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SEMIGROUP STRUCTURE IN COMPACTIFICATIONS OF ORDERED SEMIGROUPS

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1. INTRODUCTION

The natural appearance of semigroups with compact topologies in which multiplication is continuous on one side only (for example in Chapter 7 of [3] and in § 3 of [4]) suggests that an investigation of these structures may be valuable. Ruppert, in [8] and elsewhere, has already begun this task in a general setting. However, many of the examples which arise in practice appear as Stone-Čech compactifications βS of semigroups S, as in [3], and in any case, such semigroups would be expected to be, and are, in a sense universal ([4], Theorem 3.4, and (1.1) below). Work on transformation groups [3] and recent, unpublished, investigations by R. J. BUTCHER at Sheffield indicate that βZ, where Z is the additive group of integers, is a very complex semigroup. Success in determining the structure of βS will therefore only be possible if S does not contain a copy of Z. In this paper, we consider certain ordered sets S which can be made into semigroups by giving them the multiplication max (i.e. $xy = \max \{x, y\}$ for $x, y \in S$).

Before we describe our results, we will establish some terminology and recall how βS becomes a semigroup when S is discrete. We shall say that a semigroup S with a topology is a *right topological semigroup* if, whenever $x \in S$ and $y_i \to y$ in S, then $xy_i \to xy$ in S. We describe this situation by saying that multiplication in S is *continuous on the right*; but care is needed, since the mapping which is continuous is multiplication *on the left* by x, for each $x \in S$.

Now let S be discrete. Fix $y \in S$. The map $x \mapsto xy$ is trivially continuous, and so it extends in a unique way to a map of βS to βS which we again denote by $x \mapsto xy$. Now, for $x \in \beta S$, the map $y \mapsto xy$ of S into βS is continuous, and hence it extends to a uniquely determined continuous map (again denoted by $y \mapsto xy$) of βS to βS. This product xy gives the multiplication we sought. (This proof is given by ELLIS in Theorem 7.1 of [3], though he was concerned only with groups S; a more general result – about topological semigroups – is to be found in § 3 of MILNES [4].)

The first question we consider is whether this construction will extend to the case in which S is a totally ordered, separately continuous (non-discrete) semigroup. After some preliminary results about totally ordered topological semigroups, we discover that this is the case if and only if the topology of S is locally convex. The proof of the theorem (2.8) includes a full description of the structure of βS in this case.

In § 4, we determine the multiplication in βS when S is a direct product of two monotone sequences. Not surprisingly, the results depend on whether the sequences are increasing or decreasing. More worthy of note is the fact that if one sequence is increasing and the other decreasing, βS may not be idempotent (Theorem 4.3 (iv)). We indicate how an extension to finite products of monotone sequences can be obtained (though the situation becomes very complex).

In order to carry out the investigation just described, some knowledge of the compactification $\beta(N \times N)$ (where N is the set of positive integers) is necessary. Of course, $\beta(N \times N)$ is homeomorphic with βN, but by retaining the product structure we are able to show that it contains subspaces homeomorphic to $N^* \times N_P^*$, where $N^* = \beta N \setminus N$ and N_P^* is N^* with a P-topology (in the sense of [9]). We believe that this fact, besides being vital to the descriptions of the multiplications in § 4, is of interest in itself.

In our final section we show that the semigroup βS may be a useful tool for the solution of problems. To be precise, we use βS for a discrete semigroup S to show that if $T_1, \ldots, T_n$ are totally ordered topological semigroups, then the weakly almost periodic compactification of $T_1 \times \ldots \times T_n$ is just the product of the weakly almost periodic compactifications of the individual semigroups T_i. A general criterion for this to occur for arbitrary semigroups $T_1, \ldots, T_n$ has been given by J. F. Berglund and P. Milnes in a paper yet to appear, but it is not clear that their conditions give a proof easier than the one we present. It turns out that the weakly almost periodic compactification of $T_1 \times \ldots \times T_n$ coincides with its almost periodic compactification. (These results are relevant to our study of measure algebras in [7].)

We shall have occasion to use the universal property of the semigroup βS. The result is a particular case of Theorem 3.4. of [4].

