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STRUCTURE OF WEAKLY ABELIAN QUASIGROUPS

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This paper is concerned with some properties of weakly abelian quasigroups and, it is a continuation of the last section from [4]. It will be shown (among other things) that the structure of weakly abelian D-quasigroups is very similar to that of distributive quasigroups.

First we recall some notions and definitions. A quasigroup Q is called

– abelian if it satisfies the identity $ab \,.\, cd = ac \,.\, bd$ (a)
– an LWA-quasigroup if it satisfies $aa \,.\, bc = ab \,.\, ac$ (b)
– an RWA-quasigroup if it satisfies $bc \,.\, aa = ba \,.\, ca$ (c)
– a WA-quasigroup if it satisfies both (b) and (c)
– a D-quasigroup if it satisfies $ab \,.\, ca = ac \,.\, ba$ (d)
– a WAD-quasigroup if it satisfies (b), (c) and (d)
– unipotent if $aa = bb$ for all $a, b \in Q$
– idempotent if $aa = a$ for every $a \in Q$
– distributive if it is an idempotent WA-quasigroup
– triabelian if every its subquasigroup which is generated by at most three elements is abelian.

If G is a groupoid and $x \in G$ then L_x and R_x will denote the left and right transiation by x, respectively. If Q is a quasigroup and $x \in Q$ then $f(x)$ and $e(x)$ will be the left and right local unit of x, respectively. If Q is a commutative Moufang loop then $N(Q)$ denotes the nucleus of Q and a mapping g of Q into Q is said to be *nuclear* provided that $x^{-1} \,.\, g(x) \in N(Q)$ for each $x \in Q$. As is easy to see, the set of all nuclear permutations of Q is a subgroup in the symmetric group S_Q.

The following lemma is an easy consequence of [5, Theorem 2].

Lemma 1. *Let Q be a commutative loop and g a mapping of Q into Q. Then the following conditions are equivalent:*

(i) $(g(a) \,.\, a)(bc) = (g(a) \,.\, b)(ac)$ *for all* $a, b, c \in Q$.
(ii) *Q is a Moufang loop and g is nuclear.*

Theorem 1. *Let Q be a quasigroup. The following conditions are equivalent:*

(i) *Q is a WA-quasigroup and there is $a \in Q$ such that $ab \,.\, ca = ac \,.\, ba$ for all $b, c \in Q$.*

(ii) *Q is a WA-quasigroup and Q is isotopic to a commutative Moufang loop.*

(iii) *Q is a WA-quasigroup and Q is isotopic to a Moufang loop.*

(iv) *There are a commutative Moufang loop $Q(\circ)$, $\varphi, \psi \in \mathrm{Aut}\, Q(\circ)$ and $g \in Q$ such that $\varphi\psi = \psi\varphi$, $\varphi\psi^{-1}$ is a nuclear automorphism of $Q(\circ)$ and $ab = (\varphi(a) \circ \psi(b)) \circ g$ for all $a, b \in Q$.*

(v) *Q is a WAD-quasigroup.*

Proof. (i) implies (ii). If $b, c \in Q$ then $(aa \,.\, ab)(ac \,.\, aa) = (aa \,.\, ab)(aa \,.\, ca) = (aa \,.\, aa)(ab \,.\, ca) = (aa \,.\, aa)(ac \,.\, ba) = (aa \,.\, ac)(aa \,.\, ba) = (aa \,.\, ac)(ab \,.\, aa)$. Hence $(aa \,.\, x)(y \,.\, aa) = (aa \,.\, y)(x \,.\, aa)$ for all $x, y \in Q$ and we can use [4, Proposition 4.8] and Lemma 1.

The implication (ii) implies (iii) is trivial.

(iii) implies (iv). Let $x \in Q$ and $a \circ b = R_{xx}^{-1}(a) \,.\, L_{xx}^{-1}(b)$ for all $a, b \in Q$. As is proved in [4], $Q(\circ)$ is a CI-loop. However, $Q(\circ)$ is a Moufang loop, hence it is an IP-loop, and consequently $Q(\circ)$ is commutative. The rest follows from [4, Proposition 4.8, Theorem 4.9].

