Pavel Tomasta

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DECOMPOSITIONS OF GRAPHS AND HYPERGRAPHS INTO ISOMORPHIC FACTORS WITH A GIVEN DIAMETER

PAVOL TOMASTA, Bratislava

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INTRODUCTION

This paper deals with k-uniform hypergraphs for $k \geqq 2$. D. PALUMBÍNY in [2], [3] studies the problem of decomposing a complete graph into factors with equal diameters. He proved in [2] that $F_m^2(d) = 2m$ for $m \geqq 2$ and $3 \leqq d \leqq 2m - 1$, where $F_m^2(d)$ is the smallest natural number such that the complete graph with $F_m^2(d)$ vertices can be decomposed into m factors with a diameter d. Even though his aim was not to find a decomposition into isomorphic factors with the diameter equal to d, the m factors of his decomposition of the complete graph with $2m$ vertices are isomorphic for d odd.

In this paper we shall systematically study the problem of decomposing a complete k-uniform hypergraph into isomorphic factors with a given diameter. The study of decompositions of complete graphs into isomorphic factors with a given diameter was initiated by [5], where the problem of decomposing a complete graph into three isomorphic factors with a given diameter $d \geqq 2$ is considered.

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First we give some definitions. *A hypergraph* is an ordered pair of sets $G = (V, H)$, where $H \subset P(V)$ (the potence of V). The set V is called *the vertex set*, H is *the edge set* of G. *A path of length* q is a sequence $x_1, h_1, \ldots, h_q, x_{q+1}$ such that $x_1, \ldots, x_{q+1}$ are distinct vertices of V, $h_1, \ldots, h_q$ are distinct edges of H and $x_k, x_{k+1} \in h_k$ for $k = 1, 2, \ldots, q$. *The distance* $d(x, y)$ *of two vertices* x *and* y is the length of the shortest path joining them. *The diameter of a hypergraph* is defined as

$$d = \sup_{x,y \in V} d(x, y)$$

A hypergraph is said to be a *k-uniform* if for each $h \in H$ we have $|h| = k$. If the set H contains all k-element subsets of V we say that G is a *complete k-uniform hypergraph* and we denote G by $\langle n \rangle_k$, where $n = |V|$. A factor of G is *a subhyper-*

graph of G which contains all vertices of G. We shall say that $G_1 = (V_1, H_1)$ and $G_2 = (V_2, H_2)$ are *isomorphic* and write $G_1 \cong G_2$ if there exists a bijection $f : V_1 \to V_2$ such that $h \in H_1$ if and only if $f(h) \in H_2$.

Denote by $G^k_m(d)$ the smallest cardinal number such that $\langle G^k_m(d)\rangle_k$ can be decomposed into m isomorphic factors with diameter d.

A question arises whether $G^k_m(d)$ has the same property as the number $F^2_m(d)$,with the additional condition of isomorphy, i.e. whether $\langle n\rangle_k$ can be decomposed into m isomorphic factors with diameter d if and only if $n \geqq G^k_m(d)$. However, if the factors of a decomposition of $\langle n\rangle_k$ are mutually isomorphic they have the same number of edges so that m divides $\binom{n}{k}$. This implies the negative answer to our question.

We shall call the numbers n for which m divides $\binom{n}{k}$ the suitable numbers. This leads us to the following definition.

Definition 1. Let $H^k_m(d)$ be the smallest cardinal number with the following property: A decomposition of the hypergraph $\langle n\rangle_k$ into m isomorphic factors with diameter d exists if and only if $n \geqq H^k_m(d)$ and n is a suitable number.

Now we introduce a concept which makes it possible to bring a common point of view into the problems concerning decompositions.

Definition 2. Let G be an arbitrary group of automorphisms of the hypergraph $\langle n\rangle_k$ and let there exist a surjection $h : G \to R$, where R is a decomposition of the hypergraph $\langle n\rangle_k$ into isomorphic factors, with the following property:

$$x(h(y)) = h(xy) \quad \text{for every} \quad x, y \in G\,.$$

Then we shall say that R is a decomposition of $\langle n\rangle_k$ by the group G. If the mapping h is a bijection then we shall say that R is a simple decomposition of $\langle n\rangle_k$ by G. The factor $h(x)$ will be denoted by G_x.

