Czechoslovak Mathematical Journal

Milan Medveď

# On a differential game of evasion described by a class of nonlinear Volterra integrodifferential equations

*Czechoslovak Mathematical Journal*, Vol. 28 (1978), No. 3, 407–418

Persistent URL: `http://dml.cz/dmlcz/101546`

Czechoslovak Mathematical Journal, 28 (103) 1978, Praha

# ON A DIFFERENTIAL GAME OF EVASION DESCRIBED BY A CLASS OF NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

MILAN MEDVEĎ, Bratislava

(Received March 8, 1976)

## 1. INTRODUCTION

We shall consider a differential game of evasion described by the integrodifferential equation

$$(1) \qquad \dot{z}(t) + A\,z(t) + \int_0^t B(u(t-\tau), v(t-\tau))\,z(\tau)\,d\tau = \\ = f(u(t), v(t), w(t)) + \mu g(z(t), u(t), v(t), w(t)),$$

where $z \in R^m$, $f, g \in R^m$, $A$ is a square constant matrix, $f(u, v, w)$ is a continuous function with respect to all arguments $(u, v, w) \in U \times V \times W$, $U \subset R^p$, $V = V_1 \times V_2 \times \ldots \times V_m$, $V_i \subset R^{q_i}$, $i = 1, 2, \ldots, m$, $W \subset R^r$ are compact sets; $B(u, v)$ is a continuous matrix function with respect to $(u, v) \in U \times V$, $\mu \in (-\infty, \infty)$ is a parameter. We shall suppose that the function $g(z, u, v, w)$ is continuous and bounded on the set $R^m \times U \times V \times W$.

In the paper [1] a sufficient condition for existence of the strategy of evasion for a differential game described by the differential equation

$$z^{(n)} + A_1 z^{(n-1)} + \ldots + A_n z = f(u, v),$$

where $A_i$ are constant square matrices, is given. That condition is different from those obtained by L. S. PONTRYAGIN, E. F. MISHCHENKO, N. SATIMOV and by others (cf. [3], [4], [5]). The formulation and the proof of this sufficient condition is given using the technique of convolutions.

In the paper [5] the problem of evasion for the game described by an equation of the type (1) without the integral term is solved. The problem is solved as well as the result is formulated by Pontryagin's technique. We shall prove a sufficient condition of evasion for (1) of the type of the condition obtained in [1] using again the technique of convolutions.

**Definition 1.** A mapping $E_u(t; z_0)$ defined on the set of measurable controls $u(\tau)$, $0 \leqq \tau < \infty$, $u(\tau) \in U$, depending on $t \geqq 0$ and on the vector of initial conditions $z_0$ is said to be *a strategy*, if it possesses the following properties:

(1) For an arbitrary measurable function $u(\tau)$, $0 \leqq \tau < \infty$ and for an arbitrary fixed $z_0$, $E_u(t; z_0)$ as a function of $t$ is measurable with values in $V \times W$.

(2) If $u_1(\tau)$, $u_2(\tau)$, $0 \leqq \tau < \infty$ are two controls and $u_1(\tau) = u_2(\tau)$ most everywhere on $[0, T]$, where $T$ is arbitrary, then $E_{u_1}(t, z_0) = E_{u_2}(t; z_0)$ almost everywhere on $[0, T]$.

Let $E_u(t; z_0) = (V_u(t; z_0), W_u(t, z_0))$, where $V_u(t; z_0) = (V_u^1(t; z_0), \ldots, V_u^m(t; z_0)) \in \in R^{q_1} \times R^{q_2} \times \ldots \times R^{q_m}$, $W_u(t; z_0) \in R^r$.

## 2. FORMULATION OF THE PROBLEM

Let $M$ be a subspace of $R^m$ of a dimension $\leqq m - 2$. The problem is to find a strategy $E_u(t; z_0)$ such that the solution $z(t)$, $0 \leqq t < \infty$ of the equation

$$\dot{z}(t) + A\, z(t) + \int_0^t B(u(t - \tau),\ V_u(t - \tau; z_0))\, z(\tau)\, d\tau = \\ = f(u(t), E_u(t; z_0)) + \mu g(z(t), u(t), E_u(t; z_0))$$

with an initial condition $z(0) = z_0 \notin M$ does not intersect the subspace $M$ for any $t \geqq 0$, for an arbitrary control $u(t)$ and for an arbitrary vector $z_0$. We shall call such a strategy an evasion strategy.

Using the technique of convolutions (cf. [1]), we can rewrite the equation (1) in the form

$$(2) \qquad \dot{z} + \hat{A} * z + B(u(t), v(t)) * z = f(u, v, w) + \mu g(z, u, v, w)\,,$$

where $\hat{A} = D * A$, $D$ is the inverse element of $S$ ($S$ is the function identically equal to one) in the Mikusinski ring (cf. [1]).

If $x(t)$ is a differentiable function, then $D * x = \dot{x}(t) + D * x(0)$ and therefore we can rewrite the equation (2) in the form

$$(3) \qquad (D + \hat{A}) * z + B(u, v) * z = f(u, v, w) + \mu g(z, u, w) + D * z_0$$

or in the form

$$(4) \qquad (\hat{I} + S * \hat{A}) * z + S * B(u, v) * z = \\ = S * f(u, v, w) + \mu S * g(z, u, v, w) + z_0$$

where $\hat{I}$ is the unit matrix, i.e. the diagonal matrix with the element $\delta$ in the diagonal ($\delta$ is the unit element in the Mikusinski ring).

