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REGULAR POLYGONS

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In the year 1970, B. L. VAN DER WAERDEN [13] published the following theorem: *A pentagon in the space whose all sides are of the same length a and all angles of the same magnitude α, is planar.* It was my teacher K. HAVLÍČEK who called my attention to this paper [13] which had attracted the interest of the mathematical public both by the motivation which led the author to the proof of the theorem and by its contents itself. It was already in 1970 that the above quoted theorem was proved in another way by W. LÜSSY - E. TROST [9] and by H. IRMINGER [8]. Later, in 1972, new proofs of the theorem on planarity of the regular pentagon were given by J. D. DUNITZ - J. WASER [5], S. ŠMAKAL [11] and B. L. van der Warden [14]. The proof presented by B. L. van der Waerden in [14] is due to G. BOL and H. S. M. COXETER. Thanks to my correspondence with B. GRÜNBAUM I have learnt some other valuable facts concerning the development of this problem in a more general context. This concerns the references [2], [6] and [12].

A. AURIC [2] and G. VALIRON [12] in 1911 were probably the first ones to engage in the problem of existence of regular polygons in the Euclidean space E^3. Then the problem was left unnoticed for a longer period, only to be solved synthetically by A. P. GARBER V. I. GARVACKIJ and V. JA. JARMOLENKO [7] in 1961. These authors answer in their paper the question posed in a Soviet non-periodical journal Matematičeskoje prosvěščenije by V. I. ARNOLD in the following form: *For which n does there exist a spatial n-gon with all its sides of the same length a and all angles of the same magnitude α?* V. A. EFREMOVIČ and JU. S. ILJAŠENKO [6] in 1962 gave a classification of regular polygons in Euclidean spaces of general dimensions. In the paper they conclude that the immediate transfer of the definition of regularity from the planar case to spatial polygons, satisfying the requirements of V. I. Arnold, is rather far from the general understanding of regularity, and they start from the following definition: A *regular polygon* is a closed polygonal (piecewise linear) line that can be isometrically transformed into itself so that given side of the line is transformed into an arbitrarily chosen side. The paper [6] is based on a study of the group of all isometries which reproduce a closed polygonal line in the sense of the above definition. The classification itself consists in giving the projections of motions which transform a polygon onto itself. Let us add for the sake of completeness that O. BOTTEMA [3] in 1973 proved by means of Euler's (Cayley-Menger's) determinant that in the Euclidean space E^4 there exists for any $\alpha \in (\frac{1}{5}\pi, \frac{3}{5}\pi)$ an equilateral pentagon with all angles equal to α, whose vertices generate the space E^4.

The character of the problem implies that the notion of a regular polygon in a Euclidean space E^z $(z > 2)$ can be interpreted in different ways. The polygons studied in this paper could be called maximally regular, as they are the analogues of planar regular polygons in the strictest sense. Our approach to the problem solved as well as the subject itself essentially different from those used by other authors. Our method is based on the properties of the Gram determinant with a cyclic matrix and on quadratic forms – hence on rather elementary tools. The paper consists of two parts. Part 1 is devoted to regular polygons with an even number of sides, Part 2 deals with polygons with an odd number of sides. To distinguish these two cases proved to be suitable from the view point of the method chosen.

I

REGULAR POLYGONS WITH AN EVEN NUMBER OF SIDES IN EUCLIDEAN SPACES

1.1. Given a positive integer d, we shall study an n-gon $A_1A_2 \ldots A_n$ with $n = 2d + 2$ in the Euclidean space E^{2d+1}. We denote the vector $\overrightarrow{A_iA_{i+1}}$ in the n-gon by $\boldsymbol{a}_i$ ($i = 1, \ldots, 2d + 2$ cyclically) and assume that for all i and all k ($k = 1, \ldots, d + 1$)

$$\text{(1)} \qquad \boldsymbol{a}_i\boldsymbol{a}_i = w^2\,, \quad \boldsymbol{a}_i\boldsymbol{a}_{i+k} = \boldsymbol{a}_i\boldsymbol{a}_{i+2d+2-k} = w^2x_k\,.$$

An n-gon which satisfies (1) will be called a *regular n-gon* of *order* $d + 1$. Let us notice that every regular n-gon in the plane is in the above sense a regular n-gon of order $d + 1$. Without loss of generality we may always assume (with regard to homothetic transformations) that $w = 1$.

The closedness of the n-gon implies $\boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots + \boldsymbol{a}_{2d+2} = \boldsymbol{0}$. Scalar multiplication of this identity by a vector $\boldsymbol{a}_i$ yields in virtue of (1) for each i

$$\text{(2)} \qquad 1 + 2\sum_{m=1}^{d} x_m + x_{d+1} = 0\,.$$

In this section we shall derive some general relations which will be useful in the sequel.

Denote by $G_{r+1} = G(\boldsymbol{a}_i, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_{i+r})$ the Gram determinant of the vectors $\boldsymbol{a}_i, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_{i+r}$ ($r = 0, 1, \ldots, 2d + 1$). The Gram determinant G_{2d+2} has a cyclic matrix of the form

$$
\mathbf{G}_{2d+2} = \begin{bmatrix}
1, & x_1, & x_2, & \ldots, & x_{d-1}, & x_d, & x_{d+1}, & x_d, & x_{d-1}, & \ldots, & x_2, & x_1 \\
x_1, & 1, & x_1, & \ldots, & x_{d-2}, & x_{d-1}, & x_d, & x_{d+1}, & x_d, & \ldots, & x_3, & x_2 \\
x_2, & x_1, & 1, & \ldots, & x_{d-3}, & x_{d-2}, & x_{d-1}, & x_d, & x_{d+1}, & \ldots, & x_4, & x_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_d, & x_{d-1}, & x_{d-2}, & \ldots, & x_1, & 1, & x_1, & x_2, & x_3, & \ldots, & x_d, & x_{d+1} \\
x_{d+1}, & x_d, & x_{d-1}, & \ldots, & x_2, & x_1, & 1, & x_1, & x_2, & \ldots, & x_{d-1}, & x_d \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_2, & x_3, & x_4, & \ldots, & x_{d+1}, & x_d, & x_{d-1}, & x_{d-2}, & x_{d-3}, & \ldots, & 1, & x_1 \\
x_1, & x_2, & x_3, & \ldots, & x_d, & x_{d+1}, & x_d, & x_{d-1}, & x_{d-2}, & \ldots, & x_1, & 1
\end{bmatrix}.
$$

It is well known that the determinant of a cyclic matrix $\mathbf{C}$ of order n is a function of the elements of $\mathbf{C}$ which can be decomposed into n linear factors of the form $g(t) = c_1 + c_2 t + c_3 t^2 + \ldots + c_n t^{n-1}$ with the coefficients $c_1, c_2, \ldots, c_n$ forming in the given ordering the first row of the cyclic matrix $\mathbf{C}$. The n factors are obtained by substituting in $g(t)$ for the variable t successively all the n-th roots of one (cf. e. g. [4], pp. 116–117). In this way the Gram determinant G_{2d+2} assumes the form

$$
G_{2d+2} = \prod_{u=1}^{2d+2} g(\varepsilon_u), \quad \varepsilon_u = \cos\frac{\pi u}{d+1} + \mathrm{i}\sin\frac{\pi u}{d+1} \quad (u = 1, \ldots, 2d+2).
$$

The factors $g(\varepsilon_u)$ have in our case the following explicit expression:

$$
g(\varepsilon_u) = 1 + \sum_{m=1}^{d} x_m(\varepsilon_u^m + \varepsilon_u^{2d+2-m}) + (-1)^u x_{d+1} \quad (u = 1, \ldots, 2d+2).
$$

In virtue of (2) we have $g(\varepsilon_{2d+2}) = 0$. Further, it holds $\varepsilon_u^{2d+2} = 1$ and hence we obtain successively

$$
g(\varepsilon_{2d+2-k}) = 1 + \sum_{m=1}^{d} x_m[\varepsilon_1^{(2d+2-k)m} + \varepsilon_1^{(2d+2-k)(2d+2-m)}] +
$$
$$
+ (-1)^{2d+2-k} x_{d+1} = 1 + \sum_{m=1}^{d} x_m(\varepsilon_1^{-km} + \varepsilon_1^{km}) + (-1)^k x_{d+1} = g(\varepsilon_k).
$$

This yields an important identity

$$
g(\varepsilon_k) = g(\varepsilon_{2d+2-k}) \quad (k = 1, \ldots, d+1).
$$

For any k and m the units ε_k^m, ε_k^{-m} are complex conjugate. Thus each of the factors $g(\varepsilon_u)$ is real.