The following lemma makes it possible to prove a necessary and sufficient condition for the existence of a simple decomposition of $\langle n\rangle_k$ by an Abelian group of a finite order.

Lemma 1. *Let R be a simple decomposition of the hypergraph $\langle n\rangle_k$ by an Abelian group H and let a group H_1 be a subgroup of H. Then there exists a simple decomposition R_1 of $\langle n\rangle_k$ by the group H/H_1.*

Proof. Denote $H^1_0 = \bigcup_{a\in H_1} H_a$.

Let $x, y \in zH_1$ for some $z \in H$. Then

$$x(H^1_0) = \bigcup_{a\in H_1} H_{ax} = \bigcup_{b\in xH_1} H_b\,, \quad y(H^1_0) = \bigcup_{b\in H_1} H_{ay} = \bigcup_{b\in yH_1} H_b\,.$$

But $x, y \in zH_1$ if and only if $xH_1 = yH_1$ and so we have $x(H_0^1) = y(H_0^1)$ if and only if $x, y \in zH_1$ for some $z \in H$. The desired decomposition R_1 is formed by factors $x(H_0^1)$, where x are representants of the classes of H/H_1. The lemma is proved.

Theorem 1. *Let H be an Abelian group of a finite order $m > 1$ and let $k \geqq 3$ be a natural number such that $(m, k!) = 1$. Then the following two statements are equivalent*:

1. *There exists a group $H_1 \cong H$ such that a hypergraph $\langle n \rangle_k$ has a simple decomposition by the group H_1.*

2. *m divides $\binom{n}{k}$ and divides precisely one of the numbers $n, n - 1, \ldots, n - - k + 1$.*

Proof. Let R be a simple decomposition of the hypergraph $\langle n \rangle_k$ by an Abelian group H_1 of order m. It is evident that m divides $\binom{n}{k}$ because the factors are isomorphic and so each factor contains the same number of edges.

The condition $(m, k!) = 1$ implies that the number m can be written in the form $m = m_1 . m_2 . \ldots . m_k$, where m_i are mutually prime and m_i divides $n - i + 1$. Then we can express the group H_1 as the direct product of cyclic groups $H_1 = = F_1 \times F_2 \times \ldots \times F_k$, where the order of the group F_i is equal to m_i.

Let $m_t > 1$ for some $1 \leqq t \leqq k$. We shall show that $m_i = 1$ for every $i \neq t$. Lemma 1 implies the existence of a simple decomposition R_t of the hypergraph $\langle n \rangle_k$ by F_t. Let v_1 be a vertex which is not a fix-point in all elements of F_t. Thus there exists $\alpha \in F_t$ such that $\alpha(v_1) = v_2, \alpha(v_2) = v_3, \ldots, \alpha(v_{k-1}) = v_k$. Let now $\beta(v_1) = v_1$ for some $\beta \in F_t$. Then we have by induction $\beta(v_j) = \beta\alpha(v_{j-1}) = \alpha\beta(v_{j-1}) = \alpha(v_{j-1}) = v_j$ for every $j = 2, 3, \ldots, k$ and thus β is conforming with the zero element of F_t on the set $h = \{v_1, v_2, \ldots, v_k\}$. The edge h is contained in a factor G_γ of the decomposition R_t. However, $h \in \beta(G_\gamma) = G_{\beta\gamma}$. This implies $\beta\gamma = \gamma$ and thus $\beta = \varepsilon$.

From this we have that for every vertex which is not a fixpoin with regard to the group F_t there exists a set of vertices which are images of this vertex by mappings $\alpha \in F_t$ and which has a cardinality equal to m_t. These sets are either disjoint or identical. Denote by S the system of these disjoint sets. Let $u \in A \in S$ and $\zeta \in F_r$, $r \neq t$. Let $\zeta(u) \in B \in S$. Now let us have $v \in A$. Then there exists $\alpha \in F_t$ such that $\alpha(u) = v$ On the other hand $\zeta(v) = \zeta\,\alpha(u) = \alpha\,\zeta(u) \in B$ and we have $\zeta(A) \subseteq B$.