The matrix $\hat{I} + S * \hat{A}$ has the inverse entire matrix $R(S) = \hat{I} - S * \hat{A} + S^2 * * \hat{A}^2 - \ldots$ (cf. [1]). Therefore, from (4) we get

$$(\hat{I} + S * R(S) * B(u, v)) * z = \tag{5}$$

$$= S * R(S) * f(u, v, w) + \mu S * R(S) * g(z, u, v, w) + R(S) * z_0 .$$

The matrix $\hat{I} + S * R(S) * B(u, v)$ has the inverse matrix $R_{(u,v)}(S) = \hat{I} + T_{(u,v)}(S)$, where

$$T_{(u,v)}(S) = -S * R(S) * B(u, v) + S^2 * R^2(S) * B^2(u, v) - \ldots . \tag{6}$$

The series (6) considered as a series of matrix function of the variable $t$ is uniformly convergent in any circle with its center at the origin for arbitrarily given functions $u(t) \in U$, $v(t) \in V$. This fact is a consequence of the following estimates:

$$|T_{(u,v)}(S)| \leqq |S * R(S) * B(u, v)| + |S^2 * R^2(S) * B^2(u, v)| + \ldots ,$$

where $|\cdot|$ is the euclidean matrix norm. For $0 \leqq t \leqq T < \infty$, $|S * R(S) * B(u, v)| \leqq \leqq \lambda |S * R(S)| \leqq \lambda \int_0^T |R(S)| \, d\tau = n_T < \infty$ because $R(S)$ is an entire matrix and $|B(u, v)| \leqq \lambda$ for all $(u, v) \in U \times V$ for some $\lambda > 0$.

Denote $\psi(t) = S * R(S) * B(u, v)$. Then

$$|S^2 * R^2(S) * B^2(u, v)| = \left| \int_0^t \Psi(t - \tau) \, \Psi(\tau) \, d\tau \right| \leqq n_T^2 t \leqq n_T^2 T ,$$

$$|S^3 * R^3(S) * B^3(u, v)| = \left| \int_0^t \Psi(t - \tau) \, \Psi^2(\tau) \, d\tau \right| \leqq n_T^3 \frac{t^2}{2!} \leqq n_T \frac{(n_T T)^2}{2!} ,$$

$$\ldots$$

$$|S^i * R^i(S) * B^i(u, v)| = \left| \int_0^t \Psi(t - \tau) \, \Psi^{i-1}(\tau) \, d\tau \right| \leqq n_T \frac{(n_T T)^{i-1}}{(i - 1)!} ,$$

$\Psi^i(t) = (\Psi * \Psi^{i-1})(t)$ and hence

$$|T_{(u,v)}(S)| \leqq n_T \left( 1 + \frac{n_T T}{1!} + \frac{(n_T T)^2}{2!} + \ldots \right) = n_T e^{n_T T} < \infty$$

for every $(u, v) \in U \times V$.

After multiplying (5) by $R_{(u,v)}(S)$ we get a formula for the solution $z_\mu$ of the equation (1)

$$z_\mu(t) = R(S) * z_0 + R(S) * T_{(u,v)}(S) * z_0 + S * R(S) * f(u, v, w) + \tag{7}$$

$$+ T_{(u,v)}(S) * R(S) * f(u, v, w) + \mu R_{(u,v)}(S) * S * g(z, u, v, w) .$$

Let $L$ be a subspace of $R^m$ of a dimension $k \geqq 2$, which is a part of the orthogonal complement of $M \subset R^m$. Let $\pi : R^m \to R^k$ be the linear map corresponding to the orthogonal projection of $R^m$ onto $L$.

From the form of $T_{(u,v)}(S)$ it is obvious that $R(S) * T_{(u,v)}(S) = S * Q_{(u,v)}(S)$, where $Q_{(u,v)}(S) = -R^2(S) * B(u, v) + S * R^3(S) * B^2(u, v) - \dots$. Denote by $c_j(u, v)(S)$ the $j$-th column of the matrix $\hat{\pi} * Q_{(u,v)}(S)$.

## 3. ASSUMPTIONS AND THE MAIN RESULT

(1) $c_j(u, v)(S)$ does not depend on $v_{j+1}, \dots, v_m$, where $v = (v_1, v_2, \dots, v_m)$; we shall write $c_j(u, v_1, \dots, v_j)(S)$ instead of $c_j(u, v)(S)$. This condition is satisfied e.g. if $B(u, v) = \operatorname{diag}(b_1, b_2, \dots, b_m)$, where $b_j(u, v_1, v_2, \dots, v_j)$, $j = 1, 2, \dots, m$.