Let us further consider the determinant J of the matrix

$$
\mathbf{J} = \begin{bmatrix}
1, & 1, & 1, & \ldots, & 1 \\
\varepsilon_1, & \varepsilon_2, & \varepsilon_3, & \ldots, & \varepsilon_n \\
\varepsilon_1^2, & \varepsilon_2^2, & \varepsilon_3^2, & \ldots, & \varepsilon_n^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\varepsilon_1^{n-1}, & \varepsilon_2^{n-1}, & \varepsilon_3^{n-1}, & \ldots, & \varepsilon_n^{n-1}
\end{bmatrix}.
$$

When evaluating the determinant J we add $2d + 1$ last columns to the first one in the transposed matrix $\boldsymbol{J}^T$. Thus we obtain zeros in the first $2d + 1$ places and the number $2d + 2$ in the last place. Developing the determinant by the first column yields the identity $J = -(2d + 2) J_1$ where J_1 is a determinant of the form

$$J_1 = \begin{vmatrix} \varepsilon_1, & \varepsilon_1^2, & \dots, & \varepsilon_1^d, & -1, & \varepsilon_1^{d+2}, & \dots, & \varepsilon_1^{2d+1} \\ \varepsilon_2, & \varepsilon_2^2, & \dots, & \varepsilon_2^d, & 1, & \varepsilon_2^{d+2}, & \dots, & \varepsilon_2^{2d+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varepsilon_d, & \varepsilon_d^2, & \dots, & \varepsilon_d^d, & (-1)^d, & \varepsilon_d^{d+2}, & \dots, & \varepsilon_d^{2d+1} \\ -1, & 1, & \dots, & (-1)^d, & (-1)^{d+1}, & (-1)^{d+2}, & \dots, & -1 \\ \varepsilon_{d+2}, & \varepsilon_{d+2}^2, & \dots, & \varepsilon_{d+2}^d, & (-1)^{d+2}, & \varepsilon_{d+2}^{d+2}, & \dots, & \varepsilon_{d+2}^{2d+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varepsilon_{2d+1}, & \varepsilon_{2d+1}^2, & \dots, & \varepsilon_{2d+1}^d, & -1, & \varepsilon_{2d+1}^{d+2}, & \dots, & \varepsilon_{2d+1}^{2d+1} \end{vmatrix}.$$

The ones and minus ones in the $(d + 1)$-st row and column alternate regularly which is a consequence of the identity $\varepsilon_j^{d+1} = \varepsilon_{d+1}^j = (-1)^j$ $(j = 1, \dots, 2d + 1)$.

With the matrix of the determinant J_1 we proceed in the following way:

a) we subtract each m-th row $(m = 1, \dots, d)$ from the $(2d + 2 - m)$-th row,

b) we add each $(2d + 2 - m)$-th column to the m-th column.

After these changes, using moreover some well known properties of complex unities, we can write the determinant J_1 schematically in the form

$$J_1 = \left|\begin{array}{c|c|c} J_2 & \begin{matrix} -1 \\ 1 \\ \vdots \\ (-1)^d \end{matrix} & B \\ \hline -2,\ 2,\ -2,\ \dots,\ 2(-1)^d & (-1)^{d+1} & (-1)^{d+2},\ \dots,\ 1,\ -1 \\ \hline Z & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & U \end{array}\right|.$$

Here the determinant J_2 is of the form

$$J_2 = \begin{vmatrix} \varepsilon_1 + \varepsilon_1^{2d+1}, & \varepsilon_1^2 + \varepsilon_1^{2d}, & \dots, & \varepsilon_1^d + \varepsilon_1^{d+2} \\ \varepsilon_2 + \varepsilon_2^{2d+1}, & \varepsilon_2^2 + \varepsilon_2^{2d}, & \dots, & \varepsilon_2^d + \varepsilon_2^{d+2} \\ \varepsilon_3 + \varepsilon_3^{2d+1}, & \varepsilon_3^2 + \varepsilon_3^{2d}, & \dots, & \varepsilon_3^d + \varepsilon_3^{d+2} \\ \dots & \dots & \dots & \dots \\ \varepsilon_d + \varepsilon_d^{2d+1}, & \varepsilon_d^2 + \varepsilon_d^{2d}, & \dots & \varepsilon_d^d + \varepsilon_d^{d+2} \end{vmatrix}.$$

The determinant Z has a zero matrix, the determinants B and U are irrelevant from the view point of our problem. The identity $J = -(2d + 2) J_1$ together with the scheme for J_1 imply immediately $J = -(2d + 2) D_0 U$ where D_0 has the obvious meaning. Since $\varepsilon_i \neq \varepsilon_u$ for $i \neq u$, the Vandermond determinant J is different from zero and so is the determinant D_0. The complex unities ε_j $(j = 1, \ldots, 2d + 1)$ are solutions of the equation

$$\sum_{r=0}^{2d+1} y^r = 0 .$$

Thus the determinant J_2 has a symmetric matrix. If $d + 1 = 2p$ (p positive integer), then each odd row in the matrix of the determinant J_2 is antisymmetric while each even row is symmetric with respect to the p-th column. If $d + 1 = 2p + 1$, then each odd row in the matrix of the determinant J_2 is antisymmetric while each even row is symmetric with respect to an axis passing between the p-th and the $(p + 1)$-st columns. It follows from the symmetry of the determinant J_2 that the same holds for its columns. From these facts and from the equation

$$\sum_{r=0}^{2d+1} y^r = 0$$

we conclude finally that the sum of elements in each odd row of the matrix of the determinant J_2 is equal to zero while the sum of elements in each even row is equal to -2.

Let us now investigate the determinant D_0 in more detail. We have evidently

$$D_0 = 2 \left| \begin{array}{c|c} & -1 \\ J_2 & 1 \\ & -1 \\ & \vdots \\ \hline -1,\ 1,\ \ldots,\ (-1)^d & \frac{1}{2}\,(-1)^{d+1} \end{array} \right|$$

where the right hand side determinant is symmetric. Adding to the k-th row $(k = 1, \ldots, d + 1)$ of this determinant all the other rows, we conclude from the above mentioned facts about the determinant J_2 that in the k-th row of the determinant we have -1 in the first d places and $-\frac{1}{2}$ in the last place. Multiplying the k-th row by -2 and, unless $k = d + 1$, multiplying the last row by 2, we obtain the following identities in which the resulting determinant of the above operation is denoted by D_k:

$$(3) \qquad \begin{aligned} D_m &= -2D_0 \quad (m = 1, \ldots, d), \\ D_{d+1} &= -D_0 . \end{aligned}$$

Since $D_0 \neq 0$, it follows from (3) that also the determinants D_m and D_{d+1} are non-zero. This fact will be used in the next section.

1.2. In this section we find necessary conditions for the existence of a regular n-gon of order $d + 1$.

The matrix $\mathbf{G}_{2d+2}$ determines uniquely a quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$. The discriminant G_{2d+2} of this quadratic form equals zero. All the principal minors of the discriminant G_{2d+2} which are needed in order to determine the kind of the quadratic form are the Gram determinants G_{r+1} $(r = 0, 1, \ldots, 2d)$. Therefore, these minors are nonnegative. Consequently, the quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$ is positive semidefinite.

Let us first assume that in the Euclidean space E^{2d+1} there exists a regular n-gon of order $d + 1$ whose vertices $A_1, A_2, \ldots, A_n$ generate the space E^{2d+1}. In this case the quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$ has the rank $h = 2d + 1$.

