}$ in $\mathfrak{L}$. It may happen that $I \neq J$. For example, if $\mathscr{S} \subset TOP$, then $\varinjlim \mathscr{S}$ in TOP is a topological space (I, t_I^*) (here I is a set and t_I^* is a topology in it). But $\mathscr{S}$ can be regarded as a presheaf from $CLOS$. There is a closure space (J, t_J^*) which is an inductive limit of $\mathscr{S}$ in $CLOS$. Then (I, t_I^*) and (J, t_J^*) need not be the same. It can be seen that $I = J$ but the closure t_J^* may be finer than that yielded by the topology t_I^*. Thus, when necessary, we write $\mathfrak{K} \varinjlim \mathscr{S}$, $\mathfrak{L} \varinjlim \mathscr{S}$ to distinguish the inductive limits in $\mathfrak{K}$ from those in $\mathfrak{L}$.

0.8. Lemma. *Let* $\mathscr{S} = \{S_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *be a preheaf from a category* $\mathfrak{K}$, *let* $B \subset A$ *be cofinal in* $\langle A \leqq \rangle$.

A) *If* $\mathscr{G} = \{g_\gamma \mid \gamma \in B\}$ *is a compatible family between* $\mathscr{S}_B$ *and a* $\mathfrak{K}$*-object* O, *then there is a unique compatible family* $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ *between* $\mathscr{S}$ *and* O *with* $f_\gamma = g_\gamma$ *for* $\gamma \in B$. ($\mathscr{F}$ *is called an extension of* $\mathscr{G}$). *Further, we have* $f = \varinjlim \mathscr{F} = \varinjlim \mathscr{G} = g$.

B) $J = \varinjlim \mathscr{S}_B$ *exists iff there exists* $I = \varinjlim \mathscr{S}$ *and they are isomorphic.*

Proof. A: If $\alpha \in A$ then there is $\beta \in B$ with $\alpha \leqq \beta$. We put $f_\alpha = g_\beta \circ \varrho_{\alpha\beta}$. If $\gamma \in B$ is another element with $\gamma \geqq \alpha$, then there is $\varepsilon \in B$ with $\varepsilon \geqq \beta$, $\varepsilon \geqq \gamma$. From $g_\varepsilon \circ \varrho_{\alpha\varepsilon} = g_\varepsilon \circ \varrho_{\beta\varepsilon} \circ \varrho_{\alpha\beta} = g_\beta \circ \varrho_{\alpha\beta} = f_\alpha$ we get that the definition of f_α does not depend on the choice of $\beta \in B$ with $\beta \geqq \alpha$, because $\langle A \leqq \rangle$ is right directed (see 0.1). If $\alpha, \beta \in A$, $\alpha \leqq \beta$ then $f_\beta \circ \varrho_{\alpha\beta} = g_\gamma \circ \varrho_{\beta\gamma} \circ \varrho_{\alpha\beta} = g_\gamma \circ \varrho_{\alpha\gamma} = f_\alpha$, where $\gamma \in B$, $\gamma \geqq \beta$ is arbitrary. Thus $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ is the desired family. The rest of (A) is clear.

B) Let I exist and $\mathscr{G} = \{g_\gamma \mid \gamma \in B\}$ be compatible between $\mathscr{S}_B$ and a $\mathfrak{K}$-object O. By A, there is $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ between $\mathscr{S}$ and O with $f_\gamma = g_\gamma$ for $\gamma \in B$. Thus there is a unique $f : I \to O$ with $f \circ \xi_\alpha = f_\alpha$ for all $\alpha \in A$ ($\xi_\alpha : S_\alpha \to I$ are the canonical morphisms – see 0.4), so $f \circ \xi_\beta = f_\beta = g_\beta$ if $\beta \in B$. If $g : I \to O$ is a $\mathfrak{K}$-morphism with $g \circ \xi_\beta = g_\beta$ for all $\beta \in B$ then $\mathscr{H} = \{g \circ \xi_\alpha \mid \alpha \in A\}$ is an extension of $\mathscr{G}$, hence (by A) $\mathscr{F} = \mathscr{H}$. Further, $g = \varinjlim \mathscr{H} = \varinjlim \mathscr{F} = f$, which shows by the definition of inductive limits in 0.4 that $\varinjlim \mathscr{S}_B$ exists and $I = \varinjlim \mathscr{S}_B$. By 0.4, every $\varinjlim \mathscr{S}_B$ is isomorphic to I. Conversely, let J exist and $\mathscr{N} = \{\eta_\beta : S_\beta \to J \mid \beta \in B\}$ be the set of the canonical maps. By (A), there is a unique family $\mathscr{M} = \{\xi_\alpha \mid \alpha \in A\}$ between $\mathscr{S}$ and J with $\xi_\beta = \eta_\beta$ for $\beta \in B$. If $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ is compatible between $\mathscr{S}$ and O then so is $\mathscr{G} = \{f_\beta \mid \beta \in B\}$ between $\mathscr{S}_B$ and O. There is a unique $f : J \to O$ with $f_\beta = f \circ \eta_\beta$ for all $\beta \in B$. We show that $f \circ \xi_\alpha = f_\alpha$ for all $\alpha \in A$. Then by 0.4, $J = \varinjlim \mathscr{S}$. The family $\mathscr{R} = \{f \circ \xi_\beta \mid \beta \in B\}$ is compatible between $\mathscr{S}_B$ and O. Further, $\mathscr{R} = \mathscr{G}$ since $f \circ \xi_\beta = f \circ \eta_\beta = f_\beta$ for $\beta \in B$. By (A), there is a unique extension $\mathscr{H} = \{h_\alpha \mid \alpha \in A\}$ of $\mathscr{R}$. We have $h_\beta = f \circ \xi_\beta$ for $\beta \in B$. If $\alpha \in A$, then there is $\beta \in B$ with $\beta \geqq \alpha$. Then $f \circ \xi_\alpha = f \circ \xi_\beta \circ \varrho_{\alpha\beta} = h_\beta \circ \varrho_{\alpha\beta} = h_\alpha$. As $\mathscr{F}$ is the extension of $\mathscr{G}$ and $\mathscr{R} = \mathscr{G}$, we get $\mathscr{H} = \mathscr{F}$. Thus $f \circ \xi_\alpha = h_\alpha = f_\alpha$ for all $\alpha \in A$, which shows that $J = \varinjlim \mathscr{S}$ as desired.

0.9. If $\mathscr{X}$ is a closure (semiuniform, proximal, ...) space, we denote by $|\mathscr{X}|$ the set of $\mathscr{X}$, i.e. the underlying set. Thus $\mathscr{X} = (|\mathscr{X}|, t)$, where t is the closure (semiuniformity, ...) of $\mathscr{X}$. If $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ is an inductive family from *CLOS* (*TOP*, *SEM*, ...), we put $|\mathscr{S}| = \{|\mathscr{X}_\alpha| \, |\varrho_{\alpha\beta}| \, \langle A \leqq \rangle\}$. Then $|\mathscr{S}| \subset$ *SET*. An object $\mathscr{X}$ from *CLOS* (*TOP*, *SEM*, ...) is denoted by $\mathscr{X} = (X, t)$, where $X = |\mathscr{X}|$ and t is the closure (topology, ...) of $\mathscr{X}$. If $\mathscr{X} = (X, t)$ is from *SEM* or *PROX*, then the closure generated in X by t is denoted by $\text{cl}\, t$. We put $\text{cl}\, \mathscr{X} = (X, \text{cl}\, t)$. If $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ is an inductive family from *SEM* or *PROX*, then we put $\text{cl}\, \mathscr{S} = \{\text{cl}\, \mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$. Clearly $\text{cl}\, \mathscr{S}$ is from *CLOS* (i.e. $\varrho_{\alpha\beta} : \text{cl}\, \mathscr{X}_\alpha \to \text{cl}\, \mathscr{X}_\beta$ are continuous). If $\mathscr{X} = (X, t)$ is a closure space, then the finest topology in X coarser than t is denoted by mt. Setting $m\mathscr{X} = (X, mt)$, then $m\mathscr{X}$ and mt are called topological modifications of $\mathscr{X}$ and of t respectively.

If X is an object from *SET*, then it can be regarded as an object from *CLOS* (*TOP*, *SEM*, ...) with the discrete closure (topology, ...) d. We write $\text{cl}\, X = (X, d)$. Thus *SET* will be regarded as a subcategory of the mentioned categories.

0.10. An inductive category $\mathfrak{L}$ is called *inductively closed* (i.c.) if

(1) $\mathfrak{L} \subset$ *CLOS* or $\mathfrak{L} \subset$ *SEM* or $\mathfrak{L} \subset$ *PROX* (see 0.5);

(2) there is an object R in $\mathfrak{L}$ which, being regarded as an object from *CLOS* (*SEM*, *PROX*), is the real line with the usual closure (semiuniformity, proximity);

(3) given a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from $\mathfrak{L}$ and its canonical maps $\{\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{I} = \varinjlim \mathscr{S} \mid \alpha \in A\}$, then

(a) if $p \in |\mathscr{I}|$, then there is $\alpha \in A$ such that p has a representative in $|\mathscr{X}_\alpha|$ (see 0.4, 0.9);

(b) if $p \in |\mathscr{I}|$ and if $a \in |\mathscr{X}_\alpha|$, $b \in |\mathscr{X}_\beta|$ are representatives of p then there is $\gamma \geqq \alpha, \beta$ such that $\varrho_{\alpha\gamma}(a) = \varrho_{\beta\gamma}(b)$.

The point (b) readily yields that if every $\varrho_{\alpha\beta} : |\mathscr{X}_\alpha| \to |\mathscr{X}_\beta|$ is $1-1$ (see 0.9), then so is every $\xi_\alpha : |\mathscr{X}_\alpha| \to |\mathscr{I}|$ (a map $f : P \to Q$ is called $1-1$ if $f(x) \neq f(y)$ for any $x, y \in P$, $x \neq y$). Throughout the paper $\mathfrak{L}$ will mean an inductively closed category.

0.11. Let $\mathscr{X}$ be an object from an i.c. category $\mathfrak{L}$. The set of all $\mathfrak{L}$-morphisms (bounded ones) between $\mathscr{X}$ and the real line R (which is regarded as an $\mathfrak{L}$-object – see 0.12(2)) is denoted by $C(\mathscr{X} \to R \mid \mathfrak{L})$ $(C^*(\mathscr{X} \to R \mid \mathfrak{L}))$. These morphisms are functions. If C is the field of complex numbers, then $C(\mathscr{X} \to C \mid \mathfrak{L}) = \{f = f_1 + + if_2 \mid f_1, f_2 \in C(\mathscr{X} \to R \mid \mathfrak{L})\}$. Likewise we define $C^*(\mathscr{X} \to C \mid \mathfrak{L})$. Clearly, if $\mathscr{X}$ is from *SEM* or *PROX* and $F \subset C_{SEM}(\mathscr{X} \to R)$ or $F \subset C_{PROX}(\mathscr{X} \to R)$, respectively, then $F \subset C_{CLOS}(\mathrm{cl}\, \mathscr{X} \to R)$ (see 0.9). The same holds for bounded morphisms.

We introduce the i.c. categories to avoid the necessity of repeating the statements, with the same proof, for the presheaves from various categories. Now we can write only one statement which holds for all presheaves from i.c. categories for example from *CLOS*, *TOP*, *SEM*, ... Otherwise we should have to formulate the statement for each of them.

0.12. A family $\mathscr{S} = \{\mathscr{X}_\alpha \mid \varrho_{\alpha\beta} \mid \langle A \leqq \rangle\}$ is called *an inductive family* (presheaf) of $\mathfrak{L}$-objects if $\mathfrak{L}$ is i.c., $\mathscr{X}_\alpha$ is from $\mathfrak{L}$ for all $\alpha \in A$ and $|\mathscr{S}|$ (see 0.9) is an inductive family (presheaf) from *SET* (it means that $\varrho_{\alpha\beta}$ need not be $\mathfrak{L}$-morphisms but just some maps of $\mathscr{X}_\alpha$ into $\mathscr{X}_\beta$). For example, if $\mathscr{S}' = \{S_\alpha \mid \varrho_{\alpha\beta} \mid \langle A \leqq \rangle\}$ is from *SET* and each set S_α is endowed with a closure τ_α, then $\mathscr{S} = \{(S_\alpha, \tau_\alpha) \mid \varrho_{\alpha\beta} \mid \langle A \leqq \rangle\}$ is an inductive family of *CLOS*-objects. An inductive family of *CLOS* (*TOP*, *UNIF*, ...) objects is called *closured* (*topologized*, *uniformized*, ...).