(iv) implies (v). Since $\varphi\psi^{-1}$ is a nuclear mapping and $\varphi\psi = \psi\varphi$, $\varphi^2\psi^{-2} = \varphi\psi^{-1}\varphi\psi^{-1}$ is nuclear. According to Lemma 1, we can write $ab \,.\, ca = (((\varphi^2(a) \circ \varphi\psi(b)) \circ \varphi(g)) \circ ((\psi\varphi(c) \circ \psi^2(a)) \circ \psi(g))) \circ g = (((\varphi^2(a) \circ \varphi\psi(b)) \circ (\varphi\psi(c) \circ \psi^2(a))) \circ (\varphi(g) \circ \psi(g))) \circ g = (((\varphi^2(a) \circ \varphi\psi(c)) \circ \varphi(g)) \circ ((\psi\varphi(b) \circ \psi^2(a)) \circ \psi(g))) \circ g = ac \,.\, ba$ for all $a, b, c \in Q$. Now the proof of the theorem is complete, the last implication being trivial.

Let Q be a WAD-quasigroup. A tetrad $(Q(\circ), \varphi, \psi, g)$ is called *an arithmetical form of Q* if it satisfies the condition (iv) from Theorem 1.

Lemma 2. *Let Q be a WAD-quasigroup and $x \in Q$. Then there exists an arithmetical form $(Q(\circ), \varphi, \psi, g)$ of Q such that the element $xx \,.\, xx$ is equal to the unit of $Q(\circ)$ and $g = (xx \,.\, xx)(xx \,.\, xx)$.*

Proof. The lemma follows from the proof of [4, Theorem 4.9].

Proposition 1. *Let Q be a commutative WA-quasigroup. Then Q is a WAD-quasigroup and $\varphi = \psi$ for every arithmetical form $(Q(\circ), \varphi, \psi, g)$ of Q.*

Proof. Obvious.

Proposition 2. *Every unipotent WA-quasigroup is abelian.*

Proof. Let Q be a unipotent WA-quasigroup. There is $j \in Q$ such that $aa = j$ for each $a \in Q$. Put $x \circ y = R_j^{-1}(x) \,.\, L_j^{-1}(y)$ for all $x, y \in Q$. Then $Q(\circ)$ is a loop, j is the unit of $Q(\circ)$ and $(\alpha(a) \circ a) \circ (b \circ c) = (\alpha(a) \circ b) \circ (a \circ c)$ for all $a, b, c \in Q$

and $\alpha = R_j L_j^{-1}$ (see [4, Proposition 4.8]). Further, $\alpha(a) \circ a = R_j^{-1} R_j L_j^{-1}(a) \,.\, L_j^{-1}(a) = L_j^{-1}(a) \,.\, L_j^{-1}(a) = j$ for every $a \in Q$, and hence $c = \alpha(a) \circ (a \circ c)$ for every $c \in Q$. On the other hand,

$$\alpha^2(a) \circ (\alpha(a) \circ (c \circ a)) = (\alpha^2(a) \circ j) \circ (\alpha(a) \circ (c \circ a)) =$$
$$= (\alpha^2(a) \circ \alpha(a)) \circ (c \circ a) = (\alpha^2(a) \circ c) \circ (\alpha(a) \circ a) = \alpha^2(a) \circ c \,.$$

Thus $c = \alpha(a) \circ (c \circ a) = \alpha(a) \circ (a \circ c)$ and $a \circ c = c \circ a$. We have proved that $Q(\circ)$ is commutative, and therefore $Q(\circ)$ is a commutative Moufang loop by [4, Proposition 4.1]. By Lemma 1, $\alpha(a) = a^{-1}$ is a nuclear mapping, so that $a^{-2} \in N(Q(\circ))$ for every $a \in Q$. However, since $Q(\circ)$ is a commutative Moufang loop, $a \circ a \circ a \in N(Q(\circ))$ for every $a \in Q$ ([2, pg. 128]), and so $N(Q(\circ)) = Q(\circ)$. Thus $Q(\circ)$ is an abelian group and Q is an abelian quasigroup by [4, Proposition 4.3].