Throughout the paper, let $\mathbf{E}$ denote the unit matrix of the relevant order. It is well known that there exists an orthogonal transformation which transforms the quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$ into the canonical form

$$\sum_{i=1}^{2d+2} \lambda_i y_i^2$$

where λ_i are the roots of the characteristic equation $|\mathbf{G}_{2d+2} - \lambda \mathbf{E}| = 0$. We shall show that the linear factors $g(\varepsilon_u)$ are roots of this characteristic equation. Moreover, the factors $g(\varepsilon_{d+1})$ and $g(\varepsilon_{2d+2})$ are simple roots while the factors $g(\varepsilon_m)$ are double roots of the equation $(m = 1, \ldots, d)$. In the proof we shall use the matrix $\mathbf{J}$ introduced in Sec. 1.1. Evaluating the matrix product $[\mathbf{G}_{2d+2} - g(\varepsilon_m)\,\mathbf{E}]\,\mathbf{J}$, we obtain after some modifications

$$[\mathbf{G}_{2d+2} - g(\varepsilon_m)\,\mathbf{E}]\,\mathbf{J} =$$

$$= \begin{Vmatrix} [g(\varepsilon_1) - g(\varepsilon_m)], & \ldots, & 0, & \ldots, & 0, & \ldots, & -g(\varepsilon_m) \\ \varepsilon_1[g(\varepsilon_1) - g(\varepsilon_m)], & \ldots, & \varepsilon_m \,.\, 0, & \ldots, & \varepsilon_{2d+2-m} \,.\, 0, & \ldots, & -g(\varepsilon_m) \\ \varepsilon_1^2[g(\varepsilon_1) - g(\varepsilon_m)], & \ldots, & \varepsilon_m^2 \,.\, 0, & \ldots, & \varepsilon_{2d+2-m}^2 \,.\, 0, & \ldots, & -g(\varepsilon_m) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon_1^{2d+1}[g(\varepsilon_1) - g(\varepsilon_m)], & \ldots, & \varepsilon_m^{2d+1} \,.\, 0, & \ldots, & \varepsilon_{2d+2-m}^{2d+1} \,.\, 0, & \ldots, & -g(\varepsilon_m) \end{Vmatrix}.$$

The zeros in the m-th column of the resulting matrix are caused by the factor $[g(\varepsilon_m) - g(\varepsilon_m)]$ which appears in each element of the m-th column. Similarly, a factor $[g(\varepsilon_{2d+2-m}) - g(\varepsilon_m)]$ which is again zero appears in each element of the $(2d + 2 - m)$-th column. The resulting matrix includes therefore two zero columns and consequently, its rank is $h \leqq 2d$. Since multiplication by a regular matrix cannot influence the rank of a matrix, we conclude that the matrix $\mathbf{G}_{2d+2} - g(\varepsilon_m)\,\mathbf{E}$ has for all m $(m = 1, \ldots, d)$ a rank $h \leqq 2d$. Consequently, every factor $g(\varepsilon_m)$ is at least a double root of the characteristic equation $|\mathbf{G}_{2d+2} - \lambda \mathbf{E}| = 0$. It can be shown quite analogously that the factors $g(\varepsilon_{d+1})$ and $g(\varepsilon_{2d+2})$ are at least simple roots of the same characteristic equation. However, the characteristic equation has precisely $2d + 2$ roots and thus we can replace the words "at least" by the word "precisely". The closedness condition (2) implies that $g(\varepsilon_{2d+2}) = 0$. Moreover, we know already that the quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$ is positive semidefinite and its rank is $h = 2d + 1$. Therefore it follows from the Law of Inertia for quadratic forms that all the other roots of the characteristic equation $|\mathbf{G}_{2d+2} - \lambda \mathbf{E}| = 0$ are positive. Hence we have the inequalities

$$(4) \qquad g(\varepsilon_k) > 0 \quad (k = 1, \ldots, d + 1).$$

Let us now consider the numbers x_k $(k = 1, \ldots, d + 1)$ in the given ordering to be the coordinates of a point X in the space E^{d+1} and let us consider the following problem: Which are the points of the space E^{d+1} whose coordinates satisfy the inequalities (4) and the closedness condition (2)? To obtain a deeper geometrical view of the matter, let us set $g(\varepsilon_k) = 0$ $(k = 1, \ldots, d + 1)$ and denote the system of these equations by (S_0). Each equation of the system represents the equation of a hyperplane in the space E^{d+1}. The system (S_0) has the explicit form

(S_0)

$$(\varepsilon_1 + \varepsilon_1^{2d+1})\, x_1 + (\varepsilon_1^2 + \varepsilon_1^{2d})\, x_2 + \ldots + (\varepsilon_1^d + \varepsilon_1^{d+2})\, x_d + \varepsilon_1^{d+1} x_{d+1} = -1,$$
$$(\varepsilon_2 + \varepsilon_2^{2d+1})\, x_1 + (\varepsilon_2^2 + \varepsilon_2^{2d})\, x_2 + \ldots + (\varepsilon_2^d + \varepsilon_2^{d+2})\, x_d + \varepsilon_2^{d+1} x_{d+1} = -1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(\varepsilon_{d+1} + \varepsilon_{d+1}^{2d+1})\, x_1 + (\varepsilon_{d+1}^2 + \varepsilon_{d+1}^{2d})\, x_2 + \ldots + (\varepsilon_{d+1}^d + \varepsilon_{d+1}^{d+2})\, x_d + \varepsilon_{d+1}^{d+1} x_{d+1} = -1.$$

The determinant of the system (S_0) is the determinant D_0 which was introduced in Sec. 1.1. We found there that it is nonzero. The hyperplanes determined by the system (S_0) have a common point R_0 whose all coordinates are equal to one. This result is obtained immediately by substituting into the system (S_0) with regard to the fact that the unities ε_k $(k = 1, \ldots, d + 1)$ are solutions of the equation

$$\sum_{r=0}^{2d+1} y^r = 0.$$

The closedness condition (2) can be written briefly in the form $g(\varepsilon_{2d+2}) = 0$ and this equation may be considered again an equation of a hyperplane in the space E^{d+1}. If we replace successively the k-th equation $(k = 1, \ldots, d + 1)$ in the system (S_0) by the equation $g(\varepsilon_{2d+2}) = 0$ we obtain together $d + 1$ systems (S_1), (S_2), $\ldots$, (S_{d+1}) of linear equations. Here the index indicates which equation of the original system (S_0) was replaced by the equation $g(\varepsilon_{2d+2}) = 0$. The determinants of these systems are the determinants $D_1, D_2, \ldots, D_{d+1}$ introduced in (3). We know that all of them are nonzero. Each of the systems (S_k) has therefore precisely one solution which represents in the geometrical interpretation the common point of the hyperplanes determined by the given system (S_k). The hyperplanes corresponding to the system (S_0) together with the hyperplane $g(\varepsilon_{2d+2}) = 0$ determine a simplex $S = \langle R_0, R_1, \ldots$ $\ldots, R_{d+1}\rangle$ in the space E^{d+1}.

It is useful to introduce at this moment some new notions: Let the points $R_0, R_1, \ldots$ $\ldots, R_z$ generate the space E^z.

a) The simplex $S = \langle R_0, R_1, \ldots, R_z \rangle$ in the space E^z is the convex set of points

$$\sum_{j=0}^{z} \omega_j R_j \quad \text{with} \quad 0 \leqq \omega_j \leqq 1 \quad \text{and} \quad \sum_{j=0}^{z} \omega_j = 1 \, .$$

b) The face ϱ_k $(k = 0, 1, \ldots, z)$ of the simplex $S = \langle R_0, R_1, \ldots, R_z \rangle$ is the convex set of points

$$\sum_{j=0}^{z} \omega_j R_j \quad \text{with} \quad \omega_k = 0 \, , \quad 0 \leqq \omega_j \leqq 1 \quad \text{for} \quad j \neq k \quad \text{and} \quad \sum_{j=0}^{z} \omega_j = 1 \, .$$

c) The open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_z \rangle$ is the convex set of points

$$\sum_{j=1}^{z} \omega_j R_j \quad \text{where} \quad 0 < \omega_j < 1 \quad \text{and} \quad \sum_{j=1}^{z} \omega_j = 1 \, .$$

d) A boundary point of the face ϱ_0 is any point of the face ϱ_0 which does not belong to the open face ϱ_0.

Let us consider once more the simplex $S = \langle R_0, R_1, \ldots, R_{d+1} \rangle$. The coordinates of the vertex R_0 are known. By solving the systems (S_k) of linear equations we obtain the coordinates of the other vertices R_k $(k = 1, \ldots, d + 1)$. Thus we have

$$R_k = \left[\frac{\varepsilon_k + \varepsilon_k^{2d+1}}{2}, \frac{\varepsilon_k^2 + \varepsilon_k^{2d}}{2}, \frac{\varepsilon_k^3 + \varepsilon_k^{2d-1}}{2}, \ldots, \frac{\varepsilon_k^{d+1} + \varepsilon_k^{d+1}}{2} \right]$$
$$(k = 1, \ldots, d + 1) \, .$$

Passing to the goniometric representation and adding the point R_0 we have

$$(5) \qquad R_0 = [1, 1, \ldots, 1] \, ,$$

$$R_k = \left[\cos \frac{k\pi}{d+1}, \cos 2 \frac{k\pi}{d+1}, \cos 3 \frac{k\pi}{d+1}, \ldots, \cos d \frac{k\pi}{d+1}, (-1)^k \right]$$
$$(k = 1, \ldots, d + 1) \, .$$

As mentioned above, our problem would be solved by satisfying simultaneously the closedness condition (2) and the inequalities (4). In the geometrical interpretation: If there exists in the space E^{2d+1} such an n-gon of order $d + 1$ with $n = 2d + 2$ whose vertices $A_1, A_2, \ldots, A_n$ generate the space E^{2d+1}, then the point $X = [x_1, x_2, \ldots, x_{d+1}]$ whose coordinates have the same meaning as in (1) must belong to the intersection of the hyperplane $g(\varepsilon_{2d+2}) = 0$ with that part of the space E^{d+1} in which the inequalities (4) hold. We shall show that this intersection coincides with the open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_{d+1} \rangle$.