(2) (8$j$)
$$c_j(u, v_1, \dots, v_j)(S) = G_j(S) * (\Phi_0^j(u, v_1, v_2, \dots, v_j) + S * \Phi_1^j(u, v_1, v_2, \dots, v_j) + \dots) + \Phi_j(t), \quad 1 \leqq j \leqq m,$$

(9)
$$c_{m+1}(u, v, w) = \hat{\pi} * S * R(S) * f(u, v, w) + \hat{\pi} * T_{(u,v)}(S) * R(S) * f(u, v, w) = H(S) * (\Psi_0(u, v, w) + S * \Psi_1(u, v, w) + \dots) + \Psi(t),$$

where

(a) $\Psi_i(u, v, w)$, $i = 0, 1, 2, \dots$ are continuous functions on $U \times V \times W$; $\Phi_i^j(u, v_1, \dots, v_j)$, $j = 1, 2, \dots, m$; $i = 0, 1, 2, \dots$ are continuous functions on $U \times V_1 \times \dots \times V_j$;

(b) $|\Psi_i(u, v, w)|_1 \leqq \alpha_i$ for all $(u, v, w) \in U \times V \times W$, $|\Phi_i^j(u, v_1, \dots, v_j)|_1 \leqq \beta_{ij}$ for all $(u, v_1, v_2, \dots, v_j) \in U \times V_1 \times \dots \times V_j$, $i = 0, 1, 2, \dots$; $j = 1, 2, \dots, m$, where $\alpha_i$, $\beta_{ij}$ are constants, $|\cdot|_1$ is the euclidean norm in $R^k$ and the series
$$\hat{\alpha}_0 + S * \hat{\alpha}_1 + S^2 * \hat{\alpha}_2 + \dots,$$
$$\hat{\beta}_{i1} + S * \hat{\beta}_{i2} + S^2 * \hat{\beta}_{i3} + \dots, \quad i = 1, 2, \dots$$
are entire functions of the variable $t(\hat{\alpha}_i = D * \alpha_i, \hat{\beta}_{ij} = D * \beta_{ij})$;

(c) the matrices $H(S)$ and $G_j(S)$ are entire matrices over the Mikusinski ring and $\det^* H(S) \neq 0$, $\det^* G_j(S) \neq 0$ for all $j$ ($\det^*$ is calculated as a determinant in the usual formal way using the ring multiplication);

(d) the functions $\Psi(t)$, $\Phi_j(t)$, $j = 1, 2, \dots, m$ do not depend on $u, v, w$.

(e) Denote by $[\Psi_0(u, v, w)]$ the smallest linear subspace of $R^k$ containing all points $\Psi_0(u, v, w)$, $(u, v, w) \in U \times V \times W$. Let us suppose that the subspace $[\Psi_0(u, v, w)]$ has the largest possible dimension among all representations (9).

We shall say that the parameter $(v, w)$ in the expressions $(8j)$, $(9)$ has *complete maneuverability*, if

(A) for every $1 \leqq j \leqq m$, there exists a subspace $M_j$ of $R^k$ such that $c_j(u, v_1, \ldots, v_j)(S) \in M_j$ for all $(u, v_1, \ldots, v_j) \in U \times V_1 \times \ldots \times V_j$ and either $c_j(u, v_1, \ldots, v_j)(S) \equiv 0$ or the set

$$F_j = \bigcap_{(u,v_1,\ldots,v_{j-1}) \in U \times V_1 \times \ldots \times V_{j-1}} \operatorname{co}_{V_j} \Phi_0^j(u, v_1, \ldots, v_{j-1}, v_j) \tag{10}$$

contains interior points of the space $M_j$, where $\operatorname{co}_{V_j} \Phi_0^j(u, v_1, \ldots, v_{j-1}, v_j)$ denotes the convex hull of the set of all points $\Phi_0^j(u, v_1, \ldots, v_{j-1}, v_j)$, $v_j \in V_j$ for $(u, v_1, \ldots, v_{j-1}) \in U \times V_1 \times \ldots \times V_{j-1}$ fixed;

(B) neither $c_{m+1}(u, v, w)(S) \equiv 0$ nor does the set

$$F_{m+1} = \bigcap_{(u,v) \in U \times V} \operatorname{co}_W \Psi_0(u, v, w) \tag{11}$$

contain interior points of the space $L$, where $\operatorname{co}_W \Psi_0(u, v, w)$ denotes the convex hull of the set of all points $\Psi_0(u, v, w)$, $w \in W$ for $(u, v) \in U \times V$ fixed.

**Theorem 1.** *If the parameter $(v, w)$ in the expressions $(8j)$, $(9)$ has complete maneuverability, then there exists a number $\mu_1 > 0$ such that for all $\mu$, $|\mu| \leqq \mu_1$ there exists an evasion strategy. Moreover, there exist numbers $\lambda, \nu, \theta > 0$ and an integer $l$ such that for the distance $\varrho(z_\mu(t), M)$ of the point $z_\mu(t)$ from the subspace $M$ the following estimate is valid:*

$$\varrho(z_\mu(t), M) \geqq \tfrac{1}{2}\theta \left(\frac{\varrho(z_\mu(0), M)}{\lambda \nu}\right)^{l+1} \frac{1}{(1 + |z_\mu(t)|_1)^{l+1}} \tag{12}$$

*for $0 \leqq t < \infty$, where $\lambda, \nu, \theta > 0$, $l$ are constants depending neither on the course of the game nor on $z_\mu(0)$.*

Remark. The integer $l$ from Theorem 1 is equal to $\min(l_1, \ldots, l_k)$, where $H(S) = H_1(S) * \operatorname{diag}(S^{l_1}, \ldots, S^{l_k}) * H_2(S)$, $l_1 \leqq l_2 \leqq \ldots l_k$, $H_i(S)$, $i = 1, 2$ are entire invertible matrices. It was shown in [1] that an arbitrary entire matrix $H(S)$ has such a representation.