1.1. Proposition. *Let S be a discrete semigroup, and let S_0 be a semigroup with a compact topology and a right continuous multiplication. Let $\Phi : S \to S_0$ be a homomorphism, and suppose that, for each $y \in S$, $s \to s\,\Phi(y)$ is a continuous map on S_0. Then there is a unique continuous homomorphism $\hat{\Phi} : \beta S \to S_0$ which extends Φ.*

The question this proposition answers is when the unique continuous extension $\hat{\Phi}$ of Φ to βS is a homomorphism. (A proof is easy to obtain by following the steps of the construction of the multiplication in βS given above; the reader who requires a proof can turn to [4].) The conditions of the proposition are essentially necessary, for if $\hat{\Phi}$ is surjective, the continuity of $x \mapsto xy$ (resp. $x \mapsto yx$) in βS leads immediately to the continuity of $s \mapsto s\,\hat{\Phi}(y)$ (resp. $s \mapsto \hat{\Phi}(y)\,s$) in S_0.

We will now give an example to show that there can be no relaxation of the conditions of this proposition even when S is totally ordered. We do this by exhibiting a right-continuous compact semigroup S_0 containing a totally ordered subsemigroup S for which multiplication on the right by elements of S is not continuous. There can then be no continuous homomorphism $\beta S \to S_0$ which extends the identity on S.

1.2. Example. Let S be a totally ordered set with elements $a_1 < b_1 < a_2 < b_2 < \ldots$, with the discrete topology and multiplication max. Let $S_0 = \{a_1, a_2, \ldots, x\} \cup \cup \{b_1, b_2, \ldots, y\}$ be the union of two disjoint clopen sets whose compact Hausdorff topology is determined by the requirements $a_n \to x$, $b_n \to y$. Multiplication in S_0 is to induce max on S, to yield $a_n x = b_n x = x$ for all n, and all other products are to have the value y. It is easy to check that S_0 is a semigroup.

To discuss continuity of multiplication, we observe that there are essentially only two convergent sequences. Moreover, if $a_n \to x$ and m is fixed, we find $b_m a_n = a_n$ eventually, so $\lim_n (b_m a_n) = \lim_n a_n = x = b_m(\lim_n a_n)$; this and similar arguments show that multiplication is continuous on the right. On the other hand, $\lim_n (a_n b_m) = \lim_n a_n = x$, while $(\lim_n a_n)\, b_m = x b_m = y$, so that multiplication on the right by elements of S is not continuous.

2. TOTALLY ORDERED SEPARATELY CONTINUOUS SEMIGROUPS

Let T be a totally ordered set made into a commutative semigroup by giving it the multiplication max.

2.1. Notation and definitions. For $x, y \in T$ the sets $]x, y[= \{z \in T : x < z < y\}$, $]-\infty, y[= \{z \in T : z < y\}$ and $]x, \infty[= \{z \in T : x < z\}$ (which notations we use whether or not T has maximal or minimal elements) are called *open intervals*. The *closed intervals* $[x, y]$, $]-\infty, y]$ and $[x, \infty[$ are obtained by replacing $<$ by $\leqq$. A subset I of T is *convex* if $x, y \in I$ and $x < z < y$ imply $z \in I$. A *segment* U has the property that if $y \in U$ and $z < y$, then $z \in U$. The *interval topology* on T has the open intervals as a base for its open sets [2]. (A compact Hausdorff topology on T in which open intervals are open sets is necessarily the interval topology.) Any topology which has a base for its open sets consisting of convex sets is called *locally convex*.

We begin with two results about the relationship between the continuity of max and the separation properties of the topology.

2.2. Proposition. *Let T have a topology in which points are closed (ie. T has a T_1-topology). Then max is separately continuous if and only if both*

(i) $]x, \infty[$ *is open for each* $x \in T$, *and*
(ii) *if* U *is open,* $]-\infty, x[\cup U$ *is open for each* $x \in U$.

Proof. Suppose max to be separately continuous. Let $x \in T$, so that $T \setminus \{x\}$ is open. By separate continuity,

$$]x, \infty[= \{y : yx \in T \setminus \{x\}\}$$

is open, and if U is open and $x \in U$,

$$]-\infty, x[\cup U = \{y : yx \in U\}$$

is open.

To see the converse, take $x \in T$ and consider $y \mapsto yx$. We shall show this is continuous at $z \in T$. There are two cases. First, when $z > x$, we have $zx = z$. Let U be any neighbourhood of z and observe that, from (i), $V = U \cap]x, \infty[$ is again a neighbourhood of z. Then $Vx = V \subseteq U$, which shows continuity in this case. Secondly, take $z \leqq x$, so that $zx = x$. Let U be an open neighbourhood of x, and then $V =]-\infty, x[\cup U$ is an open neighbourhood of z, by (ii). Also, $Vx \subseteq U$, and so continuity is again proved.

2.3. Examples. (i) Separate (even joint) continuity does not imply that a T_1-topology must be Hausdorff.

Let T be given the topology with a base of open sets of the form $]x, \infty[\setminus F$ ($x \in T$, $F \subseteq T$ finite). Then max is jointly continuous, and the topology is T_1 but not Hausdorff.