The converse inclusion can be proved analogously and so we have $\zeta(A) = B$. Now let $A = B$. Then $\zeta(u) = \beta(u)$ for some $\beta \in F_t$. Let us have $x \in A$. Then $x = \gamma(u)$ for some $\gamma \in F_t$ and $\zeta(x) = \zeta\,\gamma(u) = \gamma\,\zeta(u) = \gamma\,\beta(u) = \beta\,\gamma(u) = \beta(x)$. This implies that ζ and β are identical on the set A. If we take now some k-tuple g from the set A we get $\beta(g) = \zeta(g)$ and thus $\beta = \zeta$, because otherwise the edge g would be included in two different factors which is a contradiction.

However, the equality $\beta = \zeta$ holds if and only if $\beta = \zeta = \varepsilon$. This implies: If $\zeta \neq \eta$ then $\zeta(A) \neq \eta(A)$, $\zeta, \eta \in F_r$, and thus for every set $A \in S$ there exists a system of cardinality m_r of disjoint sets of cardinality m_t. These systems are for different A either disjoint or identical. This implies that the system S splits into $(n - t + 1)/m_t.m_r$ subsystems. This is possible if and only if $m_r = 1$ which we want to prove.

Proof of the sufficient condition: Let H be a group of order m and let m satisfy the condition (2) in Theorem 1. Thus there exists such i that m divides $n - i$, $0 \leqq i \leqq \leqq k - 1$.

We shall define a group $H_1 \cong H$ by which we shall be able to construct the simple decomposition just found.

Choose i vertices from $\langle n \rangle_k$ and divide the remaining $n - i$ vertices into m-tuples which will be denote by $A_1, A_2, \ldots, A_z$. To every natural $x \leqq z$ and to every $\alpha \in H$ assign a vertex $u_x(\alpha) \in A_x$ such that

$$\beta\, u_x(\alpha) = u_x(\beta\alpha) \quad \text{for every} \quad \beta \in H\,.$$

In this way we define a group of automorphisms $H_1 \cong H$ on the set consisting of $n - i$ vertices. On the remaining i vertices define H_1 to be point stationary.

A simple decomposition of the hypergraph $\langle n \rangle_k$ by H_1 is constructed in the following way: We choose an arbitrary edge h_1 and insert it into the factor G_ε corresponding to the unit element of H_1. We insert the edges $\alpha(h_1)$ into the factors $G_\alpha = \alpha(G_\varepsilon)$ for every $\alpha \in H_1$. If we do not use all edges in this way, we insert an arbitrary one of them — for example h_2 — into the factor G_ε. Then we insert $\alpha(h_2)$ into $\alpha(G_\varepsilon) = G_\alpha$. We continue in this way, while we exhaust all edges. The decomposition obtained in this way is obviously a simple decomposition of $\langle n \rangle_k$ by the group $H_1 \cong H$. The proof is complete.

Remark 1. In the construction described above a weaker condition is sufficient for the existence of a simple decomposition, namely $(m, k) = 1$. We shall often use this fact in the sequel.

Theorem 2. *Let $m, k \geqq 3$, $m > k$, $(m, k) = 1$, $d \geqq 2$ be integers. Then $G_m^k(d)$ exists and*

$$G_m^k(d) \leqq m[(d-2)(k-1)+1] \quad \textit{if} \quad d \geqq 3\,,$$
$$G_m^k(2) \leqq 2m\,.$$

Proof. I. Let $d \geqq 3$. Denote $n = mt$, where $t = (d-2)(k-1)+1$. Denote the vertices of $\langle n \rangle_k$ by i_j, $1 \leqq i \leqq m$; $1 \leqq j \leqq t$. Obviously m divides $\binom{n}{k}$. Moreover, m divides n. Because $(m, k) = 1$, the sufficient condition for the existence of a simple decomposition of $\langle n \rangle_k$ by a cyclic group H of order m, generated by the element $\beta = (1_1, \ldots, m_1) \ldots \ldots (1_t, \ldots, m_t)$, is satisfied. Now we shall show that there exists a special simple decomposition of $\langle n \rangle_k$ by the group H, whose factors have the diameters equal to d. We construct the factors G_α, $\alpha \in H$ as follows:

1. Let the factor G_ε corresponding to zero of H contain a path of length $d - 2$ which is formed by the edges

$$h_i = \{1_{t_i}, \ldots, 1_{t_{i+1}}\}, \quad \text{where} \quad t_i = 1 + (i - 1)(k - 1); \quad i = 1, \ldots, d - 2;$$
$$f = \{1_1, 2_1, \ldots, k_1\}.$$

2. Let $A_i = \{2_i, \ldots, m_i\}$, $i = 1, \ldots, t$; let $\{B \cup \{2_j\}\} \in G_\varepsilon$ for an arbitrary $(k - 1)$-tuple $B \subset A_i$ and for every $j \neq i$.