For the simplicity of computations we can assume without loss of generality that the origin of the space $L$ is an interior point of the set $F_{m+1}$. Denote by $Q$ the closed $k$-dimensional cube with the center at the origin, with sides parallel to the axes and such that $Q \subset \operatorname{int} \bigcap_{(u,v) \in U \times V} \operatorname{co}_W \Psi_0(u, v, w)$ (int $P$ denotes the interior of a set $P$ relative to $L$).

**Lemma 1** (see [1], § 5). *Let $\alpha(u, v) = S^n * K(S) * (\varphi_0(u, v) + S * \varphi_1(u, v) + \ldots)$, where $K(S) = K^{(1)}(S) * \Delta(l_1, \ldots, l_k) * K^{(2)}(S)$, $K^{(1)}(S)$, $K^{(2)}(S)$ are entire invertible* *matrices,* $l_1 \leqq l_2 \leqq \ldots \leqq l_k = 1$, $\Delta(l_1, \ldots, l_k) = \operatorname{diag}(S^{l_1}, S^{l_2}, \ldots, S^{l_k})$, $\varphi_i$ *are continuous in* $(u, v) \in U \times V$, $U$, $V$ *are compact sets. Let* $\operatorname{int} \bigcap_{u \in U} \operatorname{co}_v \varphi_0(u, v) \neq \emptyset$. *Then for a sufficiently small cube* $Q \subset \operatorname{int} \operatorname{co}_v \varphi_0(u, v)$ *with its center at the origin, there exists a number* $T$ *such that for any* $\varepsilon > 0$ *there exists a measurable function* $v(t) \in V$, $0 \leqq t \leqq T$ *such that*

$$\|\alpha(u(t), v(t)) + t^{n+1}\xi\| \leqq \varepsilon \tag{13}$$

*for* $0 \leqq t \leqq T$ *and for arbitrarily prescribed* $u(t) \in U$, $\xi \in Q$, *where the calculation of* $v(t)$ *requires the values* $u(\tau)$, $0 \leqq \tau \leqq t$ *and* $\xi$ *only* ($\|p(t)\| = \sup_{t \in [0,T]} |\int_0^t p(\tau)\, d\tau|_1$).

Sketch of the proof. Let $u \in U$ be fixed. Since $\operatorname{int} \bigcap_{u \in U} \operatorname{co}_v \varphi_0(u, v) \neq 0$, there exists a finite number of points $v_0, v_1, \ldots, v_r \in V$ such that $Q \subset \operatorname{co}\{\varphi_0(u, v_0), \varphi_0(u, v_1), \ldots, \varphi_0(u, v_r)\}$ for a sufficiently small cube $Q$. The fact that $\varphi_0(u, v)$ is continuous on the compact set $U \times V$ implies that for each $u \in U$ we can choose values $v_i(u)$, $i = 0, 1, \ldots, r$ ($r$ is independent of $u$) such that $v_i(u)$ is a measurable function.

Denote $p_i(u) = \varphi_0(u, v_i(u))$, $i = 0, 1, \ldots, r$. Then

$$Q \subset \operatorname{int} \bigcap_{u \in U} \operatorname{co}\{p_0(u), p_1(u), \ldots, p_r(u)\}\,. \tag{14}$$

To find a solution of the inequality (13) it suffices to find a solution of the inequality

$$\|\beta(u(t), v) + \omega_\xi(t)\| \leqq \varepsilon \tag{15}$$

where $\beta(u(t), v) = \varphi_0(u(t), v) + S * \varphi_1(u(t), v) + \ldots$, $\omega_\xi(t) = -K^{-1}(S) * S^1 * \xi$. Denote $\beta_\mu(t) = \sum_{i=0}^{r} [\varphi_0(u(t), v_i(u(t)))\, \mu_i + S * \varphi_1(u(t), v_i(u(t)))\, \mu_i + S^2 * \varphi_2(u(t), v_i(u(t)))\, \mu_i + \ldots]$, where $\mu_i \geqq 0$ are such numbers that $\sum_{i=0}^{r} \mu_i = 1$, $\mu = (\mu_0, \mu_1, \ldots, \mu_r)$. Using the condition (14) it is possible to prove by the method of successive approximations that for $\xi \in Q$ there exist measurable functions $\mu_i(t) \geqq 0$, $\sum_{i=0}^{r} \mu_i(t) \equiv 1$ for $t \in [0, T]$ such that

$$\beta_{\mu(t)}(t) + \omega_\xi(t) = 0\,, \quad t \in [0, T] = I\,. \tag{16}$$

By [2, Lemma 4.1] there exist disjoint subintervals $I_{\alpha_i}$, $i = 0, 1, \ldots, r$, $\sum_{i=0}^{r} I_{\alpha_i} = I$ such that if we define $v(t) = v_i(t) = v_i(u(t))$ for $t \in I_{\alpha_i}$, $i = 0, 1, \ldots, r$, then

$$\|\beta(u(t), v(t)) - \beta_{\mu(t)}(t)\| \leqq \varepsilon\,, \quad t \in [0, T]\,. \tag{17}$$