(ii) Separate continuity in a T_1-topology does not imply joint continuity.

This example is achieved by giving T the topology with a base of open sets of the form $[x, \infty[\setminus S$ ($x \in T$, S either finite or a strictly decreasing sequence $\{x_n\}$ with $\inf \{x_n\} = x$).

The second example is of interest in view of the next result.

2.4. Proposition. *Let* T *be Hausdorff, and suppose max is separately continuous. Then every open interval is an open set (i.e. the topology of* T *is finer than the interval topology). Conversely, if* T *has a topology finer than the interval topology, it is Hausdorff and max is jointly continuous.*

Proof. First assume T is Hausdorff and separately continuous. We show that an open interval $]x, y[$ is a neighbourhood of each point $z \in]x, y[$. Let W, V be disjoint open neighbourhoods in T of z and y respectively. By Proposition 2.2, $]-\infty, y[\cup V$ is open, and $]x, \infty[$ is open. Hence

$$W \cap]x, y[= W \cap (]-\infty, y[\cup V) \cap]x, \infty[$$

is open. It is clearly a neighbourhood of z contained in $]x, y[$.

Conversely, the interval topology is obviously Hausdorff. To prove the continuity of multiplication, let $x, y \in T$ and let W be a neighbourhood of xy. We consider two cases. If $x = y$, continuity follows from the relation $WW = W$. If $x < y$, we take disjoint intervals U, V which are neighbourhoods of x and y respectively. Then $V \cap W$ is again a neighbourhood of $y (= xy)$, and $U(V \cap W) = V \cap W \subseteq W$.

Below, we shall be concerned with the topological properties of the suprema of segments. We look more closely at segments in the following lemma.

2.5. Lemma. *Let $U \subseteq T$ be a segment. Then either* (i) *for some x, $U =]-\infty, x[$, or* (ii) *for some x, $U =]-\infty, x]$, or* (iii) *neither* (i) *nor* (ii) *holds and $U = \bigcup_{x \in U}]-\infty, x[$ and $T \setminus U = \bigcup_{y \in T \setminus U}]y, \infty[$. If, in addition, T is separately continuous in a Hausdorff topology, then in case* (i) *U is clopen if $x \notin \overline{U}$; in case* (ii) *U is clopen if $x \notin \overline{T \setminus U}$; and in case* (iii) *$U$ is always clopen.*

Proof. If U is not of the form (ii), then for each $z \in U$ there is $x \in U$ with $z < x$; hence $U \subseteq \bigcup_{x \in U}]-\infty, x[\subseteq U$. Similarly, if U is not of the form (i), $T \setminus U$ is not of the form $[x, \infty[$, so that $T \setminus U = \bigcup_{y \in T \setminus U}]y, \infty[$.

Let $U =]-\infty, x[$. Then, using 2.4, $\overline{U} \subseteq]-\infty, x]$. Hence U is closed if and only if $x \notin \overline{U}$. Also, U, being an open interval, is open. Case (ii) is dealt with in the same way. In case (iii), both U and $T \setminus U$ are unions of open sets, so open.

We next construct an order compactification αT of a Hausdorff separately continuous semigroup T. If $U \neq \emptyset$ is a segment for which $\overline{U}$ does not contain a supremum for U, we adjoin an element $s(U)$ to T; if $U \neq T$ is a segment for which $\overline{T \setminus U}$ does not contain an infimum for $T \setminus U$, we adjoin an element $i(U)$ to T. We order the resulting set αT by writing, for example,

$$x < s(U) < i(U) < y \quad (x \in U,\ y \in T \setminus U).$$

Then αT becomes a totally ordered set in which each subset has both a supremum and an infimum. The order topology therefore makes αT a compact Hausdorff space, and when it is given the multiplication max it becomes a jointly continuous semigroup. (The space αT is in fact the order compactification mentioned in [6], especially on page 104, when T is completely regular.)