3. Let $G_\alpha = \alpha(G_\varepsilon)$ for every $\alpha \in H$.

The diameter of the factors defined in this way is obviously equal to d. This is true for G_ε: the shortest path of length d is formed by the edges $\{2_3, 3_3, \ldots, k_3, 2_1\}$, $f, h_1, h_2, \ldots, h_{d-2}$.

The other factors are isomorphic to G_ε, thus their diameters are also equal to d.

The factors G_α need not form a decomposition of the hypergraph $\langle n \rangle_k$. Let S be a system of all edges, which are not included in any factor. The group H decomposes s into disjoint sets of cardinality m. Since $m > k$, there exists in each of these sets an edge h that it does not contain the vertices 1_i, where $1 \leqq i \leqq t$. Let then $h \in G_\varepsilon$ and $\alpha(h) \in G_\alpha$ for every $\alpha \in H$.

In this way we do not change the diameter of the factors and so we obtain a simple decomposition of $\langle n \rangle_k$ by the cyclic group of order m, which implies the existence of the number $G_m^k(d)$ and at the same time its upper bound.

II. Let $d = 2$. We decompose the hypergraph $\langle 2m \rangle_k$ into m factors with diameter two.

Obviously, the sufficient condition for the existence of a simple decomposition of $\langle 2m \rangle_k$ by a group H generated by the element $\beta = (1_1, \ldots, m_1)(1_2, \ldots, m_2)$ is satisfied.

Let $A_i = \{1_i, \ldots, m_i\}$, $i = 1, 2$ and let B be an arbitrary $(k - 1)$-tuple, $B \subset A_i$. Then let $h = \{B \cup \{1_j\}\} \in G_\varepsilon$ for $i \neq j$; $i, j = 1, 2$ and $\alpha(h) \in G_\alpha$ for every $\alpha \in H$.

The diameter of the factors constructred in this way is obviously two, because $d_{G_\varepsilon}(m_1, m_2) = 2$.

The factors G_α need not form a decomposition of the hypergraph $\langle 2m \rangle_k$. Let S be a system of all edges, which are not included in any factor. The group H decomposes S into disjoint sets of cardinality m. Since $m > k$, there exists in each of these sets an edge h that it does not contain the vertices m_1, m_2. Then let $h \in G_\varepsilon$ and $\alpha(h) \in G_\alpha$ for every $\alpha \in H$.

The edges added to G_α in this way do not change the diameter of G_α. Thus the factors G_α form a simple decomposition of $\langle 2m \rangle_k$ by the cyclic group H of order m into factors with diameter two, which implies the existence of the number $G_m^k(2)$ and its upper bound. The theorem is proved.

In the following considerations the concept of a "simple decomposition by a group is not sufficient. Thus we shall use decompositions by a group of greater order than the number of factors of the decomposition.

Theorem 3. *Let $m > k$, $(m, k) > 1$, $k \geqq 3$, $d \geqq 2$ be integers. Then $G_m^k(d)$ exists and*

$$G_m^k(d) \leqq km[(d-2)(k-1)+1] \quad \text{if} \quad d \geqq 3\,,$$
$$G_m^k(2) \leqq 2mk\,.$$

Proof. I. Let $d \geqq 3$. Denote $n = kmt$, where $t = (d-2)(k-1)+1$. We shall show the existence of a decomposition of $\langle n \rangle_k$ by a cyclic group H of order mk into m factors with diameter d.