The equality (16) and the inequality (17) imply

$$\|\beta(u(t), v(t)) + \omega_\xi(t)\| \leqq \|\beta(u(t), v(t) - \beta_{\mu(t)}(t)\| + \|\beta_{\mu(t)}(t) + \omega_\xi(t)\| \leqq \varepsilon\,.$$

**Lemma 2.** *For a sufficiently small cube $Q$, there exists a number $T > 0$ such that for any $\varepsilon > 0$ there exists a measurable function $E(t) = (v_1(t), v_2(t), \ldots, v_m(t), w(t)) \in V_1 \times \ldots \times V_m \times W$, $0 \leqq t \leqq T$ such that*

$$(\mathrm{N}_j) \qquad \|c_j(u, v_1, v_2, \ldots, v_j)(S)\| \leqq \varepsilon$$

*for* $j = 1, 2, \ldots, m$,

$$(\mathrm{N}) \qquad \|c_{m+1}(u, v, w)(S) + t^1\xi\| \leqq \varepsilon$$

*for $0 \leqq t \leqq T$ and for arbitrarily prescribed $u(t) \in U$, $\xi \in Q$, where the calculation of $E(t)$ requires the values $u(\tau)$ on the interval $[0, t]$ and the point $\xi$ only.*

Proof. For $B(u, v) \equiv 0$ the lemma follows from Lemma 1. Let $B(u, v) \not\equiv 0$. By Lemma 1 there exists a measurable function $v_1(t) \in V_1$, $0 \leqq t \leqq T_1$, $T_1 > 0$ such that this function solves the inequality $(\mathrm{N}_1)$ on the interval $[0, T]$ for an arbitrarily prescribed $u(t) \in U$, where the calculation of $v_1(t)$ requires the values $u(\tau)$, $\tau \in [0, t]$ only. Denote $v_1(t) = r_1(t, u, \varepsilon)$. Now, by Lemma 1 we can find a measurable function $v_2(t) \in V_2$, $0 \leqq t \leqq T_2$, $T_2 > 0$, which solves the inequality $(\mathrm{N}_2)$ for an arbitrarily prescribed $(u(t), v_1(t)) \in U \times V_1$. Denote this solution by $v_2(t) = r_2(t, u, v_1, \varepsilon)$. Let $v_1(t) = r_1(t, u, \varepsilon)$. Then $v_2(t) = r_2(t, u, r_1(t, u, \varepsilon), \varepsilon)$ and this means that the calculation of $v_2(t)$ requires the values $u(\tau)$, $\tau \in [0, t]$ only. Proceeding step by step in this way we can find a measurable solution $v(t) = (v_1(t), v_2(t), \ldots, v_m(t)) \in V_1 \times V_2 \times \ldots \times V_m$ of the system of inequalities $(\mathrm{N}_j)$ on an interval $[0, T_3]$, $T_3 > 0$, such that the calculation of the functions $v_j(t)$, $j = 1, 2, \ldots, m$ requires the values $u(\tau)$, $\tau \in [0, t]$ only. Now, by Lemma 1, if $Q$ is sufficiently small, there exists a number $T > 0$ such that there exists a measurable function $w(t)$, $0 \leqq t \leqq T$ for which $(v_1(t), v_2(t), \ldots, v_m(t), w(t))$ is a solution of the inequality (N) and the calculation of $w(t)$ requires the values $u(\tau)$ $[0, t]$ and the point $\xi$ only. This proves Lemma 2.

Proof of Theorem 1. The formula (7) for the solution of the equation (1) and the formulas ($8j$), (9) imply $\hat{\pi} * z_\mu(t) = \hat{\pi} * R(S) * z_0 + \hat{\pi} * R(S) * T_{(u,v)}(S) * z_0 + \hat{\pi} * S * R(S) * f(u, v, w) + \hat{\pi} * T_{(u,v)}(S) * R(S) * f(u, v, w) + \mu\hat{\pi} * R_{(u,v)}(S) * S * g(z, u, w) = \hat{\pi} * R(S) * z_0 + S * c_1(u, v_1)(S) * z_1^0 + S * c_2(u, v_1, v_2)(S) * z_2^0 + \ldots + S * c_m(u, v_1, \ldots, v_m)(S) * z_m^0 + S * c_{m+1}(u, v, w)(S) + \mu\hat{\pi} * S * R(S) * R_{(u,v)}(S) * g(z, u, v, w)$, where $z_i^0$ are components of the vector $z_0$.

**Sublemma 1.** *Let $\mu_1 > 0$. If $\varrho(z_0, M) > 0$, then for $\mu$ such that $|\mu| \leqq \mu_1$ and for a sufficiently large number $\lambda$,*

$$(18) \qquad \varrho(z_\mu(t), M) \geqq \frac{\varrho(z_0, M)}{2} \quad \text{for} \quad 0 \leqq t \leqq \frac{\varrho(z_0, M)}{\lambda(1 + |z_0|_1)}.$$

The proof of this sublemma is the same as the proof of the inequality (5.4) in [1].