2.6. Proposition. *The inclusion $T \subseteq \alpha T$ is continuous. The topology αT induces on T is the finest locally convex topology coarser than the original topology of T; it has a sub-base for its open sets consisting of the open intervals, the clopen segments and complements of clopen segments.*

Proof. Lemma 2.5 together with the construction of αT shows that the topology induced on T by αT does have the sub-base asserted in the proposition, and that therefore the inclusion $T \to \alpha T$ is continuous. To see that it is the finest locally convex topology coarser than the original topology of T, notice first that it is certainly locally convex. Then, let I be convex and open in the original topology of T. Take

$x \in I$. Then both the segment $U =]-\infty, x[\cup I$ and the set $I \cup]x, \infty[= T \setminus V$, say, which is the complement of a segment V, are open in T, by 2.4. We now have three cases to consider. If U is of the form (i) of 2.5 it is clearly open in the topology induced by αT. If U is of the form (ii), say $U =]-\infty, y]$, it is closed as well as open in the original topology and so $y \notin T \setminus U = \overline{T \setminus U}$. The construction of αT then shows U is open in the induced topology. In case (iii), U is a union of open intervals, and so is open in the induced topology. Similarly, we see that $T \setminus V$ is open in the induced topology, and hence $I = U \cap (T \setminus V)$ is also open. We have proved that every open convex set in the original topology is open in the induced topology, and the proposition is now proved.

Since αT is compact, the locally convex topology described in Proposition 2.6 is completely regular. We now give two examples, the first to show that T can be Hausdorff without being completely regular, and the second to show that T can be completely regular without having the topology of 2.6.

2.7. Examples. (1) Let T be the usual interval $[0, 1]$ in the real line, and take the basic open sets in T to be of the form $I \setminus C$ where I runs through open intervals and C runs through countable sets. Then T is Hausdorff, not regular, and max is jointly continuous (Proposition 2.4).

(2) Again take $T = [0, 1]$. Let (a_n) be any fixed strictly increasing sequence in T with $a_n \nearrow 1$. Let $T \setminus \{a_1, a_2, \ldots\}$ bear the subspace topology and let $\{a_1, a_2, \ldots\}$ be clopen and bear the discrete topology. Then T is completely regular, but does not have a locally convex topology. (The finest locally convex topology coarser than the topology of T has as a base of open sets the open intervals and finite subsets of $\{a_1, a_2, \ldots\}$.)

We now consider the question of whether βT can be made into a semigroup by the procedure of § 1. Since αT is a compact topological space, there is a unique continuous map $\psi : \beta T \to \alpha T$ which is the identity on T. (There is an interesting parallel here with Proposition 10.34 of [9].)

2.8. Theorem. *Let T be Hausdorff and separately continuous. The following assertions are equivalent.*

(*i*) *βT has a right-continuous multiplication for which the elements of T commute with all elements of βT.*

(ii) *The map $\psi : \beta T \to \alpha T$ is such that $\psi^{-1}(x)$ contains only the point x for $x \in T$ (or, in other words, $\psi(\beta T \setminus T) = \alpha T \setminus T$).*

(iii) *The finest completely regular topology coarser than the original topology of T is locally convex.*

Proof. (i) implies (ii). Let $x \in T$, and suppose $y \in \beta T$ with $\psi(y) = \psi(x)$. Assume $y \neq x$. Then

$$y \in \overline{]-\infty, x[\cup]x, \infty[} = \overline{]-\infty, x[} \cup \overline{]x, \infty[}\,.$$

Suppose first $y \in \overline{]-\infty, x[}$. Then there is a net (y_j) in $]-\infty, x[$ with $y_j \to y$. As also $x \in \overline{]-\infty, x[}$, we can also find (x_i) in $]-\infty, x[$ with $x_i \to x$. We shall obtain a contradiction by proving $y = xy = x$. Indeed, for a fixed i, $\psi(y) \in]x_i, \infty[$, and so, eventually, $x_i \leqq y_j$, whence

$$x_i y = \lim_j x_i y_j = \lim_j y_j = y$$

and because $x_i, x \in T$ and so commute with elements of βT,

$$xy = yx = \lim_i y x_i = \lim_i x_i y = \lim_i y = y .$$

Then again, for each j, $y_j < x$ so that

$$xy = \lim_j x y_j = \lim_j x = x .$$

A similar contradiction is achieved if $y \in \overline{]x, \infty[}$.

(ii) implies (i). Suppose (ii) holds. Define multiplication in βT by

$$\begin{aligned} xy &= y \quad \text{if} \quad \psi(x) < \psi(y) ; \\ xy &= x \quad \text{if} \quad \psi(x) = \psi(y) \quad \text{and} \quad x(= y) \in T ; \\ xy &= x \quad \text{if} \quad \psi(x) = \psi(y) = s(U) \quad \text{for some} \quad U ; \\ xy &= y \quad \text{if} \quad \psi(x) = \psi(y) = i(U) \quad \text{for some} \quad U . \end{aligned}$$

It is then easy (but tedious) to check that multiplication has the properties required in (i).