The definition of inductive limits can be extended to inductive families of $\mathfrak{L}$-objects. Indeed, using Definition 0.4 to define the inductive limit of a presheaf from $\mathfrak{L}$, we do not use the fact that $\varrho_{\alpha\beta}$ are $\mathfrak{L}$-morphisms. Thus Def 0.4 works even for inductive families of $\mathfrak{L}$-objects.

It is known that if $\mathscr{S} = \{\mathscr{X}_\alpha \mid \varrho_{\alpha\beta} \mid \langle A \leqq \rangle\}$ is a presheaf of *CLOS* objects. then *CLOS* $\varinjlim \mathscr{S} = (I, t)$, where $I =$ *SET* $\varinjlim \mathscr{S}$ (see 0.7, 0.9) and t is inductively generated by the canonical maps $\xi_\alpha : \mathscr{X}_\alpha \to I$. The same holds in terms of *TOP*, *UNIF*, *SEM*, *PROX*.

Given two sets X, A, let f_α for each $\alpha \in A$ be a map of X into a topological space $\mathscr{X}_\alpha$ (of $\mathscr{X}_\alpha$ into X). We say a topology t in X is *projectively* (*inductively*) defined in X by the maps $\{f_\alpha \mid \alpha \in A\}$ if t is the coarsest (finest) of all topologies τ in X such that every f_α is $\tau - \tau_\alpha$ $(\tau_\alpha - \tau)$ continuous. Likewise we can proceed in terms of *CLOS*, *SEM*, ...

0.13. Let $\mathscr{X}$, $\mathscr{J}$ be objects from an i.c. category $\mathfrak{L}$, let F be a set of $\mathfrak{L}$-morphisms between $\mathscr{X}$ and $\mathscr{J}$. We say that F *is separating* (or that it distinguishes points of $\mathscr{X}$) if for any $x, y \in |\mathscr{X}|$ with $x \neq y$ there is $f \in F$ with $f(x) \neq f(y)$. We say $f \in F$ is $1-1$ if the one – point set $\{f\}$ is separating.

0.14. Given a closure (topological, uniform, ...) space (X, t), $B \subset X$, then the closure (topology ...) induced in B by t is denoted by $\mathrm{ind}_B\, t$ (shortly $\mathrm{ind}\, t$ or t/B). The set of all continuous (uniformly continuous) or bounded real functions of this sort on a closure (semiuniform) space is denoted by $C(\mathscr{X} \to R)$ $(U(\mathscr{X} \to R))$ or $C^*(\mathscr{X} \to R)$ $(U^*(\mathscr{X} \to R))$ respectively.

0.15. We say that a map $h : S \to T$ is *onto* (*into*) if $h(S) = T(h(S) \subset T)$. The *dual* map h^* is the map which assigns to every function f on T a function h^*f on S such that $h^*f = f \circ h$.

If s or t is a closure (semiuniformity, ...) in the set S or T respectively, then we say h is *a homeomorphism* into (*uniform embedding*), if $h : (S, s) \to (h(S), \text{ind } t)$ is a homeomorphism (uniformly continuous together with h^{-1}).

0.16. Lemma. *Let X, Y be two sets, $h : X \to Y$ a map, $F(X)$ and $F(Y)$ some sets of real functions on X and on Y respectively. Assume $F(X)$ is separating* (see 0.13, recall that X, Y are from *CLOS* – see 0.9). *If $F(X) \subset h^* F(Y) = \{h^*f \mid f \in F(Y)\}$ then h is* $1-1$.

Proof. If $x, y \in X$, $h(x) = h(y)$, $f \in F(X)$, then there is $g \in F(Y)$ with $f = g \circ h$. Hence $f(x) = g \circ h(x) = g \circ h(y) = f(y)$. Thus $f(x) = f(y)$ for all $f \in F(X)$. As $F(X)$ is separating, we have $x = y$.

0.17. When speaking of a compact (locally compact) space $\mathscr{X}$ we mean that $\mathscr{X}$ is hausdorff. When speaking of a closure space $\mathscr{X}$ we mean that to every $x \in |\mathscr{X}|$ here is given a filter base $\mathscr{B}x$ of subsets of $|\mathscr{X}|$ such that $x \in B$ for all $B \in \mathscr{B}x$. $\mathscr{X}$ is defined to be T_1 if the points of $\mathscr{X}$ are closed.

A uniform space (X, n) whose uniformity n is given by a filter $\mathscr{F}$ of vicinities of the diagonal Δ in $X \times X$, is called separated (or Hausdorff) if $\bigcap\{ \mid F \in \mathscr{F}\} = \Delta$.

0.18. Where useful, we shorten a complicated notation "A" by introducing another and simpler abbreviation "B" using the symbol $B \overset{*}{=} A$ or just $B = A$. For example, $G_{\beta\alpha}(\mathscr{S}, \mathscr{E}) \overset{*}{=} G$ means that $G_{\beta\alpha}(\mathscr{S}, \mathscr{E})$ is denoted by G.

0.19. Let an inductive family $\mathscr{S} = \{\mathscr{X}_\alpha = (X_\alpha, \tau_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ from $\mathfrak{L}$ be given, where $\mathfrak{L}$ is one of *CLOS*, *TOP*, *SEM*, *UNIF*, *PROX*. If the order $\leqq$ is the equality (i.e. $a, b \in A$ then $a \leqq b$ iff $a = b$) then there is $\mathscr{I} = \varinjlim \mathscr{S}$ in $\mathfrak{L}$. It is known that $\mathscr{I} = (I, t)$, where $I = \bigcup\{X_\alpha \mid \alpha \in A\}$ and t is inductively defined by the canonical embeddings $j_\alpha : \mathscr{X}_\alpha \to I$. We denote $\mathscr{I}$ by $\sum\{\mathscr{X}_\alpha \mid \alpha \in A\}$. In general, we say that a category $\mathfrak{K}$ has the sum property if every inductive family $\mathscr{S} = \{S_\alpha |\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ from $\mathfrak{K}$, over a set $\langle A \leqq \rangle$ whose order is the equality, has an inductive limit $\mathscr{I}$ in $\mathfrak{K}$. We write $\mathscr{I} = \sum\{S_\alpha \mid \alpha \in A\}$.

0.20. *Let $\mathfrak{L}$ be i.c., $\mathscr{S} = \{\mathscr{X}_\alpha |\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\} \subset \mathfrak{L}$ a presheaf, $\mathscr{I} = \varinjlim \mathscr{S}$. Then*

(i) $|\mathscr{I}| = \varinjlim |\mathscr{S}|$ (see 0.9),

(ii) *if $f : \mathscr{I} \to R$ is a real function such that $f \circ \xi_\alpha \in C(\mathscr{X}_\alpha \to R \mid \mathfrak{L})$ for all $\alpha \in A$ ($\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{I}$ are the natural maps), then $f \in C(\mathscr{I} \to R)$.*

Proof. (i): Given a set S and a compatible family $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ between $|\mathscr{S}| = \{|\mathscr{X}_\alpha| \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ and S, $p \in |\mathscr{I}|$, then there is a representative $a \in |\mathscr{X}_\alpha|$

of p (see 0.10(3a)) we set $h(p) = f_\alpha(a)$. It follows from 0.10 (3b) that $h(p)$ does not depend on the choice of α and of the representative $a \in |\mathscr{X}_\alpha|$ of p. Thus gotten map $h : |\mathscr{I}| \to S$ fulfils $h \circ \xi_\alpha = f_\alpha$ for all $\alpha \in A$, and if $\tilde{h} : |\mathscr{I}| \to S$ is another map which fulfils it then clearly $\tilde{h} = h$.

(ii): Set $\mathscr{G} = \{g_\alpha = f \circ \xi_\alpha \mid \alpha \in A\}$. Then $\mathscr{G}$ is a compatible family of $\mathfrak{L}$ – morphisms between $\mathscr{S}$ and R – see 0.12 (2), 0.6. There is a unique $g \in C(\mathscr{I} \to R \mid \mathfrak{L})$ with $g \circ \xi_\alpha = g_\alpha$ for all $\alpha \in A$. By (i) there is a unique $h : |\mathscr{I}| \to R$ with $h \circ \xi_\alpha = g_\alpha$ for all $\alpha \in A$. Also $f \circ \xi_\alpha = g_\alpha$ for all $\alpha \in A$. Thus $f = h = g$ and $f \in C(\mathscr{I} \to R \mid \mathfrak{L})$.

1. SEPARATION THEOREMS

1. A GENERAL SEPARATION THEOREM FOR INDUCTIVELY CLOSED CATEGORIES

1.1.1. Definition. A closure space (Y, c) is called *functionally separated* (shortly f.s.) if the set $C = C((Y, c) \to R)$ distinguishes points of Y, i.e. if C is separating for Y (see 0.13, 0.14). If a nonempty family $F(Y) \subset C$ distinguishes points of Y, we say that (Y, c) is f.s. by $F(Y)$. Clearly, if h is a continuous $1-1$ map of a closure space (X, t) into (Y, c) (see 0.15), where (Y, c) is f.s. by $F(Y)$, then (X, t) is f.s. by $h^* F(Y) = \{f \circ h \mid h \in F(Y)\}$. More generally, if $\mathscr{E}$ is a family of continuous maps of (X, t) into (Y, c) which distinguishes points of X (see 0.13) then (X, t) is f.s. if so is (Y, c).

This can be likewise written in terms of any inductively closed category $\mathfrak{L}$ (see 0.10). An $\mathfrak{L}$-object $\mathscr{I}$ is called functionally separated (f.s.) if $C(\mathscr{I} \to R \mid \mathfrak{L})$ separates points of $|\mathscr{I}|$ (see 0.13, 0.9, 0.11). If $F \subset C(\mathscr{I} \to R \mid \mathfrak{L})$ is nonempty and separates points of $|\mathscr{I}|$, we say that $\mathscr{I}$ is f.s. by F. If $\mathscr{E}$ is a family of $\mathfrak{L}$-morphisms of $\mathscr{X}$ into $\mathscr{I}$ which distinguishes points of $\mathscr{X}$ (see 0.13) and if $\mathscr{I}$ is f.s. by F, then $\mathscr{X}$ is f.s. by $\{f \circ h \mid f \in F,\ h \in \mathscr{E}\}$.

1.1.2. Lemma. *Let (Y, c) be a f.s. closure space. Then there is a Hausdorff topology u in Y coarser than c. This lemma holds also in terms of SEM and PROX* (see 0.5).

Proof. We define u projectively by the functions from $C = C((Y, c) \to R)$ (see 0.14). Then u is a topology coarser than c and is Hausdorff for C separates points of Y. To prove the lemma in terms of *SEM* (*PROX*), we define the uniformity (proximity) u projectively by the set of all uniformly (proximally) continuous functions on (Y, c).

Given a presheaf $S = \{(S_\alpha, \tau_\alpha) \, |\varrho_{\alpha\beta}| \, \langle A \leqq \rangle\}$ from *CLOS* (*SEM*, *PROX*, ...) and its inductive limit (I, t_I^*), Lemma 1.1.2 gives us a method for the proof of separation theorems. If we want to find when there is a Hausdorff topology (uniformity, ...)

in I, coarser than t_I^*, we try to find some conditions for the set $C((I, t_I^*) \to R)$ . . $(U((I, t_I^*) \to R), \ldots)$ to distinguish points of I (see 0.5, 0.14).

1.1.3. Notation. Let $\langle A \leqq \rangle$ be a partially ordered set, $\alpha \in A$.

A. The set $A_{[\alpha]} = \{\beta \in A \mid \beta < \alpha\}$ is called *a segment* of A. If there is a largest (smallest) element of $\langle A \leqq \rangle$ smaller (larger) than α, we denote it by $\alpha - 1$ ($\alpha + 1$). If $\langle A \leqq \rangle$ is well ordered and α is not the largest element of $\langle A \leqq \rangle$, then there is $\alpha + 1$; the smallest element of $\langle A \leqq \rangle$ exists and is denoted by 1. When speaking about $\alpha + 1$ ($\alpha - 1$), we mean that α is not the largest (smallest) element of A.