Let us denote by $\mathbf{e}$ the vector in the space E^{d+1} with all its coordinates equal to -1. The straight line $X = P + \mathbf{e}t$ passing through the origin of the coordinate system is incident with the vertex R_0 of the simplex $S = \langle R_0, R_1, \ldots, R_{d+1}\rangle$. This line intersects the hyperplane $g(\varepsilon_{2d+2}) = 0$ for $t = 1/(2d+1)$. The point of intersection

$$M = \left[-\frac{1}{2d+1}, -\frac{1}{2d+1}, \ldots, -\frac{1}{2d+1}\right]$$

lies on the halfline opposite to the halfline PR_0 since the vertex R_0 corresponds to the value of the parameter $t = -1$. It is easy to verify that the point M is a point of the open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_{d+1}\rangle$ since

$$M = \frac{2}{2d+1}\sum_{m=1}^{d} R_m + \frac{1}{2d+1} R_{d+1}\,. \tag{6}$$

Hence each point of the segment MR_0 belongs to the simplex $S = \langle R_0, R_1, \ldots, R_{d+1}\rangle$. The origin P which is an interior point of the segment MR_0, is an interior point of the simplex. Substituting the coordinates of the origin P into all linear factors $g(\varepsilon_k)$ we find $g(\varepsilon_k) = 1$ $(k = 1, \ldots, d+1)$. Hence the inequalities (4) are really satisfied in the intersection K of open halfspaces determined by the hyperplanes whose equations are $g(\varepsilon_k) = 0$, and the simplex $S = \langle R_0, R_1, \ldots, R_{d+1}\rangle$ is a part of this intersection. On the other hand, the intersection K is at the same time the only part of the space E^{d+1} in which all inequalities (4) hold simultaneously. Indeed, passing from K to any other part of the space E^{d+1}, at least one of the inequalities (4) is violated.

We shall now illustrate the general method explained above by two relatively simple examples.

1. Let us consider a regular tetragon of order 2 in E^3. The decomposition of the Gram determinant G_4 into linear factors is obtained by substituting successively all fourth roots of unity into the relation $g(t) = 1 + x_1 t + x_2 t^2 + x_1 t^3$:

$$\begin{aligned}
g(\varepsilon_1) &= g(\mathrm{i}) = 1 - x_2\,,\\
g(\varepsilon_2) &= g(-1) = 1 - 2x_1 + x_2\,,\\
g(\varepsilon_3) &= g(-\mathrm{i}) = 1 - x_2\,,\\
g(\varepsilon_4) &= g(1) = 1 + 2x_1 + x_2\,.
\end{aligned}$$

The vertex $R_0 = [1, 1]$ is the solution of the system (S_0):

$$\begin{aligned}
-x_2 &= -1\,,\\
-2x_1 + x_2 &= -1\,.
\end{aligned}$$

The vertices $R_1 = [0, -1]$, $R_2 = [-1, 1]$ result from solving the systems (S_1), (S_2) which arise from the system (S_0) by replacing first the former and then the latter equation by the closedness condition $g(\varepsilon_4) = 0$. The equations $g(\varepsilon_u) = 0$ $(u = 1, \ldots, 4)$ yield in this case straight lines in a plane, the face ϱ_0 is the segment R_1R_2, the open face ϱ_0 is the set of all interior points of this segment. We shall show later that each interior point of the segment R_1R_2 corresponds to a regular spatial tetragon of order 2, the point R_1 corresponds to a square and the point R_2 corresponds to a degenerate regular tetragon on the line which arises from "running" through the unit segment twice in one as well as in the opposite direction (see Fig. 1).

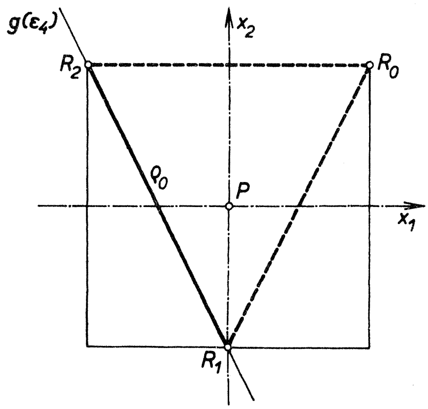

Fig. 1.

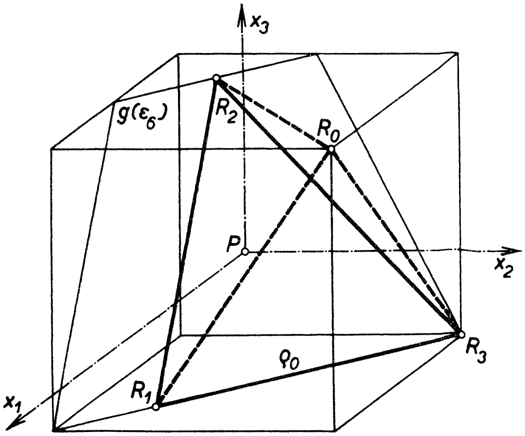

Fig. 2.

2. Consider a regular hexagon of order 3 in the space E^5. By decomposing the Gram determinant G_6 we obtain the linear factors $g(\varepsilon_u)$ $(u = 1, \ldots, 6)$ which constitute the system (S_0). This system yields the vertex $R_0 = [1, 1, 1]$ of the simplex $S = \langle R_0, R_1, R_2, R_3 \rangle$:

$$\begin{aligned} x_1 - x_2 - x_3 &= -1\,, \\ -x_1 - x_2 + x_3 &= -1\,, \\ -2x_1 + 2x_2 - x_3 &= -1\,. \end{aligned}$$

The systems $(S_1), (S_2), (S_3)$ result from replacing successively the k-th equation $(k = 1, 2, 3)$ by the equation

$$2x_1 + 2x_2 + x_3 = -1\,.$$

Solving these systems we obtain the vertices R_k $(k = 1, 2, 3)$:

$$R_1 = [\tfrac{1}{2}, -\tfrac{1}{2}, -1]\,, \quad R_2 = [-\tfrac{1}{2}, -\tfrac{1}{2}, 1]\,, \quad R_3 = [-1, 1, -1]\,.$$

The simplex $S = \langle R_0, R_1, R_2, R_3 \rangle$ lies in a space E^3 (see Fig. 2). The face ϱ_0 is the triangle $R_1R_2R_3$, the open face ϱ_0 is the set of all its interior points. It is worth noticing that all vertices of the simplex lie on the surface of a cube with the length of edge equal to 2 (for the tetragon from Example 1, all vertices lie on the perimeter of a square with the length of side equal to 2). The point of intersection of the body diagonals of this cube coincide with the origin P of the coordinate system, the faces of the cube pass through the unit points on the axes parallelly to the coordinate planes.

The fact that the vertices of the simplex lie on a perimeter of a square for the tetragon on the surface of a cube for the hexagon is not incidental but follows from (5).

We conclude the investigation of necessary conditions for the existence of a regular n-gon of order $d + 1$, whose vertices generate the space E^{2d+1}, by the following

Theorem 1. *Let the coordinates of a point* $X = [x_1, x_2, \ldots, x_{d+1}]$ *have the same meaning as in* (1). *If there exists for* $n = 2d + 2$ *a regular* n*-gon of order* $d + 1$ *whose vertices generate the space* E^{2d+1}, *then the point* X *lies in the open face* ϱ_0 *of the simplex* $S = \langle R_0, R_1, \ldots, R_{d+1} \rangle$ *whose vertices are given by* (5).