(ii) is equivalent to (iii). The topology βT induces on T is the finest completely regular topology coarser than the original one. In view of Proposition 2.6, the present equivalence follows from the following lemma

2.9. Lemma. *The topologies induced on T by αT and βT coincide if and only if* (ii) *holds.*

Proof. Let Φ be the identity map from T with the topology induced by αT to T with the topology induced by βT. Its inverse, ψ restricted to T, is continuous, so the question is whether Φ is continuous.

Suppose (ii) holds. Let $x \in T$, $x_i \to x$ in T in the topology of αT. Then $(\Phi(x_i))$ is a net in the compact set βT. Let $(\Phi(x_j))$ be any convergent subnet, say $\Phi(x_j) \to y \in \beta T$. As ψ is continuous $x_j = \psi(\Phi(x_j)) \to \psi(y)$, and as (x_j) is a subnet of (x_i), $\psi(y) = x$. By (ii), this means $y = \Phi(x)$. We conclude $\Phi(x_i) \to \Phi(x)$, and hence that Φ is continuous.

Now suppose Φ is continuous. Let $x \in T$, $y \in \beta T$, $\psi(x) = \psi(y)$. Let (x_i) be a net in T with $x_i \to y$ in βT. Then $\psi(x_i) \to \psi(y)$, and so $x_i = \Phi(\psi(x_i)) \to \Phi(\psi(y)) = \Phi(\psi(x)) = x$. Thus $y = x$, and (ii) follows.

Example 2.7(2) shows that the conditions of this theorem are not always satisfied. They are, of course, when T has the discrete topology, and the proof that (ii) implies (i) gives a complete description of the multiplication in this case. Dr. J. W. BAKER (in a personal communication to the authors) has a criterion, in terms of the compactness of sets of translates of functions, for βS to be a right continuous semigroup for a general topological semigroup S.

Later (in section 5) we shall see that αT is both the almost periodic and the weakly almost periodic compactification of T.

3. A TOPOLOGICAL CONSTRUCTION

Before we proceed to discuss in detail the multiplication in $\beta(N \times N)$, we shall need to know a little about its structure. Of course, $\beta(N \times N)$ is homeomorphic with βN (which has been the subject of many investigations [9]) but to say this is to lose information which is useful in considering semigroup properties. For our purposes it is enough to find certain subsets embedded in $\beta(N \times N)$, and this we do in Proposition 3.1; however, an explicit description of the manner of the embedding is also of interest, and this we provide in Proposition 3.3.

Below, we shall denote by $|X|$ the set underlying the topological space X.

3.1. Proposition. *Let X and Y be completely regular (which we take to include Hausdorff) topological spaces. Then there is a (unique) completely regular topology τ on $|\beta Y \times X|$ such that $\beta Y \times \{x\}$ is naturally homeomorphic with βY for each $x \in X$ and $Y \times X$ is C^*-embedded in $(|\beta Y \times X|, \tau)$ (i.e. every continuous bounded function on $Y \times X$ extends continuously to $(|\beta Y \times X|, \tau)$).*

Proof. Let f be bounded and continuous on $Y \times X$. For each $x \in X$, the map $y \mapsto f(y, x)$ extends to a unique continuous function on $\beta Y \times \{x\}$, which we denote by $\hat{f}(\cdot, x)$, such that $\sup\{f(y, x) : y \in Y\} = \sup\{\hat{f}(y, x) : y \in \beta Y\}$. We see that $\hat{f}$ is defined on $|\beta Y \times X|$ and has the same bounds as f. We give $|\beta Y \times X|$ the topology determined by the functions $\hat{f}$; it is completely regular, and the proposition follows.

We shall now give a construction for the topology of Proposition 3.1 in the case in which $Y = N$. We write $N^* = \beta N \setminus N$ (so that N^* is the 'growth' of N in the sense of [9]).

3.2. Notation. We denote by X_P the topological space whose underlying set is $|X|$ and whose topology has a base consisting of sets U such that there exists a sequence (V_n) of open sets in X with $U = \bigcap_n V_n$. (Then X_P is the P-space coreflection of X; see exercise 10B of [9]. Some properties of P-spaces are given in [5].)

It is convenient to give the definition of the topology of Proposition 3.1 in terms of neighbourhoods rather than open sets.

3.3. Proposition. *Let τ be the topology of Proposition* 3.1 *on $|\beta N \times X|$. Then, if $(n, x) \in N \times X$, a set U is a neighbourhood of (n, x) if and only if $U \cap |N \times X|$ is a neighbourhood of (n, x) in the usual topology. Again, if $(y, x) \in |N^* \times X|$ then U is a neighbourhood of (y, x) if and only if it contains a set of the form*

$$\Big(\bigcup_{n \in W \cap N} (\{n\} \times V_n)\Big) \cup \Big((W \cap |N^*|) \times \Big(\bigcap_{n \in W \cap N} V_n\Big)\Big)$$

where W is a neighbourhood of y in βN, and for each $n \in W \cap N$, V_n is a neighbourhood of x in X.