B. If $\langle A \leqq \rangle$ is a partially ordered set, $\alpha, \beta \in A$, $\alpha \leqq \beta$, we put $\langle \alpha, \beta \rangle = \{\gamma \in \in A \mid \alpha \leqq \gamma \leqq \beta\}$. Likewise we define $\langle \alpha, \beta)$, $(\alpha, \beta\rangle$, (α, β). Further we put $A(\alpha) = = \{\beta \in A \mid \beta \geqq \alpha\}$.

1.1.4. Definition. Let $\mathscr{S} = \{S_\alpha \mid \varrho_{\alpha\beta} \mid \langle A \leqq \rangle\}$ be an inductive family from an arbitrary category $\mathfrak{K}$ (see 0.2), $\alpha \in A$. We say that S_α is *limit* if $A[\alpha]$ is right directed and $S_\alpha = \mathfrak{K} \varinjlim \mathscr{S}_{A[\alpha]}$ (see 0.8, 0.4). This also means that $\{\varrho_{\beta\alpha} : S_\beta \to S_\alpha \mid \beta \in A[\alpha]\}$ are the canonical maps (see 0.4). We set $\mathscr{L}(\mathscr{S}) = \{\alpha \in A \mid S_\alpha \text{ is limit}\}$. If no misunderstanding can occur we write $\mathscr{L}$ instead of $\mathscr{L}(\mathscr{S})$. Clearly $1 \notin \mathscr{L}$ if there is the smallest element 1 of $\langle A \leqq \rangle$.

1.1.5. Definition. A. Let an inductively closed category $\mathfrak{L}$ and an inductive family $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ of $\mathfrak{L}$-objects (see 0.12) be given. If for every $\alpha \in A$ we have a set $F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L})$ (see 0.11), then we say $\mathscr{S}$ is *endowed* with the family $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$. $\mathscr{E}$ is called *separating* (*strongly separating*) if for every $\alpha \in A$ the set F_α separates points of $\mathscr{X}_\alpha$ (points and points from closed sets of cl $\mathscr{X}_\alpha$). If the points of $\mathscr{X}_\alpha$ are closed in cl $\mathscr{X}_\alpha$ – see 0.9 – for every $\alpha \in A$, then any strongly separating family is also separating.

If $\langle A \leqq \rangle$ is well ordered then $\mathscr{E}$ is called *leftward* (*rightward*) smooth if $F_\alpha \subset \subset \varrho^*_{\alpha,\alpha+1} F_{\alpha+1}$ ($\varrho^*_{\alpha,\alpha+1} F_{\alpha+1} \subset F_\alpha$) for all $\alpha \in A$. $\mathscr{E}$ is called *smooth* if it is leftward and rightward smooth.

A thread through $\mathscr{E}$ is a family $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ with $\varrho^*_{\alpha\beta} f_\beta = f_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$. (Thus the threads through $\mathscr{E}$ are the compatible families $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ with $f_\alpha \in F_\alpha$ for all $\alpha \in A$ – see 0.3). If $M \subset A$, we put $\mathscr{E}_M = \{F_\alpha \mid \alpha \in M\}$. $\mathscr{E}$ is called *connected* (*fully connected*) when the following condition holds: "If $\alpha \in A$ is such that $\alpha - 1$ does not exist, $\beta \in A[\alpha] - \mathscr{L}$ ($\beta \in A[\alpha]$) (see 1.1.4) and if $\mathscr{F} = \{f_\gamma \mid \gamma \in \in \langle \beta\alpha)\}$ is a thread through $\mathscr{E}_{\langle \beta, \alpha)}$ then there is $f \in F_\alpha$ such that $\varrho^*_{\gamma\alpha} f = f_\gamma$ for all $\gamma \in \in \langle \beta\alpha)$." Clearly, $\mathscr{E}$ is connected if $\langle A \leqq \rangle$ is well ordered and countable with the ordinal type ω_0, because then every $\alpha \in A$ has $\alpha - 1$.

In the samy way we define the notions of this definition in the complex case, i.e. if we have the field C of complex numbers instead of R.

B. If $\langle A \leqq \rangle$ is well ordered and every $\varrho^*_{\alpha\beta}$ sends F_β into F_α then $\mathscr{E}$ is connected iff it is fully connected. Indeed, let $\alpha \in A$ such that $\alpha - 1$ does not exist, $\beta \in A[\alpha]$

and a thread $\mathscr{G} = \{f_\gamma \mid \gamma \in \langle\beta\alpha)\}$ through $\mathscr{E}_{\langle\beta\alpha)}$ be given. Setting $f_\gamma = \varrho^*_{\gamma\beta} f_\beta$ for $\gamma < \beta$, $\mathscr{F} = \{f_\gamma \mid \gamma \in \langle 1, \alpha)\}$, then $\mathscr{F}$ is a thread through $\mathscr{E}_{\langle 1,\alpha)}$. As $1 \notin \mathscr{L}$, there is $f \in F_\alpha$ with $\varrho^*_{\gamma\alpha} f = f_\gamma$ for all $\gamma \in \langle 1, \alpha)$ provided $\mathscr{E}$ is connected. Thus $\varrho^*_{\gamma\alpha} f = f_\gamma$ for all $\gamma \in \langle\beta\alpha)$ whence $\mathscr{E}$ is fully connected; the converse is plain.

1.1.6. Lemma. *Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq\rangle\}$ be from an i.c. category $\mathfrak{L}$, $\langle A \leqq\rangle$ well ordered. Assume $\mathscr{S}$ is endowed with a leftward smooth and connected family $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$. Then $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ (it means that $\mathscr{E}_M$ is leftward smooth for any $M \subset A$). Thus $\varrho_{\alpha\beta}$ is $1-1$ for all $\alpha \in A$ if every F_α separates points of $\mathscr{X}_\alpha$.*

Proof. Given $\alpha, \beta \in A$, $\alpha \leqq \beta$, $f \in F_\alpha$, we prove by induction that there is a thread $\mathscr{F}_\beta = \{f_\gamma \mid \gamma \in \langle\beta\alpha\rangle\}$ through $\mathscr{E}_{\langle\beta\alpha\rangle}$ with $f_\alpha = f$. Let $\varepsilon \in A$, $\varepsilon > \alpha$ and let $\mathscr{F}_\varepsilon = \{f_\gamma \mid \gamma \in \langle\alpha, \varepsilon)\}$ be a thread through $\mathscr{E}_{\langle\alpha\varepsilon)}$ with $f_\alpha = f$. If there is $\varepsilon - 1$ then by virtue of the leftward smoothness of $\mathscr{E}$ we can find $g \in F_\varepsilon$ so that $\varrho^*_{\varepsilon-1,\varepsilon} g = f_{\varepsilon-1}$. If $\varepsilon - 1$ does not exist then the connectedness of $\mathscr{E}$ yields the existence of a $g \in F_\varepsilon$ such that $\varrho^*_{\gamma\varepsilon} g = f_\gamma$ for all $\gamma \in \langle\alpha, \varepsilon)$. In the both cases, $\mathscr{G} = \{f_\gamma \mid \gamma \in \langle\alpha, \varepsilon\rangle\}$ with $f_\varepsilon = g$ is a thread through $\mathscr{E}_{\langle\alpha,\varepsilon\rangle}$ If $\varepsilon = \beta$, we have $f = f_\alpha = \varrho^*_{\alpha\beta} f_\beta$ which with $f_\beta \in F_\beta$ proves that $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$ By 0.16, $\varrho_{\alpha\beta}$ is $1-1$. The lemma is proven.

1.1.7. Theorem. *Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq\rangle\}$ be a presheaf from an i.c. category $\mathfrak{L}$* (see 0.10), *$\langle A \leqq\rangle$ well ordered. Suppose $\mathscr{S}$ is endowed with a leftward smooth and connected separating family $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$.*

If $p, q \in \mathscr{I} = \varinjlim \mathscr{S}$, $p \neq q$, then there is $\alpha \in A - \mathscr{L}$, representatives $a \in S_\alpha$ of p and $b \in S_\alpha$ of q and a thread $\mathscr{F} = \{f_\gamma \mid \gamma \geqq \alpha\}$ through $\mathscr{E}$ such that $f_\gamma \circ \varrho_{\alpha\gamma}(a) = 1$, $f_\gamma \circ \varrho_{\alpha\gamma}(b) = 0$ for all $\gamma \geqq \alpha$. Thus for $f = \varinjlim \mathscr{F}$ we have $f(p) = 1$, $f(q) = 0$. Hence $\mathscr{I}$ is f.s. by $C(\mathscr{I} \to R \mid \mathfrak{L})$ (see 0.6b).

Proof. Let $p, q \in \mathscr{I}$, $p \neq q$. There is a smallest element $\alpha \in A$ such that there is a representative $a \in \mathscr{X}_\alpha$ of p and $b \in \mathscr{X}_\alpha$ of q. We have $\alpha \in A - \mathscr{L}$ since otherwise there would be $\beta < \alpha$ so that p, q have representatives in $\mathscr{X}_\beta$, which contradicts the choice of α. By 1.1.6, $\varrho_{\alpha\beta}$ is $1-1$ for all α, β. By the condition (3) from 0.10, the canonical maps $\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{I}$ are $1-1$ and hence the representatives are unique. As $\langle A \leqq\rangle$ is ordered, the set $A(\alpha) = \{\gamma \in A \mid \gamma \geqq \alpha\}$ is confinal in $\langle A \leqq\rangle$, hence by 0.8B, $\varinjlim \mathscr{S}$ is $\mathfrak{L}$-isomorphic to $\varinjlim \mathscr{S}_{A(\alpha)}$. Thus we may suppose $\alpha = 1$ (see 1.1.3B). Put $a_\beta = \varrho_{1\beta}(a)$, $b_\beta = \varrho_{1\beta}(b)$ for $\beta \in A$. Using transfinite induction we get a thread $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ through $\mathscr{E}$ such that $f_\alpha(a_\alpha) = 1$, $f_\alpha(b_\alpha) = 0$ for each $\alpha \in A$. If we find such a family then there is $\varinjlim \mathscr{F} = f \in C(\mathscr{I} \to R \mid \mathfrak{L})$ with $f(p) = 1$, $f(q) = 0$ and with $\xi^*_\gamma f = f_\gamma \in F_\gamma$ for any $\gamma \in A$, $\gamma \geqq \alpha$ (see 0.4, 0.15). Thus $\mathscr{I}$ is f.s. by $C(\mathscr{I} \to R \mid \mathfrak{L})$.

The construction of $\mathscr{F}$: There is $f_1 \in F_1$ with $f_1(a) = 1$, $f_1(b) = 0$. Suppose $\alpha \in A$ and let us have a thread $\mathscr{F}_\alpha = \{f_\beta \mid \beta \in A[\alpha]\}$ through $\mathscr{E}_{A[\alpha]}$ with $f_\beta(a_\beta) = 1$, $f_\beta(b_\beta) = 0$ for each $\beta \in A[\alpha]$.

(#): If $\alpha - 1$ does not exist, then there is $g \in F_\alpha$ with $\varrho^*_{\gamma\alpha} g = f_\gamma$ for all $\gamma \in A[\alpha]$. Putting $f_\alpha = g$, we have $f_\alpha(a_\alpha) = 1$, $f_\alpha(b_\alpha) = 0$.

(##): If there is $\alpha - 1$, we have $F_{\alpha-1} \subset \varrho^*_{\alpha-1,\alpha} F_\alpha$. As $f_{\alpha-1} \in F_{\alpha-1}$, there is $g \in F_\alpha$ with $\varrho^*_{\alpha-1,\alpha} g = f_{\alpha-1}$. Putting $f_\alpha = g$, we have $f_\alpha(a_\alpha) = 1$, $f_\alpha(b_\alpha) = 0$. In both cases, $\{f_\beta \mid \beta \leqq \alpha\}$ is a thread through $\mathscr{E}_{\langle 1\alpha\rangle}$. Therefore the transfinite construction gives us the desired thread $\mathscr{F}$. The theorem is proven.