Denote by i_j, $1 \leqq i \leqq km$; $1 \leqq j \leqq t$ the vertices of $\langle n \rangle_k$. Let the group H be generated by $\beta = (1_1, \ldots, (km)_1) \ldots (1_t, \ldots, (km)_t)$. The construction of the factors with diameter d proceeds as follows: Let the edges $h_j = \{m_j, (2m)_j, \ldots, (km)_j\} \in G_\varepsilon$ for $1 \leqq j \leqq t$ where G_ε is the factor corresponding to the zero element of H.

Let $g_1 \subset A_1^0 = \{1_1, \ldots, (km)_1\}$ be an adge containing the vertices m_1, $(m+1)_1$. The remaining vertices belonging to g_1 let be different from the vertices of h_1. The element β^m is obviously of order k and so let $\beta^{mr}(g_1) \in G_\varepsilon$ for $1 \leqq r \leqq k$.

Now we insert into the factor G_ε a path of length $d-2$: Denote

$$f_s = \{m_{t_s}, (2m)_{t_s+1}, \ldots, (km)_{t_{s+1}}\}\,,$$

where $t_s = 1 + s(k-1)$, $0 \leqq s < d-2$. Then let $\beta^{mr}(f_s) \in G_\varepsilon$ for every $1 \leqq r \leqq k$

Denote $A_j = \{i_j \mid 1 \leqq i \leqq km\} - h_j$ and take an arbitrary $(k-1)$-tuple $B \subset A_j$. Then insert $\beta^{mr}(B \cup (m+1)_i)$ into G_ε for every $i \neq j$; $i, j = 1, 2, \ldots, t$, $1 \leqq r \leqq k$.

The factor G_ε constructed in this way has a diameter equal to d, since

$$d_{G_\varepsilon}((m+1)_2, (m(k-q))_t) = d\,, \quad q \equiv d-3 \pmod{k}\,.$$

The shortest path of length d is formed by the edges

$$\{(m+1)_2, (m+1)_1, (m+2)_1, \ldots, (m+k-1)_1\}\,,$$
$$g_1, f_0, \beta^{m(k-1)}(f_1), \beta^{m(k-2)}(f_2), \ldots, f_k, \beta^{m(k-1)}(f_{k+1}), \ldots, \beta^{m(k-q)}(f_{d-3})\,.$$

Now define $G_{\beta^i} = \beta^i(G_\varepsilon)$, $0 \leqq i < m$. The factors G_{β^i} need not form a decomposition of $\langle n \rangle_k$. Let S be a system of all edges which are not in any factor. The group H decomposes S into disjoint sets. Since $m > k$, in each of these sets there exists an edge g that does not contain the vertices of h_j, $1 \leqq j \leqq t$. Let us insert the edges $\beta^{mr}(g)$, $1 \leqq r \leqq k$ into G_ε and their images in the mappings β^i into the factors G_{β^i}, $0 \leqq i < m$. In this way we obviously do not change the diameter of G_ε and the factors G_{β^i} form a decomposition of $\langle n \rangle_k$ into m factors with a diameter d which implies the existence of the number $G_m^k(d)$ and its upper bound.

II. If $d = 2$, the proof is analogous to the above one. It is not difficult to prove the existence of a decomposition of $\langle 2mk \rangle_k$ into m isomorphic factors with diameter two, because now we need not construct a path of length d. The theorem is proved.

In the end of this part we can say that in Theorems 2 and 3 the problem of the existence of a decomposition of a complete k-uniform hypergraph into isomorphic factors with a diameter d is affirmatively solved for d greater or equal to two and for the number of factors greater the than uniformity of the hypergraph.

DECOMPOSITIONS OF GRAPHS INTO ISOMORPHIC FACTORS WITH A GIVEN DIAMETER

In this part we shall prove the existence of the number $G_m^2(d)$ for $d \geqq 3$ and $m \geqq 4$. We also prove the existence of the number $H_m^2(d)$ for $d \geqq 3$ and for m which is a power of a prime different from two. Moreover, we shall show that the existence of the number $H_m^2(d)$ for m which is not a power of a prime, cannot be proved by the method of a simple decomposition by an Abelian group.

Theorem 4. *Let t, d and m be integers, $t > 2$, $d \in \{3, \ldots, t + 2\}$, $m > 3$ and m odd. Then the graphs $\langle mt \rangle_2$ and $\langle mt + 1 \rangle_2$ can be decomposed into m isomorphic factors with diameter d.*

Proof. I. First we prove the existence of a decomposition of $\langle mt \rangle_2$. Denote its vertices by $0_1, 1_1, \ldots, (m-1)_1, \ldots, 0_t, 1_t, \ldots, (m-1)_t$. Let $q \in \{0, 1, \ldots, t-1\}$.