Proposition 3. *Let Q be a WA-quasigroup such that the mapping $x \mapsto xx$ is a permutation. Then Q is a WAD-quasigroup.*

Proof. Let $\alpha(x) = xx$ and $a * b = \alpha^{-1}(ab)$ for all $x, a, b \in Q$. Then $(a * b) * (a * c) = \alpha^{-1}(\alpha^{-1}(ab) \,.\, \alpha^{-1}(ac)) = (\alpha^{-2}(a) \,.\, \alpha^{-2}(b))(\alpha^{-2}(a) \,.\, \alpha^{-2}(c)) = (\alpha^{-2}(a) \,.\, \alpha^{-2}(a))(\alpha^{-2}(b) \,.\, \alpha^{-2}(c)) = \alpha^{-1}(a) \,.\, (\alpha^{-2}(b) \,.\, \alpha^{-2}(c)) = a * (b * c)$, since Q is a WA-quasigroup and α is an automorphism of Q. Similarly we can show $(b * a) * (c * a) = (b * c) * a$, and hence $Q(*)$ is a distributive quasigroup. As is easy to see, α is an automorphism of $Q(*)$ and $ab \,.\, ca = (\alpha^2(a) * \alpha^2(b)) * (\alpha^2(c) * \alpha^2(a))$ for all $a, b, c \in Q$. Hence it is enough to prove that every distributive quasigroup is a D-quasigroup. However, every distributive quasigroup is triabelian, as follows from a more general theorem proved by Belousov. Here we give an other direct proof of this theorem.

Theorem. [1, pg. 147]. *Let Q be a distributive quasigroup and let $a, b, c, d \in Q$ be such that $ab \,.\, cd = ac \,.\, bd$. Then the subquasigroup generated by these elements is abelian.*

Proof. The proof is divided into several lemmas. First, it is easy to observe that the group generated by all the translations L_x, R_x, $x \in Q$, is contained in the group Aut Q. If $a, b, \ldots \in Q$ then $S(a, b, \ldots)$ will denote the subquasigroup generated by $a, b, \ldots$.

Lemma. *Let $a, b, c, d \in Q$ and $ab \,.\, cd = ac \,.\, bd$. Then $ax \,.\, yd = ay \,.\, xd$ for all $x, y \in S(a, b, c, d)$.*

Proof. If $h = R_d^{-1} L_{ac}^{-1} R_{cd} L_a$ then $au \,.\, cd = ac \,.\, h(u)\, d$ for every $u \in Q$. Further, $ab \,.\, cd = ac \,.\, bd$, $aa \,.\, cd = ac \,.\, ad$, $ac \,.\, cd = ac \,.\, cd$ and $ad \,.\, cd = ac \,.\, dd$. Hence $h(a) = a$, $h(b) = b$, $h(c) = c$ and $h(d) = d$. Since h is an automorphism, the set $P = \{x \in Q \mid h(x) = x\}$ is a subquasigroup and $S(a, b, c, d) \subseteq P$. Thus $ax \,.\, cd =$

$= ac \,.\, xd$ for every $x \in S(a, b, c, d)$. By symmetry, $ax \,.\, bd = ab \,.\, xd$ for every $x \in S(a, b, c, d)$ and the result easily follows.

Lemma. *Let* $a, b, c, d \in Q$ *and* $ab \,.\, cd = ac \,.\, bd$. *Then* $zx \,.\, yd = zy \,.\, xd$ *and* $ax \,.\, yv = ay \,.\, xv$ *for all* $x, y \in S(a, b, c, d)$, $z \in S(a, b, c)$ *and* $v \in S(b, c, d)$.

Proof. Let $x, y \in S(a, b, c, d)$. By the preceding lemma, $ax \,.\, yd = ay \,.\, xd$. The set $P = \{u \in Q \mid ux \,.\, yd = uy \,.\, xd\}$ is a subquasigroup and $a, x, y \in P$. Hence $ux \,.\, yd = uy \,.\, xd$ for every $u \in S(a, x, y)$. Now let $y = ba$. Then $b \in S(a, x, y)$, and so $bx \,.\, (ba \,.\, d) = (b \,.\, ba) \,.\, xd$. From this we obtain the equality $bp \,.\, qd = bq \,.\, pd$ for all $p, q \in (b, x, ba, d)$. If $x = c$ then $S(b, x, ba, d) = S(a, b, c, d)$ and $bp \,.\, qd = bq \,.\, pd$ for all $p, q \in S(a, b, c, d)$. Using the symmetry, we get the equality $cp \,.\, qd = cq \,.\, pd$ for all $p, q \in S(a, b, c, d)$. The rest of the proof is now clear.