1.1.8. Corollary. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *be a presheaf from an i.c. category* $\mathfrak{L}$, $\langle A \leqq \rangle$ *well ordered. Suppose* $\mathscr{S}$ *is endowed with a leftward smooth separating family* $\mathscr{E} = (F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$ *and moreover*

1. *if* $\alpha \in A$, $\alpha \neq 1$ *then either* $\alpha \in \mathscr{L}$ *or there is* $\alpha - 1$;

2. *if* $\alpha \in A$, $\alpha \neq 1$ *is such that* $\alpha - 1$ *does not exist,* $\beta \in A[\alpha] - \mathscr{L}$, *and if* $\mathscr{F} = \{f_\gamma \in F_\gamma \mid \gamma \in \langle \beta\alpha)\}$ *is a thread through* $\mathscr{E}_{\langle\beta\alpha)}$, *then* $\varinjlim \mathscr{F} = f \in F_\alpha$ *(see* 0.4; *this condition* 2 *holds namely if* $F_\alpha = C(\mathscr{X}_\alpha \to R \mid \mathfrak{L})$ *for all* $\alpha \in A$ *for which* $\alpha - 1$ *does not exist.)*

Then $\mathscr{E}$ *is connected and thus the statement of Th.* 1.1.7 *holds.*

Proof. If α, β, f, $\mathscr{F}$ are the symbols mentioned in 2, then $\varrho^*_{\alpha\gamma} f = f_\gamma$ for all $\gamma \in \langle \beta\alpha)$, thus $\mathscr{E}$ is connected. Hence Th. 1.1.7 works.

Theorem 1.1.7 is a sufficient condition for $\varinjlim \mathscr{S}$ to be f.s. Its key points are the left smoothness and connectedness of $\mathscr{E}$. We deal with that in the next two sections.

2. THE CONNECTEDNESS

In this section some conditions for the connectedness are found.

1.2.1. Proposition. Let $\mathscr{S} = \{S_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be an inductive family from SET (see 0.5). Suppose that for every $\alpha \in A$ we have a set $R_\alpha \subset S_\alpha$. Put $r_{\alpha\beta} = \varrho_{\alpha\beta}/R_\alpha$ (the restriction; we have $r_{\alpha\beta} : R_\alpha \to S_\beta$), $\mathscr{R} = \{R_\alpha \mid r_{\alpha\beta} | \langle A \leqq \rangle\}$ ($\mathscr{R}$ need not be an inductive family for it need not be $r_{\alpha\beta} R_\alpha \subset R_\beta$). A family $a = \{a_\alpha \in R_\alpha \mid \alpha \in A\}$ is called *a thread* through $\mathscr{R}$ if $r_{\alpha\beta}(a_\alpha) = a_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ (this is in accordance to the definition of the threads in 1.1.5). We denote by $P_\mathscr{R}$ the set of all threads through $\mathscr{R}$ (then $P_\mathscr{R} \subset \varinjlim \mathscr{S}$ – see 0.6). Assume $\langle A \leqq \rangle$ is right or left directed (see 0.1). Let a set S and a projective family $\mathscr{F} = \{f_\alpha : S \to S_\alpha \mid \alpha \in A\}$ between S and $\mathscr{S}$ (see 0.6) such that $R_\alpha \subset f_\alpha(S)$ for all $\alpha \in A$ be given. Setting $f = \varinjlim \mathscr{F}$ (see 0.6 – we have $f : S \to \varinjlim \mathscr{S}$), we have

A. $P_\mathscr{R} \subset f(S)$ *if each* f_α *is* $1-1$ *on* $f_\alpha^{-1}(R_\alpha)$.

B. *Let, moreover,* $\langle A \leqq \rangle$ *be left directed and let S be a topological space. If* $a = \{a_\alpha \mid \alpha \in A\}$ *is a thread through* $\mathscr{R}$, *we set* $\mathscr{B}a = \{M_\alpha = f_\alpha^{-1}(a_\alpha) \mid \alpha \in A\}$ *(*$\mathscr{B}a$ *is a filter base in S as* $\langle A \leqq \rangle$ *is left directed). Then* $P_\mathscr{R} \subset f(S)$ *if one of the* *following conditions is fulfilled:*

a) *for every $a \in P_{\mathscr{R}}$, $\mathscr{B}a$ consists of compact sets (in the topology of S).*

b) *S is uniformisable, complete, and for every $a \in P_{\mathscr{R}}$, $\mathscr{B}a$ is a base of a Cauchy filter of closed subsets of* cl S (see 0.9).

c) *A is countable, $\langle A \leqq^{\cdot} \rangle$ with the inverse order $\leqq^{\cdot}$ (i.e. $\alpha \leqq^{\cdot} \beta$ iff $\beta \leqq \alpha$) is well ordered and for every thread $a = \{a_\alpha \mid \alpha \in A\}$ through $\mathscr{R}$ there is β such that M_β is either locally compact or a complete metric space and M_γ is of Baire type G_δ and dense in M_β for all $\gamma \,{}^{\cdot}\!\geqq \beta$.*

d) *There is no countable confinal subset in $\langle A \leqq^{\cdot} \rangle$ (then A is, of course, uncountable; $\leqq^{\cdot}$ is the inverse order), $\langle A \leqq \rangle$ is ordered, S is Lindelöf space and for every thread $a = \{a_\alpha \mid \alpha \in A\}$ through $\mathscr{R}$ the base $\mathscr{B}a$ consists of closed sets.*

Proof. A. Let every f_α be $1-1$ on $f_\alpha^{-1}(R_\alpha)$, $a = \{a_\alpha \in R_\alpha \mid a \in A\} \in P_{\mathscr{R}}$. For every $\alpha \in A$ there is b_α with $f_\alpha(b_\alpha) = a_\alpha$. If $\alpha \leqq \beta$ then $f_\beta(b_\alpha) = \varrho_{\alpha\beta} f_\alpha(b_\alpha) = \varrho_{\alpha\beta}(a_\alpha) = a_\beta$. As $f_\beta(b_\beta) = a_\beta$ and f_β is $1-1$ on $f_\beta^{-1}(R_\beta)$, we get $b_\alpha = b_\beta$. As $\langle A \leqq \rangle$ is right or left directed, we have $b_\alpha = b_\beta = b$ for all $\alpha, \beta \in A$. Clearly $f(b) = a$ as desired.

B. If (a) or (b) holds then $\mathscr{B}a$ has a cluster point b. Clearly $f_\alpha(b) = a_\alpha$ for all $\alpha \in A$ so $f(b) = a$. In the case (c), if $a = \{a_\alpha \mid \alpha \in A\}$ is a thread through $\mathscr{R}$, $\alpha \,{}^{\cdot}\!\geqq \beta$ then $M_\alpha \subset M_\beta$. By the Baire category theorem [5, Chap. XI, Sec. 10, Th. 10, p. 249], [5, Chap. XIV, Sec. 4, Th. 4.1, p. 229], $B = \bigcap\{M_\gamma \mid \gamma \leqq \beta\} \neq \emptyset$ as it is a countable intersection, hence if $b \in B$ then $f(b) = a$.

In the case (d) assume that $\bigcap \mathscr{B}a = \emptyset$. Then the Lindelöf property of S yields the existence of a countable subset $\mathscr{C}$ of $\mathscr{B}a$ with $\bigcap \mathscr{C} = \emptyset$. Let, say, $\mathscr{C} = \{M_{\alpha_n} = f_{\alpha_n}^{-1}(a_{\alpha_n}) \mid n = 1, 2, \ldots\}$. As $\{\alpha_n \mid n = 1, 2, \ldots\}$ is not cofinal in $\langle A \leqq^{\cdot} \rangle$ and $\langle A \leqq^{\cdot} \rangle$ is ordered, there is $\beta \in A$ such that $\alpha_n \leqq^{\cdot} \beta$ for all n. Since $M_\beta = f_\beta^{-1}(a_\beta) \neq \emptyset$ and $\varrho_{\beta\alpha_n}(M_\beta) \subset M_{\alpha_n}$ for all n we have $\bigcap \mathscr{C} \neq \emptyset$ – a contradiction. So there is $b \in \bigcap \mathscr{B}a$ and $f(b) = a$.

1.2.2. Proposition. *Let $\mathscr{S}$ $\mathscr{R}$, $P_{\mathscr{R}}$ have the same meaning as in* 1.2.1. *Given a set S and a projective family $\mathscr{F} = \{f_\alpha : S \to S_\alpha \mid \alpha \in A\}$ between S and $\mathscr{S}$ (it need not be $R_\alpha \subset f_\alpha(S)$ for all $\alpha \in A$), $f = \varprojlim \mathscr{F} : S \to \varprojlim \mathscr{S}$ (see* 1.2.1, 0.6) *let us assume that $\langle A \leqq \rangle$ is right or left directed and there is $\beta \in A$ such that $R_\beta \subset f_\beta(S)$. If $\varrho_{\alpha\beta}$ is $1-1$ on $M_{\alpha\beta} = \varrho_\beta^{-1} \varrho_{\alpha\beta} R_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ (if it is fulfilled then $M_{\alpha\beta} = R_\alpha$), then $P_{\mathscr{R}} \subset f(S)$.*

Proof. By 1.2.1, $P_{\mathscr{R}}$ is the set of all threads through $\mathscr{R}$. Given a thread $a = \{a_\alpha \mid \alpha \in A\}$ through $\mathscr{R}$, there is β and $b \in S$ with $f_\beta(b) = a_\beta$. Then $d = \{b_\alpha = f_\alpha(b) \mid \alpha \in A\}$ is a thread through $\mathscr{S}$. Clearly $a_\gamma = b_\gamma$ for all $\gamma \geqq \beta$, and if $\delta \in A$ with $a_\delta = b_\delta$ then $a_\gamma = b_\gamma$ for all $\gamma \geqq \delta$. Let $\gamma \in A$. If $\langle A \leqq \rangle$ is right directed then there is δ with $\delta \geqq \beta$, $\delta \geqq \gamma$. Then $\varrho_{\gamma\delta}(a_\gamma) = a_\delta = b_\delta = \varrho_{\gamma\delta}(b_\gamma)$, $a_\gamma, b_\gamma \in M_{\gamma\delta}$ hence $a_\gamma = b_\gamma$. If $\langle A \leqq \rangle$ is left directed then there is δ with $\delta \leqq \gamma$, $\delta \leqq \beta$. Then $\varrho_{\delta\beta}(a_\delta) = b_\beta = \varrho_{\delta\beta}(b_\delta)$, $a_\delta, b_\delta \in M_{\delta\beta}$ hence $a_\delta = b_\delta$. Since $\delta \leqq \gamma$, we have $a_\gamma = b_\gamma$. Thus $a_\gamma = b_\gamma$ for all $\gamma \in A$, hence $d = a = f(b)$ which finishes the proof.

1.2.3. Corollary. *Let* $\mathscr{S} = \{S_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ *be an inductive family from SET,* $P = \varprojlim \mathscr{S}$. *Assume* $\langle A \leqq \rangle$ *is right or left directed. Let S be a set and* $\mathscr{F} = \{f_\alpha : S \to S_\alpha \mid \alpha \in A\}$ *a projective family between S and* $\mathscr{S}$ (see 0.6) *such that* f_α *carries S onto* S_α *for all* $\alpha \in A$. *Then* $f = \varprojlim \mathscr{F}$ *carries S onto P if either each* f_α *is* $1-1$ *or each* $\varrho_{\alpha\beta}$ *is* $1-1$.

Proof. By 0.6, P is the set of all threads through $\mathscr{S}$ (see 1.2.1). If every f_α or every $\varrho_{\alpha\beta}$ is $1-1$ then the statement follows respectively from 1.2.1 or 1.2.2 (we put $R_\alpha = S_\alpha$ so that $\mathscr{S} = \mathscr{R}$), because $P_{\mathscr{S}} = P_{\mathscr{R}} = \varprojlim \mathscr{S} = P$ (see 1.2.1, 0.6).