We construct the factor G_t^q as follows:

a) $q > 1$:

Let
$$\begin{aligned}
X_1 &= \{[0_1, 2_1], [1_1, 2_1], [2_2, 3_2]\}\,,\\
X_2 &= \{[0_i, 1_{i+1}], [0_{i+1}, 1_i] \mid 1 \leqq i < q + 1\}\,,\\
X_3 &= \{[2_i, (2k+1)_i] \mid i = 1, \ldots, t;\ 2 \leqq k \leqq \tfrac{1}{2}(m-1)\}\,,\\
X_4 &= \{[0_i, 1_i] \mid 3 \leqq i \leqq q + 1\}\,,\\
X_5 &= \{[2_i, 4_i] \mid 2 \leqq i \leqq q + 1\}\,,\\
X_6 &= \{[1_i, 2_i], [0_i, 2_i] \mid q + 1 < i \leqq t\}\,,\\
X_7 &= \{[2_i, (2k)_j] \mid i \neq j;\ i, j = 1, \ldots, t;\ k \geqq 1\}\,,\\
X_8 &= \{[2_i, 3_j] \mid |i - j| > 1;\ i, j = 1, \ldots, t\}.
\end{aligned}$$

Then let $G_t^q = \bigcup_{a=1}^{8} X_a$ and the decomposition has the form
$$R = \{p(G_t^q), \ldots, p^m(G_t^q)\}\,,$$
where p is a cyclic permutation of order m on the set of vertices of the graph $\langle mt \rangle_2$ with the orbits $\{0_i, 1_i, \ldots, (m-1)_i\}$, $i = 1, 2, \ldots, t$.

R is relly a decomposition which we can verify by simply summing the edges in the factor G_t^q and making sure that no edge repeats. This follows immediately from the construction. The diameter of G_t^q is equal to $q + 3$ because, for example, $d(3_2, 0_{q+1}) = q + 3$. The shortest path of length $q + 3$ is formed by the edges

$$[3_2, 2_2], [2_2, 2_1], [2_1, 0_1], [0_1, 1_2], [1_2, 0_3], \ldots, [1_q, 0_{q+1}] \quad \text{for } q \text{ even},$$
$$[3_2, 2_2], [2_2, 2_1], [2_1, 1_1], [1_1, 0_2], [0_2, 1_3], \ldots, [1_q, 0_{q+1}] \quad \text{for } q \text{ odd}.$$

b) $q = 0$:

Let
$$Y_1 = \{[0_i, 2_i], [1_i, 2_i] \mid i = 1, 2, \ldots, t\},$$
$$Y_2 = \{[2_i, (2k)_j], [2_i, 3_j] \mid i \neq j;\ i, j = 1, 2, \ldots, t;\ k \geqq 1\},$$
$$Y_3 = \{[2_i, (2k + 1)_i] \mid i = 1, 2, \ldots, t;\ k \geqq 2\}.$$

Then let the factor $G_t^0 = Y_1 \cup Y_2 \cup Y_3$. Now we easily obtain the required decomposition by the permutation p. The factors of the decomposition have diameter $d = q + 3 = 3$ since, for example,

$$d(0_1, 0_t) = 3.$$

c) $q = 1$:

Delete the edge $[2_1, 1_1]$ from G_t^0 and insert there the edge $[0_1, 1_1]$. Then obviously $d_{G_t^0}(1_1, 0_t) = q + 3 = 4$ and we obtain the required decomposition by the permutation p.

II. We construct a decomposition of $\langle mt + 1 \rangle_2$ as follows: We add a vertex v and the edges $[v, 2_i]$, $i = 1, 2, \ldots, t$ into the factor G_t^q. It is evident that the diameter is preserved. The other factors of the decomposition are obtained by the permutation p_1 which coincides with p on its definition area, and v is a fix-point. The theorem is proved.