**Sublemma 2.** *For a sufficiently small number $T > 0$ there exists a number $v > 0$ such that for any $z_0$*

(19) $$v(1 + |z_\mu(t)|_1) \geqq 1 + |z_0|_1$$

*for $\mu \leqq \mu_1$.*

Proof. The formula (7) implies that for $0 \leqq t \leqq T$, $|z_\mu(t)|_1 \geqq |z_0|_1 - T\chi(1 + |z_0|_1)$ for $|\mu| \leqq \mu_1$ if $\chi$ is a sufficiently large constant. Then $1 + |z_\mu(t)|_1 \geqq (1 - T\chi)$ . . $(1 + |z_0|_1)$. If $T$ is so small that $1 - T\chi > 0$, then the inequality (14) is valid for $v = 1/(1 - T\chi)$.

**Sublemma 3** (cf. [1, Lemma 5.1]). *There exists $\theta > 0$ so small that that for an arbitrary vector $z_0$ there exists a point $\xi(z_0) \in Q$ such that*

(20) $$|\hat{\pi} * R(S) * z_0 - S * t^l * \xi(z_0)|_1 \geqq \theta t^{l+1}, \quad 0 \leqq t \leqq T.$$

Now we choose a number $\sigma > 0$ which satisfies the following inequalities:

(21) $$\sigma < \tfrac{1}{2}\theta T^{l+1}, \quad \sigma < \lambda T, \quad \frac{\sigma}{2} > \theta\left(\frac{\sigma}{\lambda}\right)^{l+1}$$

where $\lambda$ is chosen large enough to satisfy Sublemma 1.

Let us suppose that at the beginning of the game at the time $t = 0$ it is $\varrho(z_\mu(0), M) >$ $> \sigma$. Choose a control $(v(t), w(t))$ arbitrarily. If for some $t = t_1$, $\varrho(z_\mu(t_1), M) = \sigma$, then define a control $(v(t), w(t))$ on the interval $[t_1, t_1 + T]$ so that

(22) $$(v(t), w(t)) = E(t - t_1\ u, \xi(z_\mu(t_1)), \varepsilon) \in V \times W$$

where $E(t, u, \xi, \varepsilon)$ is a control satisfying the inequalities (8$j$), (9) for given $\varepsilon > 0$, $u(t) \in U$ and $\xi \in Q$.

**Sublemma 4.** *If $(v(t), w(t))$ is a control defined by the equality (22), then there exists a number $\mu_1 > 0$ such that for all $\mu$, $|\mu| \leqq \mu_1$*

(23) (a) $$\varrho(z_\mu(t), M) \geqq \tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda}\right)^{l+1} \frac{1}{(1 + |z_\mu(t)|_1)^{l+1}}, \quad t_1 \leqq t \leqq t_1 + T,$$

(b) $$\varrho(z_\mu(t_1 + T), M) \geqq \sigma.$$

Proof. The estimate (18) and the inequalities (21) imply that for

$$0 \leqq t - t_1 \leqq \frac{\varrho(z_\mu(t_1), M)}{\lambda(1 + |z_\mu(t_1)|_1)} = \frac{\sigma}{\lambda(1 + |z_\mu(t_1)|_1)},$$

$$\varrho(z_\mu(t), M) \geqq \frac{\sigma}{2} \geqq \theta\left(\frac{\sigma}{\lambda}\right)^{l+1} \geqq \theta\left(\frac{\sigma}{\lambda}\right)^{l+1} \frac{1}{(1 + |z_\mu(t_1)|_1)^{l+1}} - \varepsilon.$$

Now, suppose that

$$\frac{\sigma}{\lambda(1 + |z_\mu(t_1)|_1)} \leqq t - t_1 \leqq T.$$

The boundedness of $g$ implies that there exists a constant $c > 0$ such that for $0 \leqq \leqq t \leqq t_1 + T$, $|\mu| \leqq \mu_1$,

$$|S * R(S) * R_{(u,v)}(S) * g(z_\mu, u, v, w)|_1 \leqq c$$

for all $(u, v, w) \in U \times V \times W$. From this and from the Sublemmas 1–4 we obtain the following estimates:

$$\begin{aligned} \varrho(z_\mu(t), M) &= |\hat{\pi} * z_\mu(t)|_1 = |\hat{\pi} * R(S) * z_0 - S * (t - t_1)^l * \xi(z_\mu(t_1)) + \\ &+ S * (t - t_1)^1 * \xi(z_\mu(t_1)) + S * c_{m+1}(u, v, w)(S) + \\ &+ S * c_1(u, v_1)(S) * z_1^0 + \ldots + S * c_m(u\ v_1, \ldots, v_m)(S) * z_m^0 + \\ &+ \mu S * \hat{\pi} * R(S) * R_{(u,v)}(S) * g(z_\mu, u, v, w)|_1 \geqq \\ &\geqq |\hat{\pi} * R(S) * z_0 - S * (t - t_1)^l * \xi(z_\mu(t_1))|_1 - \\ &- \|(t - t_1)^l * \xi(z_\mu(t_1)) + c_{m+1}(u, v, w)(S)\| - \sum_{j=1}^{m} \|c_j(u, v_1, \ldots, v_j)\| \, |z_j^0|_1 - \\ &- \mu c \geqq \theta(t - t_1)^{l+1} - \varepsilon - \varepsilon|z_\mu(t_1)|_1 - \mu c. \end{aligned}$$