The foregoing corollary is useful when a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ from an i.c. category is endowed with a family $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$ such that the family $\mathscr{E}^* = \{F_\alpha | \varrho_{\alpha\beta}| \langle A \leqq^\cdot \rangle\}$ is inductive ($\leqq^\cdot$ is the inverse order for $\leqq$, i.e. $a \leqq^\cdot b$ iff $b \leqq a$), i.e. when $\varrho^*_{\alpha\beta}$ carries F_β into F_α for all α, β. In this case we may use 1.2.3 to verify the connectedness of $\mathscr{E}$. But if we have only $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$, then 1.2.3 cannot be used directly. For that case we have established 1.2.1, 1.2.2. The next corollary is adopted for the direct verificationo of the connectedness of $\mathscr{E}$.

1.2.4. Corollary. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ *be a presheaf from an inductively closed category* $\mathfrak{L}$, *which is endowed with* $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$, $\langle A \leqq \rangle$ *well ordered.*

A) *Assume that for every* $\alpha \in A$ *for which* $\alpha - 1$ *does not exist there is* $\beta \in A[\alpha]$ *such that at least one of the following conditions holds*:

1) $F_\gamma \subset \varrho^*_{\gamma\alpha} F_\alpha$ *and* $\varrho^*_{\gamma\alpha}$ *is* $1-1$ *on* $F_\alpha \cap \varrho^{*-1}_{\gamma\alpha} F_\gamma$ *for all* $\gamma \in \langle \beta\alpha)$;

2) *there is a cofinal set B in* $\langle A[\alpha] \leqq \rangle$ *with* $F_\gamma \subset \varrho^*_{\gamma\alpha} F_\alpha$ *for all* $\gamma \in B$, *and* $\varrho^*_{\delta\gamma}$ *is* $1-1$ *on* $\varrho^{*-1}_{\delta\gamma} \varrho^*_{\delta\gamma} F_\gamma$ *for all* $\delta, \gamma \in \langle \beta\alpha)$, $\delta \leqq \gamma$.

Then $\mathscr{E}$ *is fully connected.*

B) *Assume that for every* $\alpha \in A$ *for which there is no* $\alpha - 1$, F_α *is a topological space such that for any* $\beta \in A[\alpha]$ *and any thread* $\mathscr{G} = \{g_\gamma \mid \gamma \in \langle \beta\alpha)\}$ *through* $\mathscr{E}_{\langle \beta\alpha)}$ *there is* $\nu \in \langle \beta\alpha)$ *such that one of the following conditions is fulfilled*:

a) *The filter base* $\mathscr{B}_{\nu\alpha} = \{M_\gamma = F_\alpha \cap \varrho^{*-1}_{\gamma\alpha} g_\gamma \mid \gamma \in \langle \nu\alpha)\}$ *consists of compact subsets of* F_α;

b) F_α *is uniformisable, complete, and* $\mathscr{B}_{\nu\alpha}$ *is a Cauchy filter base of closed subsets of* F_α;

c) $\langle \nu\alpha)$ *is countable,* M_γ *is of the Baire type* G_δ, *dense in* M_ν *for all* $\gamma \in \langle \nu\alpha)$, *and either locally compact, or a complete metric space*;

d) *there is no countable cofinal set in* $\langle \nu, \alpha)$ *and either* F_α *is a Lindelöfs space and* M_γ *are closed in* F_α *for all* $\gamma \in \langle \nu\alpha)$, *or* M_ν *is a Lindelöf space and* M_γ *are closed in* M_ν *for all* $\gamma \in \langle \nu\alpha)$.

Then $\mathscr{E}$ *is fully connected.*

Proof. Given $\alpha \in A$ such that $\alpha - 1$ does not exist, $\delta \in A[\alpha]$ and a thread $\mathscr{F} = \{f_\gamma \mid \gamma \in \langle \delta\alpha)\}$ through $\mathscr{E}_{\langle \delta\alpha)}$, we may assume $\delta = \beta$. If the condition 1 or 2 of A holds then, using respectively 1.2.1A or 1.2.2 to $\mathscr{E}_{\langle\beta\alpha\rangle}$, $\mathscr{F}$, F_α and $\{\varrho^*_{\gamma\alpha} \mid \gamma \in \langle\beta\alpha)\}$ we get that there is $f \in F_\alpha$ with $\varrho^*_{\gamma\alpha} f = f_\gamma$ for all $\gamma \in \langle\beta\alpha)$ which proves A, while B follows from 1.2.1B.

1.2.5. Lemma. *Let a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha|\varrho_{\alpha\beta}|\ \langle A \leqq\rangle\}$ be from an i.c. category $\mathfrak{L}$, $B \subset A$ let $\langle B \leqq\rangle$ be well ordered and let $\mathscr{S}_B$ be endowed with a family $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in B\}$ such that for each $\alpha, \beta \in B$, $\alpha \leqq \beta$ the pair (F_α, F_β) has a property $P_{\alpha\beta}$. Let there be a countable cofinal set C in B. Then there is a cofinal set D in $\langle C \leqq\rangle$ – and therefore cofinal in $\langle B \leqq\rangle$ – such that the family $\mathscr{E}_D$ which endows $\mathscr{S}_D$, has the following properties: (1): For every $\alpha, \beta \in D$, $\alpha \leqq \beta$ the pair (F_α, F_β) has the property $P_{\alpha\beta}$,*

(2): $\mathscr{E}_D$ is connected. Moreover, $\varinjlim \mathscr{S}_D$ is $\mathfrak{L}$-isomorphic to $\varinjlim \mathscr{S}_B$, D being cofinal in $\langle B \leqq\rangle$.

Proof. $\langle C \leqq\rangle$ is countable and well ordered; it is well known that there is a cofinal set D in C of the ordinal type ω_0. Now as each $\gamma \in D$ has $\gamma - 1$ in $\langle D \leqq\rangle$, (2) follows from 1.1.5A. The rest is clear.

1.2.6. Lemma. *Under the conditions of the foregoing lemma, if some properties* P, Q *of $\mathscr{E}$ and the connectedness of $\mathscr{E}$ yield a property* R *of $\mathscr{J} = \varinjlim \mathscr{S}$ which is stable under $\mathfrak{L}$-isomorphisms, and if* P *means that for certain elements $\alpha, \beta \in B$, $\alpha \leqq \beta$ the pair (F_α, F_β) has the property $P_{\alpha\beta}$, then in order that $\mathscr{J}$ have the property* R *it is enough that the property $P_{\alpha\beta}$ be posessed by any pair (F_α, F_β), $\alpha, \beta \in B$, $\alpha \leqq \beta$, and that* Q *be stable under making subfamilies of $\mathscr{E}$.*

Proof. We take $\mathscr{S}_D$ from the foregoing lemma. Then by the foregoing lemma, $\mathscr{E}_D$ is connected and has the properties P, Q whence $\mathscr{K} = \varinjlim \mathscr{S}_D$ has the property R. Since $\mathscr{K}$ is $\mathfrak{L}$-isomorphic to $\mathscr{J}$, the lemma follows.

As a corollary we get the following strengthening of Th. 1.1.7:

1.2.7. Corollary. *Let $\mathscr{S} = \{\mathscr{X}_\alpha|\varrho_{\alpha\beta}|\ \langle A \leqq\rangle\}$ be a presheaf from an i.c. category $\mathfrak{L}$, $\langle A \leqq\rangle$ well ordered. Suppose $\mathscr{S}$ is endowed with a leftward smooth and connected separating family $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$.*

Given $p, q \in \mathscr{J} = \varinjlim \mathscr{S}$, $p \neq q$, then there is $\alpha \in A - \mathscr{L}$, representatives $a, b \in \mathscr{X}_\alpha$ of p, q, and a thread $\mathscr{F} = \{f_\gamma \mid \gamma \geqq \alpha\}$ through $\mathscr{E}$ such that $f_\gamma \varrho_{\alpha\gamma}(a) = 1$, $f_\gamma\, \varrho_{\alpha\gamma}(b) = 0$ for all $\gamma \geqq \alpha$. Thus for $f = \varinjlim \mathscr{F}$ we have $f(p) = 1$, $f(q) = 0$ hence $\mathscr{J}$ is f.s. by $C(\mathscr{J} \to R \mid \mathfrak{L})$. If moreover there is a countable cofinal set C in A and $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ then the assumption on the connectedness of $\mathscr{E}$ may be left out.

Proof. The property P: "For every α, $\alpha + 1 \in A$ we have $P_{\alpha,\alpha+1} : \{F_\alpha \subset \varrho^*_{\alpha,\alpha+1} F_{\alpha+1}\}$" – which is the left smoothness of $\mathscr{E}$ – together with Q: "$\mathscr{E}$ is separating" and the connectedness of $\mathscr{E}$ imply the property R of $\mathscr{J}$ which is described in the assertion of 1.2.7. As $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$, P as well as R, Q fulfil the conditions of 1.2.6, hence the assertion of 1.2.7 holds for R (with $A = D$, $\mathscr{E} = \mathscr{E}_D$). 0.8A, B now completes the proof.

1.2.8. Remark. In the same way we may strengthen 1.1.8 by adding to it: If there is a countable cofinal subset in A and if $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ then the condition (2) may be left out.

3. LEFT SMOOTHNESS

In this section we seek a class of presheaves for which there is a leftward smooth family $\mathscr{E}$. To this end we study when for two closure spaces $\mathscr{X}$, $\mathscr{Y}$ and a map $h : \mathscr{X} \to \to \mathscr{Y}$ there are sets $F(\mathscr{X}) \subset C(\mathscr{X} \to R)$, $F(\mathscr{Y}) \subset C(\mathscr{Y} \to R)$ such that $F(\mathscr{X}) \subset \subset h^* F(\mathscr{Y})$. If $\mathscr{X}$, $\mathscr{Y}$ are arbitrary then h need not be even $1-1$. But if we restrict ourselves to some more reasonable spaces, we must assume more about h. This is shown by the following lemma (a topological (closure, uniform, ...) space $\mathscr{X}$ is now denoted by (X, t), where $X = |\mathscr{X}|$ and t is the topology (closure, ...)).

1.3.1. Lemma. *Let $\mathscr{X} = (X, t)$, $\mathscr{Y} = (Y, t')$ be two closure spaces, $h : X \to Y$ a map, $F(X) \subset C(\mathscr{X} \to R)$, $F(Y) \subset C(\mathscr{Y} \to R)$. Denote by τ and by τ' the topologies projectively defined in X and in Y by the functions from $F(X)$ and $F(Y)$ respectively. Further, denote by h' the map $h : (X, \tau) \to (Y, \tau')$.*

a) *If $h^* F(Y) \subset F(X)$, then h' is continuous.*

b) *If h is $1-1$ and $F(X) \subset h^* F(Y)$, then $(h')^{-1}$ is continuous* (see 0.14).

c) *If $F(X)$ separates points of X and $F(X) \cap h^* F(Y)$ is a dense subset of $F(X)$ in the usual sup-norm then h is $1-1$.*

This lemma holds also in terms of SEM and PROX.

Proof. We have this diagram (i, j are identities)

$$\begin{array}{ccc} (X, \tau) & \xrightarrow{h'} & (Y, \tau') \\ \downarrow i & & \downarrow j \\ (X, t) & \xrightarrow{h} & (Y, t') \end{array}$$

a) h' is continuous iff so is $g \circ h'$ for any $g \in F(Y)$. But $g \circ h' = h^* g \in F(X)$ so $g \circ h'$ is continuous.

c) Let $x, y \in X$, $h(x) = h(y)$. Then $h^* g(x) = g \circ h(x) = g \circ h(y) = h^* g(y)$ for any $g \in F(Y)$, hence $f(x) = f(y)$ for any $f \in F(X)$ and thus $x = y$. Thus h is 1.1 and h^{-1} exists.

b) Put $\text{ind}_{h(X)}\tau' = \tau^*$ (see 0.14). Then h^{-1} is $\tau^* - \tau$ continuous iff so is $g = f \circ h^{-1}$ for any $f \in F(X)$. There is $l \in F(Y)$ such that $f = l \circ h$, hence $g = l \circ h \circ h^{-1} = l/h(X)$, thus g is τ^*-continuous.

The proof in *SEM* and *PROX* is the same.