Remarks. (i) In neighbourhoods of the second kind, $\bigcap_n V_n \subseteq V_n$ for each n, so the above expression may be written a little more simply as

$$\Big(\bigcup_{n \in W \cap N} (\{n\} \times V_n)\Big) \cup \Big(W \times \Big(\bigcap_{n \in W \cap N} V_n\Big)\Big).$$

(ii) The topology described in the present proposition induces on $|N^* \times X|$ the topology $N^* \times X_P$.

Proof. Let $(y, x) \in |\beta N \times X|$. We shall show that if f is an extension of a continuous function on $N \times X$ as in 3.1, and $f(y, x) = 1$, then there is a neighbourhood U of the form described with $f(U) > 0$; and conversely, if U is of the form described, there is a function f with $0 \leqq f \leqq 1$, $f(y, x) = 1$ which extends a continuous map on $N \times X$ and which vanishes off U. This will prove the proposition. Both parts are easy if $(y, x) \in N \times X$, so we may assume $(y, x) \in |N^* \times X|$.

Take f to be the extension of a continuous function satisfying $f(y, x) = 1$. Let $1 > \varepsilon > 0$. Since the restriction of f to $\beta N \times \{x\}$ is continuous, we can find an open neighbourhood W of y in βN such that $f(z, x) > \varepsilon$ for $z \in W$. If $n \in W \cap N$, $f(n, x) > > \varepsilon$, so we can find a neighbourhood V_n of x in X such that $f(n, t) > \varepsilon$ for $t \in V_n$. Now consider $(z, t) \in W \times \bigcap_{n \in W \cap N} V_n$. As W is open, each $z \in W$ belongs to $\overline{W \cap N}$; as f is continuous on $\beta N \times \{t\}$, it follows that $f(z, t) \geqq \varepsilon > 0$. Hence f is strictly positive on

$$\bigcup_{n \in W \cap N} (\{n\} \times V_n) \cup \Big(W \times \bigcap_{n \in W \cap N} V_n\Big).$$

To prove the converse, we take a neighbourhood of $(y, x) \in |N^* \times X|$ of the prescribed form. We may suppose W is closed, and we shall assume the neighbourhoods V_n decrease with n (for if they do not, we may replace V_n by the finite intersection $V'_n = \bigcap\{V_m : m \leqq n,\ m \in W \cap N\}$). Define a continuous function f on $N \times X$ by taking $f(n, x) = 0$ for all x if $n \notin W$, and by taking $f(n, \cdot)$ to be a continuous function with $0 \leqq f(n, \cdot) \leqq 1$, $f(n, x) = 1$, and which vanishes off $\{n\} \times V_n$ for $n \in W \cap N$. By Proposition 3.1, f extends in a unique way to $\beta N \times X$. Since $f(n, x) = 1$ for each $n \in W \cap N$ and W is a neighbourhood of y, we see $f(y, x) = 1$. If $t \notin \bigcap_n V_n$, then $f(n, t) \neq 0$ only if $n \in W \cap N$ and $t \in V_n$, and since the V_n's decrease, this implies that $f(n, t) \neq 0$ only for a finite number of values of n. Thus, $f(z, t) = 0$ for all $z \in N^*$. Since W is closed, we also see that if $z \notin W$, $f(z, t) = 0$ for all $t \in X$. Thus f vanishes off our given neighbourhood.

We shall now give our description of $\beta(N \times N)$. To establish the notation we use in the sequel, we shall write $\omega = \{1, 2, 3, \ldots\}$ and consider ω as an ordered set (i.e. the first infinite ordinal). We have tried to represent $\beta(\omega \times \omega)$ schematically in fig. 1. The diagram should be interpreted in the following way.