Corollary 1. *Let m be an arbitrary natural power of a prime different from two. Then $H_m^2(d)$ exists for $d \geqq 3$.*

Proof. Let $m = p^n$, where p is a prime, $p \neq 2$. Let p divide $\binom{N}{2}$. Then

1) either p^n divides N,
2) or p^n divides $N - 1$

and so either $p^n t = N$ or $p^n t + 1 = N$ for some t. This implies that all suitable numbers are of the form mt or $mt + 1$. By Theorem 4 for an arbitrary $d \geqq 3$ there exists t_0 such that for every $t \geqq t_0$ there exists a decomposition of $\langle mt \rangle_2$ and $\langle mt + 1 \rangle_2$ into m isomorphic factors with a diameter d so that $H_m^2(d) \leqq mt_0$.

Remark 2. We can take $t_0 = d$. For $d \geqq 5$ we can take $t_0 = d - 2$ and thus $H_m^2(d) \leqq md - 2m$. In the paper [1] an upper bound of the number $F_m^2(d)$ is found in the form

$$F_m^2(d) \leqq md - m \quad \text{for} \quad d \geqq 3.$$

Since $F_m^2(d) \leqq H_m^2(d)$ we have $F_m^2(d) \leqq md - 2m$ for $d \geqq 5$.

Theorem 5. *Let m be an odd natural number, which is not a power of a prime. Then there exists an arbitrarily large number N such that m divides $\binom{N}{2}$ and there exists no Abelian group which simply decomposes the graph $\langle N\rangle_2$ into m isomorphic factors.*

Proof. The assumptions imply that m can be written in the form $m = m_1 \, . \, m_2$, where $m_1, m_2 \neq 1$; m_1, m_2 are coprime.

The diophantic equation

$$m_1 x - m_2 y = 1$$

has obviously an infinite number of solutions. Choose from them a solution x_0, y_0 which is sufficiently large and denote $N = m_1 x_0$. Then put $m_2 y_0 = N - 1$. It is evident that m divides neither N nor $N - 1$. Nonetheless, m divides $\binom{N}{2}$.

Now we can use the theorem proved by B. Zelinka in [4]: The graph $\langle n\rangle_2$ can be decomposed by an Abelian group of order m into m factors if and only if m is odd and

1) m divides $\frac{1}{2}(n - 1)$ or m divides n, if n is odd

or

2) m divides $\frac{1}{2}n$ or m divides $n - 1$, if n is even.

Obviously m does not satisfy the necessary condition of the existence of a simple decomposition of the graph $\langle N\rangle_2$ into m factors by an Abelian group of order m. The theorem is proved.

This implies the following statement:

Corollary 2. *The method used in the proof of Theorem 4 – i.e. a simple decomposition by an Abelian group – cannot be used for proving the existence of the number $H^2_m(d)$ in the case that m is not a power of prime.*

It remains to explore the existence of the number $G^2_m(d)$ for m even.

Theorem 6. *Let m, d be natural numbers, $m \geqq 4$ and even, $d \geqq 3$. Then the number $G^2_m(d)$ exists and*

$$G^2_m(d) \leqq 2m(d - 1) \quad \text{if} \quad d > 3\,,$$
$$G^2_m(3) \leqq 2m\,.$$

Proof. Let $d \geqq 4$ and $m = 2k$, $m \geqq 4$. Denote by i_j the vertices of the graph $\langle 2m(d - 1)\rangle_2$, where $1 \leqq i \leqq 2m$; $1 \leqq j \leqq d - 1$. Put

$$X_1 = \{[2_i, a_i], [(m + 2)_i, (m + a)_i] \mid 2 \leqq i \leqq d - 1;\ 3 \leqq a \leqq m\}\,,$$
$$X_2 = \{[1_i, 1_{i+1}], [(m + 1)_i, (m + 1)_{i+1}] \mid 1 \leqq i \leqq d - 2\}\,,$$

$X_3 = \{[1_1, 1_i] \mid 2 \leqq i \leqq d - 1\}$,
$X_4 = \{[1_1, a_1], [(m + 1)_1, (m + a)_1] \mid 2 \leqq a \leqq m\} \cup \{[1_1, (m + 1)_1]\}$,
$X_5 = \{[1_1, a_i], [(m + 1)_1, (m + a)_i] \mid 2 \leqq a \leqq m;\ 2 \leqq i \leqq d - 1\}$.