1.3.2. Corollary. *Let $\mathscr{X} = (X, t)$, $Y = (Y, t')$ be completely regular topological spaces, $h : X \to Y$ a map, $F(\mathscr{X}) = C(\mathscr{X} \to R)$, $F(Y) = C(\mathscr{Y} \to R)$.*

(1) *If $h(X) = Y$ then these are equivalent:*

a) *h^* carries $F(Y)$ onto $F(X)$;*

b) *h is a homeomorphism of $\mathscr{X}$ into $\mathscr{Y}$.*

(2) *Let h be an arbitrary (we do not assume $h(X) = Y$) $1-1$ map and let $h^{-1} : (h(X), \text{ind } t') \to (X, t)$ be continuous. If every $g \in C((h(X), \text{ind } t') \to R)$ has an extension $g' \in F(Y)$, then $F(X) \subset h^* F(Y)$. If h is continuous then $F(X) = h^* F(Y)$.*

Proof. As $\mathscr{X}, \mathscr{Y}$ are completely regular, t and t' are projectively defined by $F(X)$ and $F(Y)$, respectively. Now (1) follows from 1.3.1. To prove (2), take $g \in F(X)$. If $g' \in F(Y)$ is an extension of $g \circ h^{-1}$, then $g = h^*g'$. The rest is clear.

We have shown that if $F(X)$ separates points, then $F(X) \not\subset h^* F(Y)$ unless h is $1-1$ and h^{-1} continuous. Thus, if we want to find when $F(X) \subset h^* F(Y)$, it is natural to assume that h^{-1} is ind $t' - t$ continuous (or uniformly continuous, ... if we work in *SEM*, ...).

1.3.3. Notation. A) Let $\mathscr{X} = (X, t)$ be a closure space. The set of all continuous or bounded continuous real functions on $\mathscr{X}$ is denoted by $C(\mathscr{X} \to R)$ or $C^*(\mathscr{X} \to R)$ respectively.

B) Let $\mathscr{X} = (X, n)$ be a semiuniform space. The set of all uniformly continuous or uniformly continuous bounded functions on $\mathscr{X}$ is denoted by $U(\mathscr{X} \to R)$ or $U^*(\mathscr{X} \to R)$ respectively.

C) Let $\mathscr{X} = (X, t)$ be a locally convex real topological linear space. The set of all continuous linear functions on $\mathscr{X}$ is denoted by $L(\mathscr{X} \to R)$.

D) Let $\mathscr{X} = (X, t)$ be a metrisable space with a metric d. The set of all Hölder-Lipschitz functions (shortly H.L.) on (X, d) is denoted by $HL((X, d) \to R)$ ($f : (X, d) \to R$ is a H.L. function if there is α, $0 < \alpha \leqq 1$ such that $|f(x) - f(y)| \leqq d^\alpha(x, y)$ for all $x, y \in X$).

Where it is useful we write $C((X, t) \to R), \ldots$ instead of $C(\mathscr{X} \to R), \ldots$.

1.3.4. Lemma. *Let $\mathscr{X} = (X, t)$, $\mathscr{Y} = (Y, t')$ be two closure spaces, $h : X \to Y$ a $1-1$ map, $F(X) \subset C(\mathscr{X} \to R)$, $F(Y) \subset C(\mathscr{Y} \to R)$. Then $F(X) \subset h^* F(Y)$ if one of the following conditions is fulfilled:*

a) $\mathscr{Y}$ *is a normal topological space,* $h(X)$ *is closed in it,* $h^{-1} : (h(X), \text{ind } t') \to \to \mathscr{X}$ *continuous, and* $F(X) \subset C(\mathscr{X} \to R)$, $F(Y) \supset C^*(\mathscr{Y} \to R)$;

b) $\mathscr{X}$ *and* $\mathscr{Y}$ *are uniformizable with some uniformities* n, n', h^{-1} *is* $n' - n$ *uniformly continuous and* $F(X) \subset U^*(\mathscr{X} \to R)$, $F(Y) \supset U^*(\mathscr{Y} \to R)$;

c) $\mathscr{X}, \mathscr{Y}$ *are locally convex topological linear spaces,* $h : \mathscr{X} \to \mathscr{Y}$ *linear and* $F(X) \subset L(\mathscr{X} \to R)$, $F(Y) \supset L(\mathscr{Y} \to R)$;

d) (X, t), (Y, t') *are metrizable by some metrices* d, d', $h^{-1} : (h(X), \text{ind } d') \to \to (X, d)$ *is a Hölder-Lipschitz map* (i.e. $d(h^{-1}(y), h^{-1}(z)) \leqq [d'(y, z)]^\alpha$ *for some* α, $0 < \alpha \leqq 1$) *and* $F(X) \subset HL(X, d) \to R)$, $F(Y) \supset HL((Y, d') \to R)$.

Proof. (a) follows from the Tietze theorem [5, Chap. VII, Sec. 5, Th. 5.1, p. 474], (b) from the theorems concerning uniformly continuous extensions of uniformly continuous functions [3, Chap. IV, Sec. 25, Th. 25 F2, p. 474. (c) from the Hahn-Banach theorem [10, Chap. 3, Sec. 17, p. 190] (d) from [11].

1.3.5. Remark. Only the case (b) from 1.3.4 is essential. If we may use (a), (c) or (d), then we may always use (b). Indeed, if (c) or (d) holds, then $\mathscr{X} = (X, t)$, $\mathscr{Y} = = (Y, t')$ are canonically uniformizable with some uniformities n, n', which yield t, t' so that h^{-1} – being linear or Hölder-Lipschitz – is ind $n' - n$ uniformly continuous. Thus we may use (b). If (a) holds, we may define n and n' projectively by $F(X)$, $F(Y)$. By 1.3.1b, the map h^{-1} is $n' - n$ uniformly continuous, so we may use (b) again.

1.3.6. Lemma. *Let* $\mathscr{X} = (X, t)$, $\mathscr{Y} = (Y, t')$ *be two closure spaces,* $h : X \to Y$ *a map,* $F(X) \subset C^*(\mathscr{X} \to R)$, $F(Y) \supset C^*(\mathscr{Y}, R)$. *Let us have a continuous map* $g : \mathscr{Y} \to \mathscr{X}$ *such that* $g \circ h$ *is identical on* X *(it is always fulfilled if* h *is* $1-1$, h^{-1} ind $t' - t$ *continuous and if there is a continuous map* g_1 *of* $\mathscr{Y}$ *onto* $(h(X)$, ind $t')$ *which is identical on* $h(X)$). *Then* $F(X) \subset h^*(F(Y)$.

The proof of 1.3.6 is obvious. In 1.3.6 we need not the normality of $\mathscr{Y}$. By 1.3.1b, under the conditions of 1.3.6, Lemma 1.3.4b may be used again.

The last assumption of Th. 1.1.7 which deserves to be dealt with is that the family $\mathscr{E}$ is separating. To finish this section we shall investigate when it is fulfilled.

1.3.7. Lemma. *Let* $\mathscr{X} = (X, t)$ *be a closure space. Then the set* $F(X) \subset C(\mathscr{X} \to R)$ *separates points of* X *in each of the following cases:*

(a) $\mathscr{X}$ *is a completely regular topological* T_1*-space (i.e. the points of* $\mathscr{X}$ *are closed) and* $F(X) \supset C^*(\mathscr{X} \to R)$.

(b) $\mathscr{X}$ *is uniformizable with a separated uniformity* n *and* $F(X) \supset U^*((X, n) \to R)$.

(c) $\mathscr{X}$ *is a Hausdorff locally convex topological linear space and* $F(X) \supset L(\mathscr{X} \to \to R)$.

(d) $\mathscr{X}$ *is metrizable with a metric* d *and* $F(X) \supset HL((X, d) \to R)$. *In the cases* (a), (b), (d) *the set* $F(X)$ *distinguishes points from closed sets of* $\mathscr{X}$.

Proof. We prove (b). Let $M \subset X$ be t-closed, $a \in X - M$. If n is a separated uniformity which yields t, then the function $f = 1$ on M, $f(a) = 0$ is uniformly continuous on $(M \cup \{a\}$, ind $n)$. There is $g \in U = U^*((X, n) \to R)$ with $g = f$ on $M \cup \{a\}$. Thus U separates points from closed sets. As n is separated, the points of X are t-closed, which proves (b). Likewise the cases (a), (c), (d) follow from the extension theorems used in 1.3.4.

4. SOME PROPERTIES OF $\mathscr{L}(\mathscr{S})$

We have defined $\mathscr{L}(\mathscr{S})$ in 1.1.4. Here we deal with some properties of this set that will be used later.

1.4.1. Lemma. (1) $\mathscr{L}(\mathscr{S}_{A[\alpha]}) = \mathscr{L}(\mathscr{S}) \cap A[\alpha]$,

(2) $\mathscr{L}(\mathscr{S}_{A(\beta)}) = \mathscr{L}(\mathscr{S}) \cap A(\beta)$ *if* $\langle A \leqq \rangle$ *is ordered and* $\beta \notin \mathscr{L}(\mathscr{S})$ (see 1.1.3),

(3) $\mathscr{L}(\mathscr{S}_{\langle\beta\alpha)}) = \mathscr{L}(\mathscr{S}) \cap \langle\beta\alpha)$ *if* $\langle A \leqq \rangle$ *is ordered and* $\beta \notin \mathscr{L}(\mathscr{S})$,

(4) $\varrho_{\beta,\beta+1} : S_\beta \to S_{\beta+1}$ *is a* $\mathfrak{K}$-*isomorphism if* $\beta + 1$ *exists and* $\beta + 1 \in \mathscr{L}$,

(5) $\varrho_{\alpha\beta} : S_\alpha \to S_\beta$ *is a* $\mathfrak{K}$-*isomorphism if* $\langle \alpha, \beta \rangle \subset N \subset \mathscr{L}$, *where* $\langle N \leqq \rangle$ *is well ordered.*

Proof. (1) follows from 1.1.4. By 1.1.4A, if $\alpha \in \mathscr{L}(\mathscr{S}_{A(\beta)})$ then $\alpha > \beta$ and $S_\alpha = \varinjlim \mathscr{S}_{\langle\beta\alpha)}$. By 0.9B, we have $S_\alpha = \varinjlim \mathscr{S}_{A[\alpha]}$ as $\langle\beta\alpha)$ is cofinal in $A[\alpha]$; hence $\alpha \in \mathscr{L}(\mathscr{S}) \cap A(\beta)$. Conversely, if $\alpha \in \mathscr{L}(\mathscr{S}) \cap A(\beta)$ and $\beta \notin \mathscr{L}$, then we have $\beta < \alpha$. As $\langle\beta\alpha)$ is cofinal in $A[\alpha]$, we get $S_\alpha = \varinjlim \mathscr{S}_{A[\alpha]} = \varinjlim \mathscr{S}_{\langle\beta\alpha)}$, thus $\alpha \in \mathscr{L}(\mathscr{S}_{A(\beta)})$ which proves (2). To prove (3), we recall that $\langle\beta\alpha) = (A[\alpha])(\beta)$. By (1), if $\beta \notin \mathscr{L}(\mathscr{S})$ then $\beta \notin \mathscr{L}(\mathscr{S}_{A(\alpha)})$. Thus by (1), (2) we get $\mathscr{L}(\mathscr{S}_{\langle\beta\alpha)}) = \mathscr{L}(\mathscr{S}_{A[\alpha]}) \cap \langle\beta\alpha) = \mathscr{L}(\mathscr{S}) \cap \cap A[\alpha] \cap \langle\beta\alpha) = \mathscr{L}(\mathscr{S}) \cap \langle\beta\alpha)$ as desired. (4) follows from 0.8B as β is the largest element of $M = A[\beta + 1]$. Indeed, M is directed for $\beta + 1 \in \mathscr{L}$. Thus if $\gamma \in M$, then there is $\delta \in M$ with $\delta \geqq \beta$, $\delta \geqq \gamma$ which yields $\delta = \beta$, hence $\gamma \leqq \beta$ as desired. We prove (5): Let $\gamma \in \langle \alpha, \beta)$ and let $\varrho_{\alpha\varepsilon} : \mathscr{X}_\alpha \to \mathscr{X}_\varepsilon$ be a $\mathfrak{K}$-isomorphism for any $\varepsilon \in \langle \alpha, \gamma)$. Then $\langle\alpha\gamma)$ is cofinal in $A[\gamma]$ because $\gamma \in \mathscr{L}$ and $A[\gamma]$ is right directed (see 1.1.4). By 0.8B, $\mathscr{X}_\gamma = \varinjlim \mathscr{S}_{A[\gamma]} = \varinjlim \mathscr{S}_{\langle\alpha\gamma)}$. It can be seen from 0.4 that $\mathscr{X}_\alpha = \varinjlim \mathscr{S}_{\langle\alpha\gamma)}$ and that $\{\varrho_{\alpha\varepsilon}^{-1} : \mathscr{X}_\varepsilon \to \mathscr{X}_\alpha \mid \varepsilon \in \langle\alpha\gamma)\}$ are the canonical maps. By 0.4 we get that $\varrho_{\alpha\gamma}$ is a $\mathfrak{K}$-isomorphism. Transfinite induction gives us the desired result.