Lemma. *Q is a D-quasigroup.*

Proof. Since $aa \,.\, bc = ab \,.\, ac$, $ab \,.\, ca = ac \,.\, ba$ by the preceding lemma.

Now to the proof of the theorem itself. Let $a, b, c, d \in Q$ and $ab \,.\, cd = ac \,.\, bd$. By the preceding lemmas, $zx \,.\, yd = zy \,.\, xd$ and $dx \,.\, yd = dy \,.\, xd$ for all $x, y \in S(a, b, c, d)$ and $z \in S(a, b, c)$. Hence $ux \,.\, yd = uy \,.\, xd$ for all $u, x, y \in S(a, b, c, d)$. Similarly, $ax \,.\, yu = ay \,.\, xu$ for all $u, x, y \in S(a, b, c, d)$. In particular, $ad \,.\, bc = ab \,.\, dc$ and $ac \,.\, db = ad \,.\, cb$. Hence, as was proved above, $ux \,.\, yc = uy \,.\, xc$ and $ux \,.\, yb = uy \,.\, xb$ for all $u, x, y \in S(a, b, c, d)$. From this, $dc \,.\, ab = da \,.\, cb$, so that $ux \,.\, ya = uy \,.\, xa$ for all $u, x, y \in S(a, b, c, d)$ and the result follows easily.

Let Q be a quasigroup. A mapping g (h) of Q into Q is called *left* (*right*) *regular* if there exists a mapping g^* (h^*) such that $g(xy) = g^*(x) \,.\, y$ $(h(xy) = x \,.\, h^*(y))$. A mapping k is called *middle regular* if there is a mapping k^* such that $k(x) \,.\, y = x \,.\, k^*(y)$. By L_Q we shall denote the set of all the left regular mappings and L_Q^* will be the set of all the corresponding mappings g^*. Similarly we define R_Q, R_Q^*, F_Q and F_Q^*. As is easy to see, mappings from L_Q, L_Q^*, R_Q, R_Q^*, F_Q and F_Q^* are permutations and all these sets are subgroups in S_Q.

Lemma 3. *Let Q be a WAD-quasigroup and let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q. Then*

(i) $L_Q = L_Q^* = R_Q = R_Q^* = F_Q = F_Q^*$,

(ii) *if k is a mapping of Q into Q then $k \in L_Q$ iff there is $a \in N(Q(\circ))$ such that $k(x) = x \circ a$ for every $x \in Q$.*

Proof. Let $k \in L_Q$. Then $k((\varphi(a) \circ \psi(b)) \circ g) = (\varphi\, k^*(a) \circ \psi(b)) \circ g$ for all $a, b \in Q$. Substituting $\psi^{-1}(g^{-1})$ for b, we obtain the equality $k\, \varphi(a) = \varphi\, k^*(a)$. Hence $k((a \circ b) \circ g) = (k(a) \circ b) \circ g$ for all $a, b \in Q$, so that $k(a \circ g) = k(a) \circ g$ and $k(a \circ b) = k(a) \circ b$. Thus $k(b) = k(j) \circ b$ and the equality $k(j) \circ (a \circ b) = (k(j) \circ a) \circ b$ yields $k(j) \in N(Q(\circ))$. The rest is clear.

Let Q be a commutative Moufang loop. We shall say that Q is 3-elementary if $x^3 = j$ for every $x \in Q$, where j is the unit of Q.

Proposition 4. *Let Q be a commutative WA-quasigroup. The following conditions are equivalent:*

(i) $aa \,.\, ax = bb \,.\, bx$ *for all* $a, b, x \in Q$.

(ii) $aa \,.\, ax = xx \,.\, xx$ *for all* $a, x \in Q$.

(iii) *Q is isotopic to a commutative 3-elementary Moufang loop.*

(iv) *Every commutative Moufang loop isotopic to Q is 3-elementary.*

Proof. The implication (i) implies (ii) is trivial.