Then let $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \subseteq G_\varepsilon$.

The group H generated by $\alpha = (1_1, \ldots, (2m)_1), \ldots, (1_{d-1}, \ldots, (2m)_{d-1})$ decomposes the set of all edges of $\langle 2m(d - 1)\rangle_2$ into disjoint sets of cardinality at least m. Since $m > 2$, in each of these sets there exists an edge that does not contain the vertices of the form $1_i, m_i$, where $1 \leqq i \leqq d - 1$. Insert this edge into G_ε. Then $\alpha^j(G_\varepsilon) = G_{\alpha^j}$, $j = 0, 1, \ldots, m$ evidently form the required decomposition, because for example $d_{G_\varepsilon}(2_1, (m + 1)_{d-1}) = d$.

II. If $d = 3$ put $G_\varepsilon = X_4$. It is evident that $d_{G_\varepsilon}(2_1, (m + 2)_1) = 3$. The proof is complete.

Remark 3. In [2] it was proved that $G_m^2(3) = 2m$.

*

In all above considerations we were not concerned with decompositions into smaller number of factors than the uniformity of the hypergraph. The following theorem gives a sufficient explanation.

The number $F_m^k(d)$ – if it exists – is the smallest number for which there exists a decomposition of $\langle F_m^k(d)\rangle_k$ into m factors with diameter d.

Theorem 7. *Let m, k, d be natural numbers $m \leqq k$, $k \geqq 3$, $d \geqq 4$. Then $F_m^k(d)$ does not exist.*

Proof. Evidently it is sufficient to prove our statement for $m = k$. So let $m = k$ and let $F_m^m(d)$ exist. Then the hypergraph $\langle F_m^m(d)\rangle_m$ can be decomposed into m factors with diameter d. Denote these factors by $F_1, F_2, \ldots, F_m$.

Let x, y be vertices of F_m. Let a, b be such vertices that their distance is d in F_1. Then the distance between x and either a or b is greater than one. Let this be the case for x and a. The second case is analogous. In the factor F_i there exists a vertex v_i such that the distance between x and v_i is greater than one for every $i = 2, \ldots, m - 1$. Then the edge $h = \{x, a, v_2, \ldots, v_{m-1}\} \in F_m$. The distance either between y and a or between y and b in the factor F_1 is greater than one. Let w_i be such vertex that the distance between y and w_i in F_i is greater than one for every $i = 2, \ldots, m - 1$. Then it is evident that

$$f = \{a, y, w_2, \ldots, w_{m-1}\} \in F_m \quad \text{or} \quad g = \{b, y, w_2, \ldots, w_{m-1}\} \in F_m .$$

If $f \in F_m$, then the distance between x and y in the factor F_m is less than or equal to two.

Let now $f \in F_1$ and $g \in F_m$. Consider the edge $p = \{x, y, v_2, \ldots, v_{m-1}\}$. Two cases are possible:

1. $p \in F_m$. Then $d_{F_m}(x, y) = 1$.

2. $p \in F_1$. If $q = \{b, v_2, \ldots, v_{m-1}\} \in F_m$ then $d_{F_m}(x, y) \leqq 2$. If $q \in F_1$ then $d_{F_1}(a, b) \leqq 3$ since $q, g, f \in F_1$, which is a contradiction.

Since the vertices were chosen arbitrarily, we proved that the diameter of F_m is smaller than or equal to two, which contradicts the assumption $d \geqq 4$. The theorem is proved.

Theorem 1 involves the assumption $(m, k!) = 1$. A necessary and sufficient condition for the existence of a simple decomposition by an Abelian group was found also if $(m, k) = 1$ and the proof will appear in a future paper.

It remains an unsolved problem whether the number $H_m^k(d)$ exists for not a power of a prime. Another unsolved problem is whether $G_m^k(d) = H_m^k(d)$ for $m, k \geqq 2$ and $d \geqq 1$. In [5] it was conjectured that $G_m^2(d) = H_m^2(d)$. A further problem is to find an example that $F_m^k(d) \neq G_m^k(d)$.

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Author's address: 886 25 Bratislava, Obrancov mieru 49, ČSSR (Matematický ústav SAV).