1.4.2. Lemma. *Let* $\mathscr{S} = \{S_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *be a presheaf from a category* $\mathfrak{K}$, $\mathscr{N} \subset \mathscr{L}$. *Then* $\varinjlim \mathscr{S}$ *and* $\varinjlim \mathscr{S}_{A-\mathscr{N}}$ *are* $\mathfrak{K}$-*isomorphic if either* $\langle A \leqq \rangle$ *is ordered and* $\langle \mathscr{N} \leqq \rangle$ *well ordered, or if* $A - \mathscr{N}$ *is cofinal in* $\langle A \leqq \rangle$. (*I.e. if one of the limits exists then the other exists as well and they both are* $\mathfrak{K}$-*isomorphic*). *Thus* $\varinjlim \mathscr{S}_{A[\alpha]-\mathscr{L}}$ *is* $\mathfrak{K}$-*isomorphic to* $\varinjlim \mathscr{S}_{A[\alpha]}$ *for all* $\alpha \in A$ *if* $\langle A \leqq \rangle$ *is well ordered.*

Proof. If $A - \mathcal{N}$ is cofinal then the statement follows from 0.8B. If $A - \mathcal{N}$ is not cofinal, then $\langle \mathcal{N} \leqq \rangle$ is well ordered and there is $\beta \in A$ such that $\{\alpha \in A \mid \alpha \geqq \beta\} \subset \subset \mathcal{N}$. Let M be the set of all such β and let p be the smallest element of M. Then $p \in \mathcal{N}$ and since $\mathcal{N} \subset \mathcal{L}$ so $\varinjlim \mathcal{S}_{A[p]} = S_p$. Further, $A - \mathcal{N}$ is cofinal in $A[p]$ for $\langle A \leqq \rangle$ is ordered. Thus $S_p = \varinjlim \mathcal{S}_{A-\mathcal{N}}$. We have gotten that $\varinjlim \mathcal{S}_{A-\mathcal{N}}$ exists if $\langle \mathcal{N} \leqq \rangle$ is well ordered. By 1.4.1(5), $\varrho_{\alpha\beta} : S_\alpha \to S_\beta$ is a $\mathfrak{K}$-isomorphism for all $\alpha, \beta \geqq p$, hence $\varinjlim \mathcal{S}$ is $\mathfrak{K}$-isomorphic to S_p is desired. Let $\langle A \leqq \rangle$ be well ordered, $\alpha \in A$. Using 1.4.2 to $\mathcal{S}_{A[\alpha]}$ and $\mathcal{N} = \mathcal{L}(\mathcal{S}_{A[\alpha]}) = \mathcal{L} \cap A_{[\alpha]}$ (which is well ordered), we get the last statement.

1.4.3. Lemma. *Let $\mathcal{S} = \{\mathcal{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be an inductive family from a category $\mathfrak{K}$, $\alpha \in \mathcal{L}$.*

A. *Let $\langle A - \mathcal{L}, \leqq \rangle$ be well ordered. Then either $A(\alpha) = \{\beta \in A \mid \beta \geqq \alpha\} \subset \mathcal{L}$ or there is $l(\alpha) \in A - \mathcal{L}$, $\alpha < l(\alpha)$ such that (1) $\langle \alpha, l(\alpha)) \subset \mathcal{L}$, (2) if $\beta \in A$, $\beta > \alpha$, $\langle \alpha\beta) \subset \mathcal{L}$ then $\beta < l(\alpha)$ if $\langle A \leqq \rangle$ is ordered, (3) $l(\alpha) \leqq \beta$ if $\beta \in A - \mathcal{L}$, $\beta > \alpha$ (4) $\gamma = l(\alpha)$ iff $\gamma \in A - \mathcal{L}$, $\alpha < \gamma$ and $\langle \alpha\gamma) \subset \mathcal{L}$, (5) if $\gamma \in A - \mathcal{L}$ then $A[\gamma] - \mathcal{L}$ is cofinal in $A[\gamma]$ iff $\gamma \neq l(\beta)$ for all $\beta \in \mathcal{L}$, (6) $\mathcal{X}_\alpha$ is $\mathfrak{K}$-isomorphic to $\varinjlim \mathcal{S}_{A[l(\alpha)]}$.*

B. *Let $\langle \mathcal{L} \leqq \rangle$ be well ordered. Then there is $s(\alpha) \in \mathcal{L}$, $s(\alpha) \leqq \alpha$ such that (7) $\langle s(\alpha), \alpha \rangle \subset \mathcal{L}$, (8) if $\beta \in \mathcal{L}$, $\beta \leqq \alpha$ and $\langle \beta\alpha \rangle \subset \mathcal{L}$ then $s(\alpha) \leqq \beta$, (9) if there is $\omega = s(\alpha) - 1$, then $\omega \notin \mathcal{L}$ and $\varrho_{\omega s(\alpha)}$ is a $\mathfrak{K}$-isomorphism, (10) if $\langle A \leqq \rangle$ is well ordered and 1 and λ are the smallest elements of A and of $\mathcal{L}$ respectively, then $1 < \lambda$, (11) the set $A[\alpha] - \mathcal{L}$ is cofinal in $A[\alpha]$ iff $\alpha = s(\alpha)$.*

Proof. If $A(\alpha) \not\subset \mathcal{L}$, then $M = \{\beta \in A - \mathcal{L} \mid \beta > \alpha\} \neq \emptyset$. Let $l(\alpha)$ be the smallest element of M. Clearly $l(\alpha)$ fulfils (1)–(4) while (5) follows from (2) and (4). (6) follows from 1.4.1.(5). As for B, let $s(\alpha)$ be the smallest element of $\{\beta \in \mathcal{L} \mid \beta > \gamma$ for any $\gamma \in A[\alpha] - \mathcal{L}\} \neq \emptyset$. Clearly $s(\alpha)$ satisfies (7), (8). As $s(\alpha) \in \mathcal{L}$, (9) follows from 1.4.1(4). By 1.1.4, we have $1 \notin \mathcal{L}$, hence $\lambda > 1$ which proves (10). As $A[\alpha] - \mathcal{L}$ is cofinal in $A[\alpha]$ iff $\langle s(\alpha), \alpha \rangle$ is a one point set, (11) follows forthwith.

1.4.4. Lemma. *Let $\mathcal{S} = \{\mathcal{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be from an i.c. category $\mathfrak{L}$ (see 0.10), $\langle A \leqq \rangle$ well ordered, $\alpha \in \mathcal{L}$, $\beta \in A[\alpha] - \mathcal{L}$. Suppose $\mathcal{G} = \{g_\gamma \in C(\mathcal{X}_\gamma \to R \mid \mathfrak{L}) \mid \gamma \in \in \langle \beta\alpha) - \mathcal{L}\}$ is a compatible family for $\mathcal{S}_{\langle\beta\alpha) - \mathcal{L}}$ (see 0.3). Then there is a unique compatible family $\mathcal{F} = \{f_\gamma \mid \gamma \in A[\alpha]\}$ for $\mathcal{S}_{A[\alpha]}$ with $f_\gamma = g_\gamma$ for all $\gamma \in \langle \beta\alpha) - \mathcal{L}$. Further we have $f_\gamma = \varprojlim \{f_\delta \mid \delta \in A[\gamma]\}$ if $\gamma \in \mathcal{L}$; $\varprojlim \mathcal{G} = f \in C(\mathcal{X}_{s(\alpha)} \to R \mid \mathfrak{L})$, $\varprojlim \mathcal{F} = g \in C(\mathcal{X}_\alpha \to R \mid \mathfrak{L})$ and $\varrho^*_{s(\alpha)\alpha} g = f$.*

Proof. Put $f_\gamma = \varrho^*_{\gamma\beta} f_\beta$ for $\gamma \leqq \beta$. Let $\gamma \in \langle \beta\alpha)$ and let us have a compatible family $\mathcal{F}_\gamma = \{f_\delta \mid \delta \in A[\gamma]\}$ for $\mathcal{S}$ with $f_\delta = g_\delta$ for $\delta \in A[\gamma] - \mathcal{L}$ so that $f_\delta = \varprojlim \{f_\varepsilon \mid \varepsilon \in \in A[\delta]\}$ for $\delta \in \mathcal{L} \cap A[\gamma]$. If $\gamma \notin \mathcal{L}$, we put $f_\gamma = g_\gamma$. If $\gamma \in \mathcal{L}$, we put $f_\gamma = \varprojlim \mathcal{F}_\gamma$.

Clearly $\{f_\delta \mid \delta \in \langle 1, \gamma \rangle\}$ is compatible for $\mathscr{S}_{\langle 1,\gamma \rangle}$. Transfinite construction yields the desired family. Using 1.3.4(11) to $\mathscr{S}_{\langle \beta\alpha \rangle}$, we get that $\varrho_{s(\alpha)\alpha}$ is an $\mathfrak{L}$-isomorphism which completes the proof.

5. SEPARATION THEOREMS

In Section one we have proven Th. 1.1.7, which is an abstract sufficient condition for the functional separatedness of $\varprojlim \mathscr{S}$. In the present section some separation theorems for specific presheaves are proven with the help of Th. 1.1.7.

1.5.1. Theorem. *Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be a presheaf from an i.c. category $\mathfrak{L}$ and let a set $B \subset A$ be such that $\langle B \leqq \rangle$ is well ordered. Suppose that*

(1) *Either B is cofinal in $\langle A \leqq \rangle$, or $\langle A \leqq \rangle$ is ordered, $\langle A - B, \leqq \rangle$ well ordered and $A - B \subset \mathscr{L}$*

(2) *$\mathscr{S}_B$ is endowed with a leftward smooth and connected separating family $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$* (see 1.1.5). *Then $\mathscr{J} = \varprojlim \mathscr{S}_B$ and $\mathscr{I} = \varprojlim \mathscr{S}$ are functionally separated by $D = \{f \in C(\mathscr{I} \to R \mid \mathfrak{L}) \mid$ there is $\beta \in B$ such that $\xi_\gamma^* f \in F_\gamma$ for all $\gamma \in B$, $\gamma \geqq \beta\}$ (here $\{\xi_\gamma : \mathscr{X}_\gamma \to \mathscr{I} \mid \gamma \in A\}$ are the canonical maps).*

If there is a countable confinal subset C in B and if $F_\alpha \subset \varrho_{\alpha\beta}^*(F_\beta)$ for all $\alpha, \beta \in B$, $\alpha \leqq \beta$ then the connectedness of $\mathscr{E}$ may be left out.

Proof. Using 1.1.7 to $\mathscr{S}_B$ and $\mathscr{E}$, and bearing in mind that $\mathscr{I}$ and $\mathscr{J}$ are $\mathfrak{L}$-isomorphic by 1.4.2, we get by 1.4.2 that there even is $\beta \in B - \mathscr{L}(\mathscr{S}_B)$ such that the statement of 1.5.1 holds. To get the last assertion we use 1.2.6 to $\mathscr{S}_B$, $\mathscr{E}$ and to the properties Q: "$\mathscr{E}$ is separating", P: "For every α, $\alpha + 1 \in B$ the pair $(F_\alpha, F_{\alpha+1})$ has the property $P_{\alpha,\alpha+1} : \{F_\alpha \subset \varrho_{\alpha,\alpha+1}^* F_{\alpha+1}\}$" – which is the left smoothness of $\mathscr{E}$ – and to R: "The assertion of 1.5.1". If $P_{\alpha\beta}$ is fulfilled for any $\alpha, \beta \in B$, $\alpha \leqq \beta$ then because Q is stable under making subbamilies of $\mathscr{E}$ and R is stable under $\mathfrak{L}$-isomorphisms, the assertion follows.