(ii) implies (iii). Let $(Q(\circ), \varphi, \varphi, g)$ be an arithmetical form of Q (see Proposition 1) and j the unit of $Q(\circ)$. Then

$$(((\varphi^2(a) \circ \varphi^2(a)) \circ \varphi^2(a)) \circ (\varphi(g) \circ \varphi(g))) \circ g = aa \,.\, aj = jj \,.\, jj =$$
$$= (\varphi(g) \circ \varphi(g)) \circ g \,.$$

Hence $\varphi^2(a) \circ \varphi^2(a) \circ \varphi^2(a) = j$, so that $a \circ a \circ a = j$.

(iii) implies (iv). As is well known, isotopic commutative Moufang loops are isomorphic.

(iv) implies (i). Let $(Q(\circ), \varphi, \varphi, g)$ be an arithmetical form of Q. Then $aa \,.\, ax = (((\varphi^2(a) \circ \varphi^2(a)) \circ (\varphi^2(a) \circ \varphi^2(x))) \circ (\varphi(g) \circ \varphi(g))) \circ g = (\varphi^2(x) \circ (\varphi(g) \circ \varphi(g))) \circ g$ by (iv) and with respect to the diassociativity of $Q(\circ)$. Thus $aa \,.\, ax = bb \,.\, bx$.

A commutative WA-quasigroup satisfying the equivalent conditions of the preceding proposition will be called primitive.

Proposition 5. *Let Q be a WAD-quasigroup. Define a binary relation r on Q by $a \mathrel{r} b$ iff $a = k(b)$ for some $k \in L_Q$. Then*

(i) *if $(Q(\circ), \varphi, \psi, g)$ is an arithmetical form of Q and $a, b \in Q$ then $a \mathrel{r} b$ iff $b = a \circ x$ for some $x \in N(Q(\circ))$,*

(ii) *r is a normal congruence relation of Q,*

(iii) *the factorquasigroup Q/r is a primitive commutative WA-quasigroup,*

(iv) *if a class A of r is a subquasigroup then A is an abelian quasigroup,*

(v) *if $a \mathrel{r} aa$ for an $a \in Q$ then the class $A = \{x \in Q \mid x \mathrel{r} a\}$ is an abelian subquasigroup of Q.*

Proof. (i) is obvious from Lemma 3.

(ii) Let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q and $a, b, c \in Q$. If $a \mathrel{r} b$ then $b = a \circ x$ for an $x \in N(Q(\circ))$ and $bc = (\varphi(a \circ x) \circ \psi(c)) \circ g = ((\varphi(a) \circ \psi(c)) \circ g) \circ \varphi(x) = ac \circ \varphi(x)$, since $\varphi(x) \in N(Q(\circ))$. The rest can be proved similarly.

(iii) Let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q and $k(a) = \psi(a^{-1} \circ \varphi\psi^{-1}(a))$ for every $a \in Q$. Since $\varphi\psi^{-1}$ is a nuclear mapping, $k(a) \in N(Q(\circ))$. On the other hand, $ba \circ k(a) = ((\varphi(b) \circ \psi(a)) \circ g) \circ k(a) = (\varphi(b) \circ (\psi(a) \circ k(a))) \circ g = (\varphi(b) \circ \varphi(a)) \circ g =$

$= ab \circ k(b)$. Hence ab r ba for all $a, b \in Q$ and Q/r is commutative. Finally, $x \circ x \circ \circ x \in N(Q(\circ))$ for each $x \in Q$, r is a normal congruence of $Q(\circ)$, $Q(\circ)/r$ is 3-elementary and $Q(\circ)/r$ is isotopic to Q/r. According to Proposition 4, Q/r is primitive.

(iv) Let $x \in A$ and $j = xx \,.\, xx$. As A is a subquasigroup, $j \in A$. Consider $(Q(\circ), \varphi, \psi, g)$, the arithmetical form corresponding to j in the sense of Lemma 2. Then j is the unit of $Q(\circ)$ and (i) yields the equality $A = N(Q(\circ))$. However, $g = jj \in A$, and so $(A(\circ), \varphi/A, \psi/A, g)$ is an arithmetical form of A. Since $A(\circ)$ is an abelian group, A is an abelian quasigroup. Now the proof is complete, because (v) is a straightforward consequence of (iv).

Corollary 1. (i) *Every simple WAD-quasigroup is either abelian or commutative and primitive.*

(ii) *Every finite simple WA-quasigroup is a WAD-quasigroup.*

Proof. (i) follows immediately from Proposition 5.