Let us now assume that there exists for $n = 2d + 2$ a regular n-gon of order $d + 1$ in E^{2d+1} whose vertices generate a subspace of dimension h where $h < 2d + 1$. Then a nonzero Gram determinant of order h can be selected from the matrix $\mathbf{G}_{2d+2}$ and the quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$ has the rank h. However, the rank of the quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$ is determined by the number of nonzero roots $\lambda_k = g(\varepsilon_k)$ ($k = 1, \ldots, d + 1$) of the corresponding characteristic equation $|\mathbf{G}_{2d+2} - \lambda \mathbf{E}| = 0$. One zero root is obtained from the closedness condition (2). If $h = 2d + 1 - 2v$, then also the double roots $\lambda_{\tau_1} = g(\varepsilon_{\tau_1}), \ldots, \lambda_{\tau_v} = g(\varepsilon_{\tau_v})$ are equal to zero, where $\{\tau_1, \ldots, \tau_v\}$ is an arbitrary nonvoid subset of the set $\{1, \ldots, d\}$. All other roots satisfy the inequalities (4). The point $X = [x_1, \ldots, x_{d+1}]$ whose coordinates satisfy all the above requirements belongs to the intersection of the faces $\varrho_0, \varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ and hence is a boundary point of the face ϱ_0. A similar situation occurs, from the view point of the result, if $h = 2d - 2v$. In this case the simple root $\lambda_{d+1} = g(\varepsilon_{d+1})$ as well as the double roots $\lambda_{\tau_1} = g(\varepsilon_{\tau_1}), \ldots, \lambda_{\tau_v} = g(\varepsilon_{\tau_v})$ are necessarily equal to zero, $\{\tau_1, \ldots, \tau_v\}$ being an arbitrary proper subset of the set $\{1, \ldots, d\}$. The other roots are positive according to (4). Now the point X belongs to the intersection of the faces $\varrho_0, \varrho_{d+1}, \varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ and is again a boundary point of the face ϱ_0.

Theorem 2. *Let the coordinates of a point* $X = [x_1, x_2, \ldots, x_{d+1}]$ *have the same meaning as in* (1). *If there exists for* $n = 2d + 2$ *a regular* n*-gon of order* $d + 1$ *in the space* E^{2d+1} *whose vertices generate a subspace of a dimension* $h < 2d + 1$, *then the point* X *is a boundary point of the face* ϱ_0 *of the simplex* $S = \langle R_0, R_1, \ldots, R_{d+1} \rangle$ *whose vertices are given by* (5).

1.3. In this section we shall show that to every point $X = [x_1, x_2, \ldots, x_{d+1}]$ lying on the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_{d+1} \rangle$ in the space E^{d+1}, there corresponds a regular n-gon of order $d + 1$. A way how to construct this n-gon will also be suggested.

Let us choose an arbitrary point $X = [x_1, x_2, \dots, x_{d+1}]$ which belongs to the face ϱ_0 of the simplex $S = \langle R_0, R_1, \dots, R_{d+1}\rangle$. The point X can be easily associated with a quadratic form whose matrix $\mathbf{G}^*_{2d+2}$ is formally coincident with the matrix $\mathbf{G}_{2d+2}$. The orthogonal transformation $\mathbf{D} = \mathbf{A}^T\mathbf{G}^*_{2d+2}\mathbf{A}$ brings the quadratic form $G^*_{2d+2}(\mathbf{y}, \mathbf{y})$ to the canonical form

$$D(\mathbf{y}, \mathbf{y}) = \sum_{u=1}^{2d+2} g(\varepsilon_u) y_u^2 .$$

We have proved in Sec. 1.2 that each point $X = [x_1, x_2, \dots, x_{d+1}]$ which belongs to the face ϱ_0 satisfies $g(\varepsilon_k) \geqq 0$ $(k = 1, \dots, d + 1)$. Hence the quadratic form $D(\mathbf{y}, \mathbf{y})$ is positive semidefinite and the Law of Inertia of quadratic forms implies that the same holds for the quadratic form $G^*_{2d+2}(\mathbf{y}, \mathbf{y})$. The relation $\mathbf{D} = \mathbf{A}^T\mathbf{G}^*_{2d+2}\mathbf{A}$ implies further that $\mathbf{G}^*_{2d+2} = \mathbf{A}\mathbf{D}\mathbf{A}^T$.

Let us denote by $\mathbf{W}$ the matrix which results from the matrix $\mathbf{D}$ by replacing each element $g(\varepsilon_k)$ by the element $[g(\varepsilon_k)]^{1/2}$ $(k = 1, \dots, d + 1)$ and the element $g(\varepsilon_{2d+2})$ by $[g(\varepsilon_{2d+2})]^{1/2}$. Obviously $\mathbf{D} = \mathbf{W}^2$. The orthogonal transformation $\mathbf{Q} = \mathbf{A}\mathbf{W}\mathbf{A}^T$ transforms the matrix $\mathbf{W}$ into the matrix $\mathbf{Q}$ and

$$\mathbf{Q}^2 = \mathbf{A}\mathbf{W}\mathbf{A}^T\mathbf{A}\mathbf{W}\mathbf{A}^T = \mathbf{A}\mathbf{W}^2\mathbf{A}^T = \mathbf{A}\mathbf{D}\mathbf{A}^T = \mathbf{G}^*_{2d+2} .$$

The elements of the i-th row of the matrix $\mathbf{Q}$ can be interpreted as coordinates of a vector $\mathbf{a}_i$ in the vector space of E^{2d+2} with an orthonormal basis $(i = 1, \dots, 2d + 2)$. The matrix $\mathbf{Q}$ is symmetric and the vectors $\mathbf{a}_i$ fulfil both the closedness condition and the conditions (1). The discriminant G^*_{2d+2} is in fact the Gram determinant G_{2d+2}. Consequently, given a fixed point $X \in \varrho_0$ we can construct in a unique way up to the location a regular n-gon of order $d + 1$ whose sides are suitable locations of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{2d+2}$.

If we choose an arbitrary point Z lying in the hyperplane $g(\varepsilon_{2d+2}) = 0$ but not belonging to the face ϱ_0 of the simplex $S = \langle R_0, R_1, \dots, R_{d+1}\rangle$, then at least one of the coefficients $g(\varepsilon_k)$ $(k = 1, \dots, d + 1)$ of the quadratic form $D(\mathbf{y}, \mathbf{y})$ is negative. Neither the quadratic form $D(\mathbf{y}, \mathbf{y})$ nor the quadratic form $G^*_{2d+2}(\mathbf{y}, \mathbf{y})$ obtained by the inverse transformation are positive definite and hence the discriminant G^*_{2d+2} is not in this case the Gram determinant.

To each point X of the face ϱ_0 of the simplex $S = \langle R_0, R_1, \dots, R_{d+1}\rangle$ there corresponds a regular n-gon of order $d + 1$. We have now to determine the space generated by the vertices of this n-gon. (2) implies $g(\varepsilon_{2d+2}) = 0$. If the point X belongs to the open face ϱ_0 then all the coefficients $g(\varepsilon_k)$ $(k = 1, \dots, d + 1)$ are positive. The quadratic form $D(\mathbf{y}, \mathbf{y})$ has the rank $h = 2d + 1$. A Gram determinant of order $h = 2d + 1$ selected from the matrix $\mathbf{G}_{2d+2}$ is nonzero and the vertices of the n-gon generate in this case a space of dimension $h = 2d + 1$. If the point X belongs to the intersection of the faces $\varrho_0, \varrho_{\tau_1}, \dots, \varrho_{\tau_v}$ where $\{\tau_1, \dots, \tau_v\}$ is an arbitrary nonvoid subset of the set $\{1, \dots, d\}$, then the coefficients $g(\varepsilon_{\tau_1}), \dots, g(\varepsilon_{\tau_v})$ of the quadratic form $D(\mathbf{y}, \mathbf{y})$ are zero and its rank is $h = 2d + 1 - 2v$. The vertices of the n-gon generate a space of dimension $h = 2d + 1 - 2v$ since a certain Gram determinant of order $h = 2d + 1 - 2v$ is in this case different from zero. If the point X belongs to the intersection of the faces $\varrho_0, \varrho_{d+1}, \varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ where $\{\tau_1, \ldots, \tau_v\}$ is an arbitrary proper subset of the set $\{1, \ldots, d\}$, then the rank of the quadratic form $D(\mathbf{y}, \mathbf{y})$ is $h = 2d - 2v$ and hence a certain Gram determinant of order $h = 2d - 2v$ selected from the matrix $\mathbf{G}_{2d+2}$ is positive. The vertices of the regular n-gon of order $d + 1$ generate in this case a space of dimension $h = 2d - 2v$.

In particular, the regular n-gons corresponding to the vertices R_m $(m = 1, \ldots, d)$ of the simplex are planar. The rank of the quadratic form $D(\mathbf{y}, \mathbf{y})$ at the vertex R_{d+1} is $h = 1$. If we admit this singularity, this represents a "regular n-gon of order $d + 1$" which results from running through a chosen segment alternately $(d + 1)$-times in each of the both directions.

The investigation of the existence problem being completed, we are ready to formulate the results.