If $\mu$ and $\varepsilon$ are so small that

$$0 < \varepsilon(1 + |z_\mu(t_1)|_1) + \mu c < \min\left(\tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda}\right)^{l+1} \frac{1}{(1 + |z_\mu(t_1)|_1)^{l+1}}, \tfrac{1}{2}\theta T^{l+1}\right),$$

then

$$\begin{aligned} \varrho(z_\mu(t), M) &\geqq \theta\left(\frac{\sigma}{\lambda}\right)^{l+1} \frac{1}{(1 + |z_\mu(t_1)|_1)^{l+1}} - [\varepsilon(1 + |z_\mu(t_1)|_1) + \mu c] \geqq \\ &\geqq \tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda}\right)^{l+1} \frac{1}{(1 + |z_\mu(t_1)|_1)^{l+1}} \end{aligned}$$

for $t_1 \leqq t \leqq t_1 + T$ and $\varrho(z_\mu(t_1 + T), M) \geqq \frac{1}{2}\theta T^{l+1} > \sigma$. From this and from the inequality (19) we conclude

$$\text{(24)} \qquad \varrho(z_\mu(t), M) \geqq \tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda v}\right)^{l+1} \frac{1}{(1 + |z_\mu(t)|_1)^{l+1}} \quad \text{for} \quad t_1 \leqq t \leqq t_1 + T$$

and $\varrho(z_\mu(t_1 + T), M) \geqq \sigma$.

Since at the end of the evasion maneuver the point $z_\mu(t_1 + T)$ is outside the $\sigma$-neighborhood of the space $M$ and the number $T$ is fixed for the next menauvers, we can extend the game for an arbitrarily long time, where for all $t$ we get the estimates (23). This proves Theorem 1.

**Example.** Let the game be described by the system of equations

$$(25)\qquad \dot{x}(t) + a_1\, x(t) + \int_0^t v_1(t-\tau)\, x(\tau)\, d\tau = \quad w + \mu g_1(x, y, z, u, v, w)\,,$$

$$\dot{y}(t) + a_2\, y(t) + \int_0^t v_2(t-\tau)\, y(\tau)\, d\tau = -w + \mu g_2(x, y, z, u, v, w)\,,$$

$$\dot{z}(t) + a_3\, z(t) + \int_0^t v_3(t-\tau)\, z(\tau)\, d\tau = \quad w + \mu g_3(x, y, z, u, v, w)\,,$$

$\dim x = \dim y = \dim z = 1$, $u \in U$, $v \in V$, $w \in W$; $U, V, W$ are compact sets, $g_i : R^3 \times U \times V \times W \to R^1$, $i = 1, 2, 3$ are continuous and bounded, $\mu \in (-\infty, \infty)$ is a parameter, $a_i$, $i = 1, 2, 3$ are constants.

Let $M = \{Z = (x, y, z) \in R^3 \mid x - y = 0,\ z = 0\}$. Then the orthogonal complement of $M$ is the set $M^\perp = \{Z = (x, y, z) \in R^3 \mid x + y = 0\}$. Let $L = M^\perp$. Then the matrix of the orthogonal projection onto $L$ is

$$\pi = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{\pi} = \frac{1}{2}\begin{bmatrix} \delta & -\delta & 0 \\ -\delta & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix}.$$

if we denote

$$Z(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix},$$

then we get the system of the form (1), where

$$A = \operatorname{diag}(a_1, a_2, a_3)\,, \quad B(u, v) = \operatorname{diag}(v_1, v_2, v_3)\,, \quad f(u, v, w) =$$

$$= \begin{bmatrix} w \\ -w \\ w \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}.$$

$$R(S) = \hat{I} - S * \hat{A} + S^2 * \hat{A}^2 - \ldots = \operatorname{diag}(R_1(S), R_2(S), R_3(S))\,,$$

where

$$R_i(S) = \delta - S * \hat{a}_i + S^2 * \hat{a}_i^2 - \ldots, \quad i = 1, 2, 3\,.$$

Then

$$Q_{(u,v)}(S) = \operatorname{diag}(Q^{(1)}_{(u,v)}(S), Q^{(2)}_{(u,v)}(S), Q^{(3)}_{(u,v)}(S))\,,$$

where

$$Q^{(i)}_{(u,v)}(S) = -R_i^2(S) * v_i + S * R_i^3(S) * v_i^2 - \ldots\,.$$

$$\hat{\pi} * Q_{(u,v)}(S) = \frac{1}{2}\begin{bmatrix} \delta & -\delta & 0 \\ -\delta & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix} * \begin{bmatrix} Q^{(1)}_{(u,v)}(S) & 0 & 0 \\ 0 & Q^{(2)}_{(u,v)}(S) & 0 \\ 0 & 0 & Q^{(3)}_{(u,v)}(S) \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} Q^{(1)}_{(u,v)}(S) & -Q^{(2)}_{(u,v)}(S) & 0 \\ -Q^{(1)}_{(u,v)}(S) & Q^{(2)}_{(u,v)}(S) & 0 \\ 0 & 0 & Q^{(3)}_{(u,v)}(S) \end{bmatrix}$$

i.e.