(ii) Let Q be a finite simple WA-quasigroup and $k(x) = xx$ for every $x \in Q$. Since k is an endomorphism of Q and Q is simple, k is one-to-one or $k(x) = k(y)$ for all $x, y \in Q$. In the first case, $k(Q) = Q$ (because of the finiteness of Q) and Q is a WAD-quasigroup by Proposition 3. In the second case, Q is unipotent and hence abelian by Proposition 2.

Theorem 2. *Let Q be a WA-quasigroup. Then it is a WAD-quasigroup, provided at least one of the following conditions holds:*

(i) *Q is commutative.*

(ii) *Q is unipotent.*

(iii) *The mapping $x \mapsto xx$ is biunique.*

(iv) *Q is finite and simple.*

(v) *Q is idempotent.*

Proof. Apply Propositions 1, 2, 3 and Corollary 1.

Proposition 6. *Let Q be a WAD-quasigroup with an idempotent $j \in Q$ and let $(Q(\circ), \varphi, \psi, j)$ be the arithmetical form corresponding to j in the sense of Lemma 2. For all $a \in Q$ let $k(a) = \psi(a^{-1} \circ \varphi\psi^{-1}(a))$. Then*

(i) *k is an endomorphism of $Q(\circ)$ and $k(a) \in N(Q(\circ))$ for every $a \in Q$,*

(ii) *k is an endomorphism of Q and $k(Q)$ is an abelian quasigroup,*

(iii) *$k(a) = k(b)$ iff $ab = ba$,*

(iv) *the set $A = \{a \in Q \mid aj = ja\}$ is a normal commutative subquasigroup of Q.*

Proof. (i) Clearly, $\varphi(a) = \psi(a) \circ k(a)$ for each $a \in Q$. Hence $(\psi(a) \circ \psi(b)) \circ \circ k(a \circ b) = \psi(a \circ b) \circ k(a \circ b) = \varphi(a \circ b) = \varphi(a) \circ \varphi(b) = (\psi(a) \circ k(a)) \circ (\psi(b) \circ \circ k(b)) = (\psi(a) \circ \psi(b)) \circ (k(a) \circ k(b))$, since both $k(a)$ and $k(b)$ belong to $N(Q(\circ))$. Thus $k((a \circ b) = k(a) \circ k(b)$.

(ii) Let $a \in Q$. Then $k\varphi = \varphi k$, as follows from the definition of k, and consequently

$$\psi\, k(a) \circ k^2(a) = \varphi\, k(a) = k\, \varphi(a) = k\, \psi(a) \circ k^2(a)\,.$$

Thus $\psi k = k\psi$ and $k(ab) = k(\varphi(a) \circ \psi(b)) = \varphi\, k(a) \circ \psi\, k(b) = k(a)\, .\, k(b)$. Let $B = k(Q)$. As k is an endomorphism of both $Q(\circ)$ and Q, $B(\circ)$ is a subloop and B is a subquasigroup. However, $B(\circ) \subseteq N(Q(\circ))$ and $\varphi(B) \subseteq B$, $\psi(B) \subseteq B$. Now it is obvious that $(B(\circ), \varphi \mid B, \psi \mid B, j)$ is an arithmetical form of B and that B is abelian.

(iii) If $ab = ba$ then

$$(\psi(a) \circ \psi(b)) \circ k(a) = \varphi(a) \circ \psi(b) = \varphi(b) \circ \psi(a) = (\psi(b) \circ \psi(a)) \circ k(b)\,,$$

so that $k(a) = k(b)$. Conversely, if $k(a) = k(b)$ then the equality $ab \circ k(b) = ba \circ \circ k(a)$ yields $ab = ba$.

(iv) This is obvious from (ii) and (iii).

A quasigroup Q is called *anticommutative* if $ab \neq ba$, whenever $a, b \in Q$ and $a \neq b$.