Theorem 3. *Let $X = [x_1 \ x_2, \ldots, x_{d+1}]$ be an arbitrary point of the open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_{d+1}\rangle$ whose vertices are given by (5). Then there exists for $n = 2d + 2$ a regular n-gon of order $d + 1$ whose vertices generate the space E^{2d+1} and the coordinates of the point X have the same meaning as in (1).*

Theorem 4. *Let $X = [x_1, x_2, \ldots, x_{d+1}]$ be a boundary point of the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_{d+1}\rangle$ whose vertices are given by (5).*

a) *If the point X belongs to the intersection of faces $\varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ where $\{\tau_1, \ldots, \tau_v\}$ is an arbitrary nonvoid subset of the set $\{1, \ldots, d\}$ and $X \notin \varrho_{d+1}$, then there exists for $n = 2d + 2$ a regular n-gon of order $d + 1$ whose vertices generate a space $E^{2d-2v+1}$ and the coordinates of the point X have the same meaning as in (1).*

b) *If the point X belongs to the intersection of faces $\varrho_{d+1}, \varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ where $\{\tau_1, \ldots, \tau_v\}$ is an arbitrary proper subset of the set $\{1, \ldots, d\}$, then there exists for $n = 2d + 2$ a regular n-gon of order $d + 1$ whose vertices generate a space E^{2d-2v} and the coordinates of the point X have the same meaning as in (1).*

Theorems 3 and 4 enable us to conclude the existence problem by the following

Theorem 5. *In a Euclidean space E^z there exists for every even number $n > z$ a regular n-gon of order $\frac{1}{2}n$ whose vertices generate the space E^z.*

II

REGULAR POLYGONS WITH AN ODD NUMBER OF SIDES IN EUCLIDEAN SPACES

2.1. We shall now study an n-gon $A_1A_2 \ldots A_n$ in a Euclidean space E^{2d} $(d \geqq 2)$ with an odd number $n = 2d + 1$ of sides. We shall apply the same method as in the preceding case of regular polygons with an even number of sides. Except for necessary modifications we shall preserve also the notation. We shall proceed more briefly if only the nature of the problem allows it. On the other hand, we shall pay more attention to all points in which the two cases differ. We shall show that *regular n-gons of order d* with an odd number $n = 2d + 1$ of sides (which we are going to define immediately) do not exist in spaces of odd dimensions and generating them.

We shall assume that the vectors of the sides $\boldsymbol{a}_i = \overrightarrow{A_i A_{i+1}}$ of the given n-gon satisfy for all i ($i = 1, \ldots, 2d + 1$ cyclically) and all k ($k = 1, \ldots, d$) the relations

$$\boldsymbol{a}_i \boldsymbol{a}_i = w^2 \,, \quad \boldsymbol{a}_i \boldsymbol{a}_{i+k} = \boldsymbol{a}_i \boldsymbol{a}_{i+2d+1-k} = w^2 x_k \,. \tag{7}$$

An n-gon satisfying (7) will be called a *regular n-gon of order d*. Every regular planar n-gon with an odd number of sides $n = 2d + 1$ is in the sense of the above definition a regular n-gon of order d.

We shall again simplify our reasoning by assuming $w = 1$. The closedness condition is now

$$1 + 2 \sum_{m=1}^{d} x_m = 0 \,. \tag{8}$$

The Gram determinant G_{2d+1} is associated with a cyclic matrix

$$\boldsymbol{G}_{2d+1} = \left\| \begin{array}{ccccccccccc}
1, & x_1, & x_2, & \ldots, & x_{d-1}, & x_d, & x_d, & x_{d-1}, & \ldots, & x_2, & x_1 \\
x_1, & 1, & x_1, & \ldots, & x_{d-2}, & x_{d-1}, & x_d, & x_d, & \ldots, & x_3, & x_2 \\
x_2, & x_1, & 1, & \ldots, & x_{d-3}, & x_{d-2}, & x_{d-1}, & x_d, & \ldots, & x_4, & x_3 \\
\multicolumn{11}{c}{\dotfill} \\
x_{d-1}, & x_{d-2}, & x_{d-3}, & \ldots, & 1, & x_1, & x_2, & x_3, & \ldots, & x_d, & x_d \\
x_d, & x_{d-1}, & x_{d-2}, & \ldots, & x_1, & 1, & x_1, & x_2, & \ldots, & x_{d-1}, & x_d \\
x_d, & x_d, & x_{d-1}, & \ldots, & x_2, & x_1, & 1, & x_1, & \ldots, & x_{d-2}, & x_{d-1} \\
\multicolumn{11}{c}{\dotfill} \\
x_2, & x_3, & x_4, & \ldots, & x_d, & x_{d-1}, & x_{d-2}, & x_{d-3}, & \ldots, & 1, & x_1 \\
x_1, & x_2, & x_3, & \ldots, & x_d, & x_d, & x_{d-1}, & x_{d-2}, & \ldots, & x_1, & 1
\end{array} \right\| .$$

It is

$$G_{2d+1} = \prod_{u=1}^{2d+1} g(\varepsilon_u)$$

where

$$\varepsilon_u = \cos \frac{2\pi u}{2d + 1} + \mathrm{i} \sin \frac{2\pi u}{2d + 1} \quad (u = 1, \ldots, 2d + 1) \,.$$

An explicit formula for the factors $g(\varepsilon_u)$ is

$$g(\varepsilon_u) = 1 + \sum_{m=1}^{d} x_m (\varepsilon_u^m + \varepsilon_u^{2d+1-m}) \quad (u = 1, \ldots, 2d + 1).$$

In virtue of (2) we have $g(\varepsilon_{2d+1}) = 0$. Further,

$$g(\varepsilon_k) = g(\varepsilon_{2d+1-k}) \quad (k = 1, \ldots, d).$$

Arranging the determinant J of the regular matrix

$$J = \begin{Vmatrix} 1, & 1, & \ldots, & 1, & 1 \\ \varepsilon_1, & \varepsilon_2, & \ldots, & \varepsilon_{2d}, & \varepsilon_{2d+1} \\ \varepsilon_1^2, & \varepsilon_2^2, & \ldots, & \varepsilon_{2d}^2, & \varepsilon_{2d+1}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varepsilon_1^{2d}, & \varepsilon_2^{2d}, & \ldots, & \varepsilon_{2d}^{2d}, & \varepsilon_{2d+1}^{2d} \end{Vmatrix}$$

leads to the identity $J = (2d + 1) D_0 U$. Hence we see immediatelly that $D_0 \neq 0$ and the determinant D_0 is written explicitly in the form

$$D_0 = \begin{vmatrix} \varepsilon_1 + \varepsilon_1^{2d}, & \varepsilon_1^2 + \varepsilon_1^{2d-1}, & \ldots, & \varepsilon_1^d + \varepsilon_1^{d+1} \\ \varepsilon_2 + \varepsilon_2^{2d}, & \varepsilon_2^2 + \varepsilon_2^{2d-1}, & \ldots, & \varepsilon_2^d + \varepsilon_2^{d+1} \\ \ldots & \ldots & \ldots & \ldots \\ \varepsilon_d + \varepsilon_d^{2d}, & \varepsilon_d^2 + \varepsilon_d^{2d-1}, & \ldots, & \varepsilon_d^d + \varepsilon_d^{d+1} \end{vmatrix}.$$

Complex unities ε_j $(j = 1, \ldots, 2d)$ are solutions of the equation

$$\sum_{r=0}^{2d} y^r = 0.$$

Therefore the sum of elements of each row of the matrix of the determinant D_0 equals -1. By symmetry, the same holds for the columns. Adding to the k-th row $(k = 1, \ldots, d)$ of the matrix of the determinant D_0 all the other rows, then multiplying the k-th row by -2 and denoting the determinant of the resulting matrix by D_k we obtain

(9) $$D_k = -2D_0 \quad (k = 1, \ldots, d).$$

Since $D_0 \neq 0$, all the determinants D_k are nonzero as well.

2.2. The matrix $\mathbf{G}_{2d+1}$ determines a quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$ which is positive semidefinite and has the canonical form

$$D(\mathbf{y}, \mathbf{y}) = \sum_{u=1}^{2d+1} g(\varepsilon_u)\, y_u^2.$$

These assertions can be verified in the same way as the analogous properties of the quadratic form $G_{2d+2}(\mathbf{y}, \mathbf{y})$ in Sec. 1.2. Since $g(\varepsilon_{2d+1-k}) = g(\varepsilon_k)$ for all k $(k = 1, \ldots, d)$, all coefficients $g(\varepsilon_k)$ appear twice in the canonical form of the quadratic form while the coefficient $g(\varepsilon_{2d+1})$ appears only once.