1.5.2. Theorem. *Given a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from UNIF* (see 0.5) *and a set $B \subset A$ such that the uniformity in $\mathscr{X}_\alpha$ is separated for every $\alpha \in B$, let us denote by $\alpha + 1 (\alpha - 1)$ the follower (predecessor – if it exists) of $\alpha \in B$ in $\langle B \leqq \rangle$ and let us assume the condition (1) of Th. 1.5.1. is fulfilled and*

(2) (a) *$\varrho_{\alpha,\alpha+1} : \mathscr{X}_\alpha \to \mathscr{X}_{\alpha+1}$ is a uniform embedding into $\mathscr{X}_{\alpha+1}$* (see 0.15) *for all $\alpha \in B$,*

(b) *if $\alpha \in B$ is such that $\alpha - 1$ does not exist, then $U^*(\mathscr{L}_\alpha = \varprojlim \mathscr{S}_{B[\alpha]} \to R) \subset \subset \lambda_\alpha^* U^*(\mathscr{X}_\alpha \to R)$ (here $\lambda_\alpha : \mathscr{L}_\alpha \to \mathscr{X}_\alpha$ is the canonical map* – see 1.1.3).

Then $\mathscr{J} = \varprojlim \mathscr{S}_B$ and $\mathscr{I} = \varprojlim \mathscr{S}$ are f.s. by $U^*(\mathscr{I} \to R)$. The condition (2b) *may be left out if there is a countable cofinal set C in B and if $\varrho_{\alpha\beta} : \mathscr{X}_\alpha \to \mathscr{X}_\beta$ is a uniform embedding for all $\alpha, \beta \in B$, $\alpha \leqq \beta$.*

Proof. We show that the conditions of Th. 1.5.1 are fulfilled. If $\alpha \in B$ then we put $F_\alpha = U^*(\mathscr{X}_\alpha \to R)$, $\mathscr{E} = \{F_\alpha \mid \alpha \in B\}$. By 1.3.4b and 1.3.7b, $\mathscr{E}$ is leftward smooth and separating. If $\alpha \in B$ is such that $\alpha - 1$ does not exist, $\beta \in B[\alpha]$, $\mathscr{F} = \{f_\gamma \in F_\gamma \mid \gamma \in \langle\beta\alpha) \cap B\}$ a thread through $\mathscr{E}_{\langle\beta\alpha)\cap B}$, then $\varprojlim \mathscr{F} = f \in U^*(\mathscr{L}_\alpha \to R)$. By the assumption (2b) and by 1.3.5b, there is $g \in F_\alpha$ with $\lambda_\alpha^* g = f$. Clearly $\varrho_{\gamma\alpha}^* g = f_\gamma$ for all $\gamma \in \langle\beta\alpha) \cap B$, hence $\mathscr{E}$ is connected. By 1.5.1, our theorem follows. If all the $\varrho_{\alpha\beta}$ are uniform embeddings then $F_\alpha \subset \varrho_{\alpha\beta}^* F_\beta$ for all $\alpha, \beta \in B$, $\alpha \leqq \beta$; so the assertion follows from that of 1.5.1.

1.5.3. Remark. We have the condition (2b) of the foregoing theorem to get the connectedness of $\mathscr{E}$.

The inclusion in (2b) is equivalent to the uniform continuity of $\lambda_\alpha^{-1} : (\lambda_\alpha(\mathscr{L}_\alpha), \text{ind}\, \tau'_\alpha) \to (\mathscr{L}_\alpha, t'_\alpha)$, where τ'_α and t'_α are the uniformities projectively defined in $|\mathscr{X}_\alpha|$ and in $|\mathscr{L}_\alpha|$ by $U^*(\mathscr{X}_\alpha \to R)$ and $U^*(\mathscr{L}_\alpha \to R)$ respectively – see 0.9, 1.3.1B.

The inclusion in (2b) is equivalent to the connectedness of $\mathscr{E}$. Indeed, if $f \in U^*(\mathscr{L}_\alpha \to R)$ then for each canonical maps $\{\varrho'_{\gamma\alpha} : \mathscr{X}_\gamma \to \mathscr{L}_\alpha \mid \gamma \in B[\alpha]\}$ we have $\varrho'^*_{\gamma\alpha} f = f_\gamma \in F_\gamma$. So $\{f_\gamma \mid \gamma \in B[\alpha]\}$ is a thread through $\mathscr{E}_{B[\alpha]}$. If $\mathscr{E}$ is connected then there is $g \in F_\alpha$ with $\varrho_{\gamma\alpha}^* g = \varrho'^*_{\gamma\alpha} \circ \lambda_\alpha^* g = f_\gamma$. By 0.4, we have $\lambda_\alpha^* g = f$ and hence $U^*(\mathscr{L}_\alpha \to R) \subset \lambda_\alpha^* F_\alpha$. The converse has been shown in the proof of 1.5.2.

1.5.4. Corollary. *Let $\mathscr{S} = \{\mathscr{X}_\alpha = (X_\alpha, \tau_\alpha) \, |\varrho_{\alpha\beta}| \, \langle A \leqq \rangle\}$ be a presheaf from CLOS* (see 0.5), *$B \subset A$. Suppose that for every $\alpha \in B$ there is a separated uniformity n_α in X_α which yields a topology v_α coarser than τ_α and such that $\tilde{\mathscr{S}} = \{\mathscr{N}_\alpha = (X_\alpha, n_\alpha) \, |\varrho_{\alpha\beta}| \, \langle A \leqq \rangle\}$ is from UNIF* (i.e. *$\varrho_{\alpha\beta} : \mathscr{N}_\alpha \to \mathscr{N}_\beta$ are uniformly continuous*). *Denoting by $\alpha + 1$ ($\alpha - 1$) the follower (predecessor – if ir exists) of $\alpha \in B$ in $\langle B \leqq \rangle$, we assume that the condition* (1) *of Th.* 1.5.1 *is fulfilled and*

(2) (a) *$\varrho_{\alpha,\alpha+1} : \mathscr{N}_\alpha \to \mathscr{N}_{\alpha+1}$ is a uniform embedding for all $\alpha \in B$,*

(b) *if $\alpha \in B$ is such that $\alpha - 1$ does not exist then $U^*(\mathscr{L}_\alpha = \varprojlim \tilde{\mathscr{S}}_{B[\alpha]} \to R) \subset \lambda_\alpha^* U^*(\mathscr{N}_\alpha \to R)$* (*here $\lambda_\alpha : \mathscr{L}_\alpha \to \mathscr{N}_\alpha$ is the canonical map* – see 0.4).

Then $\mathscr{J} = \varprojlim \mathscr{S}_B$ and $\mathscr{I} = \varprojlim \mathscr{S}$ are functionally separated by $C^*(\mathscr{I} \to R)$ (see 1.1.3). *The condition* (2b) *may be left out if there is a countable cofinal set C in B and if $\varrho_{\alpha\beta} : \mathscr{N}_\alpha \to \mathscr{N}_\beta$ is a uniform embedding for all $\alpha, \beta \in B$, $\alpha \leqq \beta$.*

Proof. The presheaf $\tilde{\mathscr{S}}$ fulfils the conditions of 1.5.2 (the set B mentioned in 1.5.2 is now whole B for $\tilde{\mathscr{S}}$), thus $\mathscr{K} = (K, n) = \varprojlim \tilde{\mathscr{S}}$ is f.s. by $U^*(\mathscr{K} \to R)$ (n is the uniformity in $\mathscr{K}$, $K = |\mathscr{K}|$). Let t be the topology generated in K by n. Then (K, t) is f.s. by $C^*((K, t) \to R)$. If $\{\xi_\alpha : \mathscr{N}_\alpha \to \mathscr{K} \mid \alpha \in B\}$ are the cannonical maps then $\xi_\alpha : (X_\alpha, v_\alpha) \to (K, t)$ together with the identity $e_\alpha : \mathscr{X}_\alpha \to (X_\alpha\, v_\alpha)$ are continuous for all $\alpha \in B$. Hence the same holds for all $\xi_\alpha \circ e_\alpha : \mathscr{X}_\alpha \to (K, t)$. By 0.10(3), ξ_α are $1-1$, thus there is a continuous $1-1$ map $h : \mathscr{J} \to (K, t)$. By 1.1.1, $\mathscr{J}$ is f.s. by $C^*(\mathscr{J} \to R)$. Now we use 1.4.2.

1.5.5. Corollary. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha = (X_\alpha, \tau_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ *be a presheaf from CLOS* (see 0.5), *and* $B \subset A$ *a set such that* $\mathscr{X}_\alpha$ *is a normal topological space for every* $\alpha \in B$. *Denoting by* $\alpha + 1$ $(\alpha - 1)$ *the follower (predecessor – if it exists) of* $\alpha \in B$ *in* $\langle B \leqq \rangle$, *we assume that the condition* (1) *of Th.* 1.5.1 *is fulfilled and*

(2) (a) $\varrho_{\alpha,\alpha+1} : \mathscr{X}_\alpha \to \mathscr{X}_{\alpha+1}$ is a homeomorphism into $\mathscr{X}_{\alpha+1}$ such that $\varrho_{\alpha,\alpha+1}(X_\alpha)$ is closed in $\mathscr{X}_{\alpha+1}$,

(b) if $\alpha \in B$ is such that $\alpha - 1$ does not exist then the canonical map $\lambda_\alpha : \mathscr{L}_\alpha = \varprojlim \mathscr{S}_{B[\alpha]} \to \mathscr{X}_\alpha$ is a homeomorphism into $\mathscr{X}_\alpha$ such that $\lambda_\alpha \mathscr{L}_\alpha$ is closed in $\mathscr{X}_\alpha$. Then $\mathscr{J} = \varprojlim \mathscr{S}_B$ and $\mathscr{I} = \varprojlim \mathscr{S}$ are functionally separated by $C^*(\mathscr{I} \to R)$. The condition (2b) may be left out if there is a countable confinal set C in B and if (2a) is fulfilled for any pair $\alpha, \beta \in B$ with $\alpha \leqq \beta$ instead of for $\alpha, \alpha + 1$ only.

Proof. The family $\mathscr{E} = \{F_\alpha = C^*(\mathscr{X}_\alpha \to R) \mid \alpha \in B\}$ endowes $\mathscr{S}_B$ and is separating by 1.3.7A, leftward smooth by (2a) and 1.3.4a, and connected by (2b) and 1.3.4a. If (2a) holds for any pair $\alpha, \beta \in B$, $\alpha \leqq \beta$ then $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$ for all $\alpha, \beta \in B$, $\alpha \leqq \beta$. Now we use 1.5.1.

1.5.6. Remark. A. The condition (a) of 1.5.4 is fulfilled namely if for $\alpha \in B$ the map $\varrho_{\alpha,\alpha+1}$ is $1-1$ and continuous and $\mathscr{X}_\alpha$ compact.

B. The statements 1.5.2–5 have better form if B is countable and if the assumption ensuring the left smoothness of the saparating families is fulfilled for all $\alpha, \beta \in B$, $\alpha \leqq \beta$. Then we may leave out the awkward assumption about those $\alpha \in B$ for which $\alpha - 1$ does not exist. For example, we get the following statement from 1.5.4 and 1.5.5:

Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be a presheaf from *TOP*, $B \subset A$ such that $\langle B \leqq \rangle$ is well ordered and that there is a countable cofinal set $C \subset B$. Assume that either B is cofinal in $\langle A \leqq \rangle$ or $\langle A \leqq \rangle$ is ordered, $\langle A - B \leqq \rangle$ well ordered and $A - B \subset \mathscr{L}$. If $\mathscr{X}_\alpha$ is compact and $\varrho_{\alpha\beta}$ is $1-1$ for all $\alpha, \beta \in B$, then $\mathscr{I} = \varprojlim \mathscr{S}$ is f.s.

In the general case the propositions 1.2.1, 1.2.2 can be sometimes used to verify the connectedness of $\mathscr{E}$.

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