$$c_1(u, v_1)(S) = \frac{1}{2}\begin{bmatrix} Q^{(1)}_{(u,v)}(S) \\ -Q^{(1)}_{(u,v)}(S) \\ 0 \end{bmatrix} \in M_1 = \{Z = (x, y, z) \in R^3 \mid x + y = 0,\ z = 0\},$$

$$c_2(u, v_1, v_2)(S) = \frac{1}{2}\begin{bmatrix} -Q^{(2)}_{(u,v)}(S) \\ Q^{(2)}_{(u,v)}(S) \\ 0 \end{bmatrix} \in M_2 = M_1,$$

$$c_3(u, v_1, v_2, v_3)(S) = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ Q^{(3)}_{(u,v)}(S) \end{bmatrix} \in M_3 = \{Z = (x, y, z) \in R^3 \mid x = 0,\ y = 0\}.$$

The functions $Q^{(i)}_{(u,v)}(S)$ have the form

$$Q^{(i)}_{(u,v)}(S) = v_i + S * \Psi^{(1)}_1(u, v_1, \ldots, v^i) + \ldots, \quad i = 1, 2, 3$$

and therefore

$$\Psi^1_0(u, v_1) = \begin{bmatrix} v_1 \\ -v_1 \\ 0 \end{bmatrix}, \quad \Psi^2_0(u, v_1, v_2) = \begin{bmatrix} -v_2 \\ v_2 \\ 0 \end{bmatrix},$$

$$\Psi^3_0(u, v_1, v_2, v_3) = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}. \ F_1 = \bigcap_{u \in U} \operatorname{co}_{V_1} \Psi^1_0(u, v_1) = \operatorname{co}_{V_1} \begin{bmatrix} v_1 \\ -v_1 \\ 0 \end{bmatrix},$$

$$F_2 = \bigcap_{(u,v_1) \in U \times V_1} \operatorname{co}_{V_2} \Psi^2_0(u, v_1, v_2) = \operatorname{co}_{V_2} \begin{bmatrix} -v_2 \\ v_2 \\ 0 \end{bmatrix},$$

$$F_3 = \bigcap_{(u,v_1,v_2) \in U \times V_1 \times V_2} \operatorname{co}_{V_3} \Psi^3_0(u, v_1, v_2, v_3) = \operatorname{co}_{V_3} \begin{bmatrix} -0 \\ 0 \\ v_3 \end{bmatrix}.$$

It is clear that if co $V_i \not\ni \emptyset$, $i \approx 1, 2, 3$, where co $V_i$ denote the convex hull of the set $V_i$, then all the sets $F_i$, $i = 1, 2, 3$ contain interior points of the set $M_i$, respectively. Now, we shall show that if int co $W \neq \emptyset$, then the set $F_4 = \bigcap_{(u,v)\in U\in V} \text{co}_W \Psi_0(u, v, w)$ contains interior points of the set $L$. Indeed,

$$\hat{\pi} * R(S) * R_{(u,v)}(S) * f(u, v, w) =$$

$$\frac{1}{2}\begin{bmatrix} \delta & -\delta & 0 \\ -\delta & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix} * [R(S) + S * R(S) * Q_{(u,v)}(S)] * f(u, v, w)$$

and hence

$$\Psi_0(u, v, w) = \frac{1}{2}\begin{bmatrix} \delta & -\delta & 0 \\ -\delta & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix} * \begin{bmatrix} w \\ -w \\ w \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2w \\ -2w \\ w \end{bmatrix}.$$

Therefore if co $W$ contains interior points, then

$$F_4 = \text{co}_W \begin{bmatrix} 2w \\ -2w \\ w \end{bmatrix}$$

contains interior points, too. All assumptions of Theorem 1 are fulfilled and therefore for sufficiently small $\mu$ there exists an evasion strategy.

### *References*


[1] *R. V. Gamkrelidze* and *G. L. Kharatishvili:* A Differential Game of Evasion with Nonlinear Control, SIAM Journal on Control, Vol. *12*, No. 2 (1974).

[2] *R. V. Gamkrelidze:* On some extremal problems in the theory of differential equations, SIAM Journal on Control, Vol. *3*, No. 1 (1965), 106—128.

[3] *Л. С. Понтрягин:* Линейная дифференциальная игра убегания, Труды Матем. института им. В. А. Стеклова АН СССР, Том *112* (1971).

[4] *Л. С. Понтрягин, Е. Ф. Мищенко:* Задача об убегании одного управляемого обьекта от другого, Доклады АН СССР (1969).

[5] *Н. Сатимов:* Задача убегания для одного класса нелинейных дифференциальных игр. Дифф. Урав., Том *11*, Но. 4 (1975).



*Author's address:* 886 25 Bratislava, ul. Obrancov mieru 49, ČSSR (Matematický ústav SAV).