First we shall assume that there exists for $n = 2d + 1$ a regular n-gon of order d in the Euclidean space E^{2d} whose vertices generate the space E^{2d}. Then it follows from the properties of the Gram determinant that the quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$ has the rank $h = 2d$. According to (8) we have $g(\varepsilon_{2d+1}) = 0$, hence all the other coefficients of the positive semidefinite quadratic form $D(\mathbf{y}, \mathbf{y})$ are positive. Hence

$$(10) \qquad g(\varepsilon_k) > 0 \quad (k = 1, \ldots, d).$$

The numbers x_k $(k = 1, \ldots, d)$ in the ordering given by the indices are interpreted again as coordinates of a point X in the space E^d. We ask which points of the space E^d have the property that their coordinates fulfil the inequalities (10) and the closedness condition (8). Setting $g(\varepsilon_k) = 0$ $(k = 1, \ldots, d)$ we have a system of equations (S_0) in the form

$$(S_0) \qquad \begin{aligned}
&(\varepsilon_1 + \varepsilon_1^{2d})\, x_1 + (\varepsilon_1^2 + \varepsilon_1^{2d-1})\, x_2 + \ldots + (\varepsilon_1^d + \varepsilon_1^{d+1})\, x_d = -1\,, \\
&(\varepsilon_2 + \varepsilon_2^{2d})\, x_1 + (\varepsilon_2^2 + \varepsilon_2^{2d-1})\, x_2 + \ldots + (\varepsilon_2^d + \varepsilon_2^{d+1})\, x_d = -1\,, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&(\varepsilon_d + \varepsilon_d^{2d})\, x_1 + (\varepsilon_d^2 + \varepsilon_d^{2d-1})\, x_2 + \ldots + (\varepsilon_d^d + \varepsilon_d^{d+1})\, x_d = -1\,.
\end{aligned}$$

The system (S_0), being an analogue of the identically denoted system from Sec. 1.2, has a nonzero determinant D_0. The hyperplanes of the space E^d corresponding to the system (S_0) have an only point in common, namely R_0 with all coordinates equal to one.

We replace successively the k-th equation of the system (S_0) by the closedness condition $g(\varepsilon_{2d+1}) = 0$. Thus we obtain altogether d systems $(S_1), (S_2), \ldots, (S_d)$ of linear equations with determinants D_k from (9) which we know to be nonzero. Each of the systems (S_k) has therefore a unique solution which represents in the geometrical interpretation the common point R_k of the hyperplanes determined by the corresponding system (S_k). The hyperplanes given by the system (S_0) together with the hyperplane $g(\varepsilon_{2d+1}) = 0$ determine again a simplex, in this case a simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$ in the space E^d. The vertices R_k are now of the form

$$R_k = \left[\frac{\varepsilon_k + \varepsilon_k^{2d}}{2}, \frac{\varepsilon_k^2 + \varepsilon_k^{2d-1}}{2}, \frac{\varepsilon_k^3 + \varepsilon_k^{2d-2}}{2}, \ldots, \frac{\varepsilon_k^d + \varepsilon_k^{d+1}}{2} \right] \quad (k = 1, 2, \ldots, d)\,.$$

Arranging the formulas and adding the point R_0 we obtain

$$(11) \qquad R_0 = [1, 1, \ldots, 1]\,,$$

$$R_k = \left[\cos \frac{2k\pi}{2d+1}, \cos 2\, \frac{2k\pi}{2d+1}, \cos 3\, \frac{2k\pi}{2d+1}, \ldots, \cos d\, \frac{2k\pi}{2d+1} \right].$$

Let us remind that our aim is to find those points $X = [x_1, x_2, \ldots, x_d]$ in the space E^d whose coordinates fulfil both the closedness condition (8) and the inequalities (10). The closedness condition (8) holds for every point of the hyperplane $g(\varepsilon_{2d+1}) = 0$. We shall show that both the conditions simultaneously are fulfilled only for the points $X = [x_1, x_2, \ldots, x_d]$ belonging to the open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$.

Let us denote by $\mathbf{e}$ the vector in the space E^d whose all coordinates are equal to -1. The straight line $X = P + \mathbf{e}t$ passing through the origin of the coordinate system is incident with the vertex R_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$. This line intersects the hyperplane $g(\varepsilon_{2d+1}) = 0$ for $t = 1/2d$. The point of intersection $T = [-1/2d, -1/2d, \ldots, -1/2d]$ lies on a halfline opposite to the halfline PR_0 since the vertex R_0 corresponds to the value of the parameter $t = -1$. The point T is the center of gravity of the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$ which is an immediate consequence of (11):

$$T = \frac{1}{d} \sum_{m=1}^{d} R_m \text{ *)} . \tag{12}$$

Hence the point T belongs to the open face ϱ_0 and each point of the segment TR_0 is a point of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$. The origin P of the coordinate system is an interior point of the segment TR_0 and consequently, it is also an interior point of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$. Substituting the coordinates of the origin P into the linear factors $g(\varepsilon_k)$ we find that $g(\varepsilon_k) = 1$ $(k = 1, \ldots, d)$. Hence the inequalities (10) hold in the intersection K of open halfspaces determined by the hyperplanes $g(\varepsilon_k) = 0$ to which the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$ belongs. On the other hand, the intersection K is the only part of the space E^d in which all inequalities (10) hold. Indeed, when passing from the intersection K into another part of the space E^d always at least one of the inequalities (10) is violated. This fact together with the closedness condition (8) leads to the open face ϱ_0.

Let us introduce an illustrative example.

Example. Let us consider a regular pentagon of order 2 in the space E^4. Linear factors are obtained by substituting for t into the relation $g(t) = 1 + x_1 t + x_2 t^2 +$

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*) B. Míšek in 1959 published a paper [10] in which he studies the volume of the convex hull of an $(n + 1)$-gon in the Euclidean space E^n. B. Míšek looks for an $(n + 1)$-gon in the space E^n with a given perimeter L, whose convex hull has the maximal volume V, and derives the inequality

$$V \leqq \frac{a^n}{n!} \left[\frac{(n+1)^{n-1}}{n^n} \right]^{1/2}$$

where $a = L/(n + 1)$. B. Míšek proved that the extremal case of this inequality occurs for the equilateral $(n + 1)$-gon with the property that any two sides of the polygon form the same angled with $\cos \alpha = 1/n$. This extremal $(n + 1)$-gon corresponds to the point M from (6) or to the point T from (12). The point M may be interpreted as the center of gravity of the system of vertices R_k from (5) where the vertices R_m $(m = 1, \ldots, d)$ have the weight 2 while R_{d+1} has the weight 1. The point T is analogously the center of gravity of the homogeneous system of vertices R_k from (11).

$+ x_2t^3 + x_1t^4$ successively all the fifth roots of one:

$$g(\varepsilon_1) = g(\varepsilon_4) = g\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right) = 1 + \frac{-1+\sqrt{5}}{2}x_1 - \frac{1+\sqrt{5}}{2}x_2\,,$$

$$g(\varepsilon_2) = g(\varepsilon_3) = g\left(\cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5}\right) = 1 - \frac{1+\sqrt{5}}{2}x_1 + \frac{-1+\sqrt{5}}{2}x_2\,,$$

$$g(\varepsilon_5) = g(1) = 1 + 2x_1 + 2x_2\,.$$

The vertex $R_0 = [1, 1]$ of the simplex $S = \langle R_0, R_1, R_2\rangle$ is the solution of the system of equations (S_0)

$$\tfrac{1}{2}(-1+\sqrt{5})\,x_1 - \tfrac{1}{2}(1+\sqrt{5})\,x_2 = -1\,,$$
$$-\tfrac{1}{2}(1+\sqrt{5})\,x_1 + \tfrac{1}{2}(-1+\sqrt{5})\,x_2 = -1\,.$$

The coordinates of the vertices $R_1 = [\frac{1}{4}(-1+\sqrt{5}), \frac{1}{4}(-1-\sqrt{5})]$, $R_2 = [\frac{1}{4}(-1-\sqrt{5}), \frac{1}{4}(-1+\sqrt{5})]$ are the solutions of the systems (S_1) and (S_2) which result by replacing the first and then the second equation in the system (S_0) by the closedness condition $g(\varepsilon_5) = 0$. Evidently it is possible to determine the coordinates of the vertices directly from (11). The lines with equations $g(\varepsilon_1) = 0$, $g(\varepsilon_2) = 0$ and $g(\varepsilon_5) = 0$ determine a simplex $S = \langle R_0, R_1, R_2\rangle$ in the plane, the face ϱ_0 is the

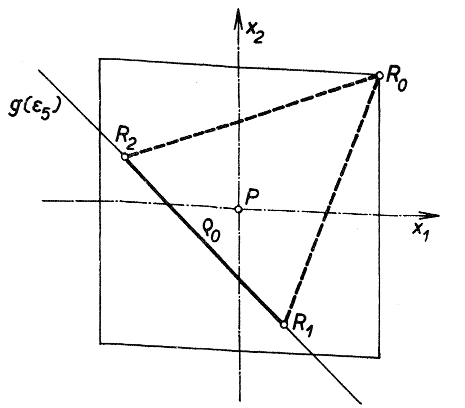

Fig. 3.

segment R_1R_2, the open face ϱ_0 is the set of all interior points of this segment (see Fig. 3). It will be shown later that the vertices R_1, R_2 correspond to regular planar pentagons (a convex pentagon corresponds to R_1, a starlike pentagon to R_2) while the interior points of the segment R_1R_2 correspond to regular pentagons of order 2 which generate the space E^4.