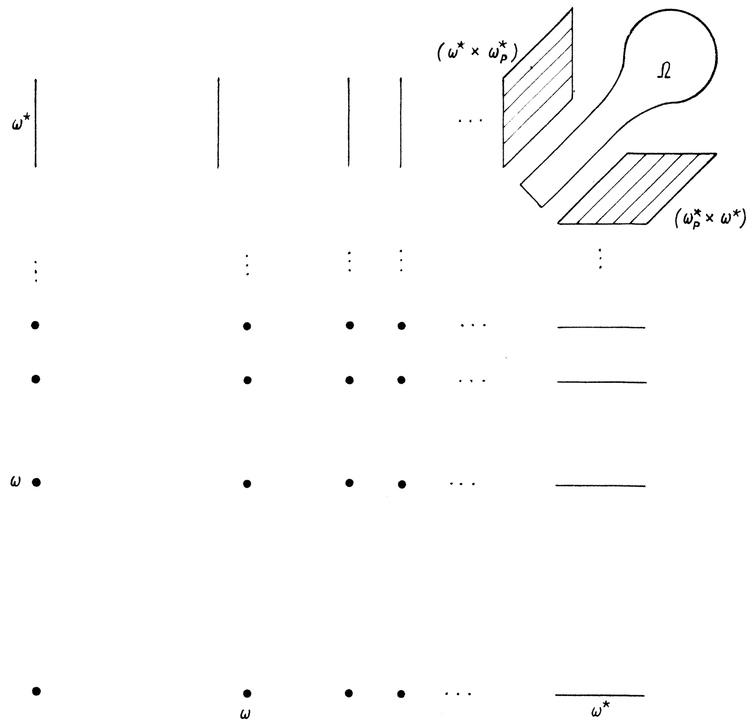

Fig. 1.

The subspace $(\omega^* \times \omega)$ together with the subspace labelled $(\omega_P^* \times \omega^*)$ is supposed to be the set $|\omega^* \times \beta\omega|$ with the topology of Proposition 3.1. The discrete space $\omega \times \omega$ is affixed to this by giving the union the finest topology which induces the topology of $\beta\omega$ on $(\omega \cup \omega^*) \times \{n\}$. The subspaces $\omega \times \omega^*$ and $\omega^* \times \omega_P^*$ are adjoined in the same way (or equivalently, by reflecting in the diagonal). The remainder, Ω, is just what has to be added to the space we have already to obtain its Stone-Čech compactification (i.e. its growth [9]); but before we can assert that the diagram gives an accurate picture of this compactification we must check that the space we are compactifying is completely regular.

3.4. Lemma. *The topology on* $(\omega \times \omega) \cup (\omega^* \times \omega) \cup (\omega_P^* \times \omega^*) \cup (\omega \times \omega^*) \cup \cup (\omega^* \times \omega_P^*)$ *is completely regular.*

Proof. Discarding those parts of the lemma which are obvious and using the symmetry of the space, it will be enough to show that if U is an open neighbourhood of $(x, y) \in (\omega^* \times \omega) \cup (\omega_P^* \times \omega^*)$ there is a continuous f on the space which is 1 at (x, y) and vanishes off U. Using Proposition 3.1, we can find f on $(\omega^* \times \omega) \cup \cup (\omega_P^* \times \omega^*)$ with these properties. Define f to be zero on $\{(m, n) \in \omega \times \omega : m < n\}$ and on $(\omega \times \omega^*) \cup (\omega^* \times \omega_P^*)$; this part of f is clearly continuous on its domain. Now, for each $n \in \omega$, $f(x, n)$ is defined for $x = m < n$ and for $x \in \omega^*$; we write also $f(x, n) = 0$ if $(x, n) \notin U$. Then $f(\cdot, n)$ is defined and continuous on a closed subspace of the normal (in fact compact) space $\beta\omega \times \{n\}$, and so it extends continuously to $\beta\omega \times \{n\}$. The resulting function on the whole space is the one required.

3.5. Proposition. *The space of figure* 1 *is* $\beta(\omega \times \omega)$.

Proof. We must show that $(\omega \times \omega)$ is C^*-embedded in

$$(\omega \times \omega) \cup (\omega^* \times \omega) \cup (\omega_P^* \times \omega^*) \cup (\omega \times \omega^*) \cup (\omega^* \times \omega_P^*) .$$

Now $(\omega \times \omega)$ is C^*-embedded in $\beta\omega \times \omega$. By construction (Proposition 3.1), $\omega^* \times \omega$ is C^*-embedded in $(\omega^* \times \omega) \cup (\omega_P^* \times \omega^*)$. Putting these assertions together and using symmetry yields the result.

4. THE SEMIGROUPS $\beta(\omega \times \omega)$, $\beta(\tilde{\omega} \times \tilde{\omega})$, $\beta(\omega \times \tilde{\omega})$

In this section, ω denotes the ordered semigroup $\{1, 2, \dots\}$ with the multiplication max. The opposite order type, $\tilde{\omega}$, we will realize as $\{\dots, -3, -2, -1\}$ and again use the multiplication max. We shall give complete descriptions of the multiplications in the Stone-Čech compactifications of products of ω and $\tilde{\omega}$, using the representation of section 3.