Corollary 2. *Every anticommutative WAD-quasigroup is abelian.*

Proof. Let Q be an anticommutative WAD-quasigroup, $x \in Q$ and $a * b = L_x^{-1}(a)\, .\, L_x^{-1}(b)$ for all $a, b \in Q$. Then $Q(*)$ is a WAD-quasigroup with a left unit and $(a * b) * (c * d) = L_{xx}^{-1} L_{xx.xx}^{-1}(ab\, .\, cd)$ for all $a, b, c, d \in Q$. As is easy to see, $Q(*)$ is anticommutative, $Q(*)$ has an idempotent element and $Q(*)$ is abelian iff Q is so. Hence we can assume that Q contains at least one idempotent element. Let k be the endomorphism of Q defined in Proposition 6. Then $k(a) = k(b)$ iff $ab = ba$ and $k(Q)$ is an abelian quasigroup. Since Q is anticommutative, k is one-to-one, and therefore Q is isomorphic to $k(Q)$.

Proposition 7. *Let Q be a WAD-quasigroup with an idempotent element j, $A = \{x \in Q \mid ax\, .\, bc = ab\, .\, jc$ for some $a, b, c \in Q\}$ and let P be the subquasigroup of Q generated by A. Then*

(i) *P is a normal subquasigroup of Q,*
(ii) *the factorquasigroup Q/P is abelian,*
(iii) *P is a primitive commutative WAD-quasigroup.*

Proof. The proof is similar to that of [1, Theorem 8.7]. Let $(Q(\circ), \varphi, \psi, j)$ be the arithmetical form of Q corresponding to j. If $a, b, c \in Q$ then there is a uniquely determined element $h(a, b, c) \in Q$ such that $(a \circ b) \circ c = (a \circ h(a, b, c)) \circ (b \circ c)$. Let $B(\circ)$ be the subloop of $Q(\circ)$ generated by all the elements $h(a, b, c)$, $a, b, c \in Q$. Since $k(h(a, b, c)) = h(k(a), k(b), k(c))$ for every endomorphism k of $Q(\circ)$, $B(\circ)$ is a normal subloop in $Q(\circ)$ and B is a normal subquasigroup in Q. The factorloop $Q(\circ)/B(\circ)$ is clearly an abelian group, and hence the factorquasigroup Q/B is abelian.

Further, according to [1, Lemma 8.6],

$$\varphi h(a, b, c) = h(\varphi(a), \varphi(b), \varphi(c)) =$$
$$= h(\psi(a) \circ k(a), \psi(b) \circ k(b), \psi(c) \circ k(c)) = \psi h(a, b, c)$$

(k is the endomorphism defined in Proposition 6) and $x \circ x \circ x = j$ for every $x \in B$. Hence B is a commutative primitive WAD-quasigroup. Now it remains to prove that $B = P$. To this purpose it suffices to show that $ab \,.\, jc = ah(a, b, c) \,.\, bc$ for all $a, b, c \in Q$. Indeed, let $ax \,.\, bc = ab \,.\, jc$. Then

$$(\varphi^2(a) \circ \varphi\psi(b)) \circ \psi^2(c) = (\varphi^2(a) \circ \varphi\psi(x)) \circ (\varphi\psi(b) \circ \psi^2(c)) \,.$$

However, $\varphi^2(a) = \varphi(\psi(a) \circ k(a)) = \varphi\psi(a) \circ \varphi\, k(a)$, $\psi^2(c) = \varphi\psi(c) \circ \psi\, k(c^{-1})$ and $\varphi\, k(a)$, $\psi\, k(c^{-1})$ belong to $N(Q(\circ))$. Thus $(a \circ x) \circ (b \circ c) = (a \circ b) \circ c$.

If Q is a quasigroup then the multiplication group $A(Q)$ of Q is the subgroup of S_Q generated by all the translations L_x, R_x, $x \in Q$. In [3], it is proved that $A(Q)$ is a solvable group, if Q is a finite distributive quasigroup. The following proposition is a generalization of this result.

Proposition 8. *Let Q be a finite WAD-quasigroup. Then $A(Q)$ is a solvable group.*

Proof. Let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q, $G = A(Q(\circ))$ and let H be the subgroup of S_Q generated by $G \cup \{\varphi, \psi\}$. Since φ, ψ are automorphisms of $Q(\circ)$ and $\varphi\psi = \psi\varphi$, G is a normal subgroup in H and H/G is an abelian group. On the other hand, the multiplication group of a finite commutative Moufang loop is nilpotent (see [2, pg. 106]), and consequently H is solvable. Finally, as is easy to see, $A(Q) \subseteq H$ and the proof is complete.

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