It is of interest that in the case of a regular n-gon of order d with an odd number $n = 2d + 1$ of sides the vertices $R_1, \ldots, R_d$ of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$ do not lie on the surface of a certain cube as was the case with regular n-gons of order $d + 1$ with an even number $n = 2d + 2$ of sides.

We formulate the results of the above considerations in

Theorem 6. *Let the coordinates of a point $X = [x_1, x_2, \ldots, x_d]$ have the same meaning as in (7). If there exists for $n = 2d + 1$ a regular n-gon of order d whose vertices generate the space E^{2d}, then the point X belongs to the open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$ whose vertices are given by (11).*

Let us now start from the assumption that there exists for $n = 2d + 1$ a regular n-gon of order d in the space E^{2d} whose vertices $A_1, A_2, \ldots, A_n$ generate a subspace of a dimension h, where $h < 2d$. Then there exists a Gram determinant of order h selected from the matrix $\mathbf{G}_{2d+1}$ which is different from zero and the quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$ has the rank h. The rank of the quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$ coincides with the number of nonzero coefficients of the quadratic form

$$D(\mathbf{y}, \mathbf{y}) = \sum_{u=1}^{2d+1} g(\varepsilon_u)\, y_u^2 \,.$$

Here we arrive at the essential contradistinction in comparison with the situation occuring for regular polygons with an even number of sides. Each of the factors $g(\varepsilon_k)$ $(k = 1, \ldots, d)$ is a double root of the characteristic equation $|\mathbf{G}_{2d+1} - \lambda \mathbf{E}| = 0$ and hence it appears twice in the canonical form $D(\mathbf{y}, \mathbf{y})$ of the quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$. Consequently, the quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$ cannot have the rank $h = 2k - 1$. The closedness condition (8) gives one zero coefficient. If $h = 2d - 2v$, then there are zero coefficients $g(\tau_1), \ldots, g(\tau_v)$ where $\{\tau_1, \ldots, \tau_v\}$ is such a subset of the set $\{1, \ldots, d\}$ that $1 \leqq v \leqq d - 1$. For all other coefficients of the quadratic form $D(\mathbf{y}, \mathbf{y})$, the number of which is $h = 2d - 2v$, we have necessarily the inequalities (10). The point X whose coordinates satisfy all the above conditions lies in the intersection of the faces $\varrho_0, \varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ and is a boundary point of the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$.

Theorem 7. *Let the coordinates of a point $X = [x_1, x_2, \ldots, x_d]$ have the same meaning as in (7). If there exists for $n = 2d + 1$ a regular n-gon of order d in the space E^{2d} whose vertices generate a subspace of a dimension $h < 2d$, then h is even and the point X is a boundary point of the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots$ $\ldots, R_d \rangle$ whose vertives are given by (11).*

2.3. In the same way as in Sec. 1.3 it can be shown that to each point X belonging to the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$, and only to such a point, there corresponds a regular n-gon of order d with an odd number $n = 2d + 1$ of sides. Formally, the cylic matrix $\mathbf{G}^*_{2d+2}$ from Sec. 1.3 is replaced by the cylic matrix $\mathbf{G}^*_{2d+1}$

and it is shown that the latter coincides with the matrix $\mathbf{G}_{2d+1}$. The resulting matrix $\mathbf{Q}$ which satisfies the identity $\mathbf{Q}^2 = \mathbf{G}^*_{2d+1}$ has obviously the order $2d + 1$. The elements of the i-th row of the matrix $\mathbf{Q}$ are again interpreted as coordinates of a vector $\boldsymbol{a}_i$ in the vector space of E^{2d+1} with an orthonormal basis ($i = 1, \ldots, 2d + 1$).

However, other conclusions are to be expected when considering the problem of the space generated by the vertices of the n-gon $A_1 A_2 \ldots A_n$.

First, let us choose a point X in the open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots \ldots, R_d \rangle$. Then $g(\varepsilon_{2d+1}) = 0$ and all coefficients $g(\varepsilon_k)$ ($k = 1, \ldots, d$) of the quadratic form

$$D(\mathbf{y}, \mathbf{y}) = \sum_{u=1}^{2d+1} g(\varepsilon_u)\, y_u^2$$

are obviously positive. The quadratic form $D(\mathbf{y}, \mathbf{y})$ and hence also the quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$ have the rank $h = 2d$. Therefore a certain Gram determinant of order $2d$ selected from the matrix $\mathbf{G}_{2d+1}$ is positive. Consequently, to each point X belonging to the open face ϱ_0 corresponds a regular n-gon of order d whose vertices generate a space of dimension $h = 2d$.

Now let us choose a point X to be a boundary point of the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$. Such a point X lies in the intersection of faces $\varrho_0, \varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ where $\{\tau_1, \ldots, \tau_v\}$ is a subset of the set $\{1, \ldots, d\}$ with $1 \leqq v \leqq d - 1$. This means that the coefficients $g(\varepsilon_{2d+1}), g(\varepsilon_{\tau_1}), \ldots, g(\varepsilon_{\tau_v})$ equal zero and the rank of the quadratic form $G_{2d+1}(\mathbf{y}, \mathbf{y})$ is $h = 2d - 2v$. A certain Gram determinant of order $h = 2d - 2v$ selected from the matrix $\mathbf{G}_{2d+1}$ is positive. The vertices of the n-gon generate in this case always a space of an even dimension $h = 2d - 2v$.

The extremal case occurs at the vertices R_k ($k = 1, \ldots, d$) where $v = d - 1$ and $h = 2$. Each vertex R_k corresponds to a regular planar n-gon with an odd number $n = 2d + 1$ of sides for which the regularity of order d coincides with the regularity in the current sense if the word.

Theorem 8. *Let $X = [x_1, x_2, \ldots, x_d]$ be an arbitrary point in the open face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$ whose vertices are given by* (11). *Then there exists for $n = 2d + 1$ a regular n-gon of order d whose vertices generate the space E^{2d} and the coordinates of the point X have the same meaning as in* (7).

Theorem 9. *Let $X = [x_1, x_2, \ldots, x_d]$ be a boundary point of the face ϱ_0 of the simplex $S = \langle R_0, R_1, \ldots, R_d \rangle$ whose vertices are given by* (11). *If the point X belongs to the intersection of faces $\varrho_{\tau_1}, \ldots, \varrho_{\tau_v}$ where $\{\tau_1, \ldots, \tau_v\}$ is a subset of the set $\{1, \ldots, d\}$ with $1 \leqq v \leqq d - 1$, then there exists for $n = 2d + 1$ a regular n-gon of order d whose vertices generate the space E^{2d-2v} and the coordinates of the point X have the same meaning as in* (7).

Taking into account the trivial case of an equilateral triangle omitted until now for the sake of simpler formulation, we can summarize the results concerning the existence problem as follows:

Theorem 10. *In a Euclidean space E^{2z} there exists for every odd number $n > 2z$ a regular n-gon of order $\frac{1}{2}(n-1)$ whose vertices generate the space E^{2z}.*

In a Euclidean space E^{2z-1} there exists for any odd number n no regular n-gon of order $\frac{1}{2}(n-1)$ whose vertices generate the space E^{2z-1}.

The theorem on planarity of a regular pentagon in E^3 published in 1970 by B. L. van der Waerden [13] is a special case of our Theorem 10. Actually, B. L. van der Waerden [13] as well as (in a more general setting) V. I. Arnold [1] require only that any two sides have the same length and that the angles of any pair of adjacent sides be equal. Nonetheless, it is seen from the relation (3) in our paper [11] that in this special case the assumptions guarantee the regularity of order 2. This is not true for n-gons with an odd number of sides which are regular in the sense of Arnold's definition provided $n \geqq 7$. As was shown by A. P. Garber, V. I. Garvackij and V. Ja. Jarmolenko [7] who solved Arnold's problem, such regular polygons in the space E^3 do exist.

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