The compact semigroup $\alpha\omega$ (see just before Proposition 2.6) is simply the one-point compactification $\{1, 2, 3, \dots, \infty\}$. The inclusion $\omega \times \omega \subseteq \alpha\omega \times \alpha\omega$ extends in a natural way to a continuous homomorphism $\Phi : \beta(\omega \times \omega) \to \alpha\omega \times \alpha\omega$ (Proposition 1.1). Observe that, in the notation of figure 1, $\Phi^{-1}(\infty, \infty) = (\omega_P^* \times \omega^*) \cup \cup (\omega^* \times \omega_P^*) \cup \Omega$, etc. We also notice that the natural projection

$$\mathrm{pr}_i : \omega \times \omega \to \omega \; ; \quad \mathrm{pr}_i(n_1, n_2) = n_i \quad (i = 1, 2)$$

extends in a unique way to a continuous map $\mathrm{pr}_i : \beta(\omega \times \omega) \to \beta\omega$. We use these notations in the following theorem.

4.1. Theorem. *The multiplication in* $\beta(\omega \times \omega)$ *is as follows*:

(i) *if* $\Phi(xy) \in \omega \times \omega$, *then* $xy \in \omega \times \omega$ *and the product is the usual one*;

(ii) *if* $\Phi(xy) = (n, \infty)$ *where* $n \in \omega$, *then* $xy \in \omega \times \omega^*$, $n = \mathrm{pr}_1(xy) = \max\{\mathrm{pr}_1 x, \mathrm{pr}_1 y\}$, *and*

$$\mathrm{pr}_2(xy) = \begin{cases} \mathrm{pr}_2 y & \text{if } \Phi(x) \in \omega \times \omega \\ \mathrm{pr}_2 x & \text{if } \Phi(x) \in \omega \times \{\infty\}\,; \end{cases}$$

(iii) *if* $\Phi(xy) = (\infty, n)$, *where* $n \in \omega$, *results are obtained from* (ii) *by symmetry*;

(iv) *if* $\Phi(xy) = (\infty, \infty)$, *then* (a) *if* $\Phi(x) = (\infty, \infty)$, $xy = x$; (b) *if* $\Phi(x) = (\infty, n)$ *where* $n \in \omega$, *then* $xy = (\mathrm{pr}_1 x, \mathrm{pr}_2 y) \in \omega_P^* \times \omega^*$; (c) *if* $\Phi(x) = (n, \infty)$ *where* $n \in \omega$, *then* $xy = (\mathrm{pr}_1 y, \mathrm{pr}_2 x) \in \omega^* \times \omega_P^*$; (d) *if* $\Phi(x) \in \omega \times \omega$, *then* $xy = y$.

Proof. Let $x, y \in \beta(\omega \times \omega)$ and let (m_i, n_i), (p_j, q_j) be nets in $\omega \times \omega$ with $(m_i, n_i) \to x$, $(p_j, q_j) \to y$. Then

$$x(p_j, q_j) = \lim_i (m_i, n_i)(p_j, q_j) = \lim_i (\max\{m_i, p_j\}, \max\{n_i, q_j\}),$$

and therefore

$$xy = \lim_j \lim_i (\max\{m_i, p_j\}, \max\{n_i, q_j\}).$$

We shall repeatedly use this formula.

Part (i) of the theorem needs no proof. For part (ii), observe that we may take m_i, p_j fixed so that indeed $n = \max\{m_i, p_j\} = \max\{pr_1 x, pr_1 y\}$. If also (n_i) is eventually constant, then for large enough j, $q_j \geqq n_i$, so that $\mathrm{pr}_2(xy) = \lim_j q_j = \mathrm{pr}_2 y$. On the other hand, if $\Phi(x) \in \omega \times \{\infty\}$, then $n_i \to \infty$, so that for fixed j, eventually $n_i \geqq q_j$, and so $\mathrm{pr}_2(xy) = \lim_j \lim_i \max\{n_i, q_i\} = \lim_i n_i = \mathrm{pr}_2 x$.

Part (iii) is proved in the same way.

For part (iv) (a), notice that for fixed j, eventually $m_i \geqq p_j$ and $n_i \geqq q_j$ (for $\Phi(m_i, n_i) \to (\infty, \infty)$). Hence

$$xy = \lim_j \lim_i (m_i, n_i) = x.$$

In (iv) (b), we may take $n_i = n$ for all i, and then eventually we will have $q_j \geqq n$. But again, for fixed j, $m_i \geqq p_j$ eventually. Thus, we have for large j, referring to figure 1,

$$x(p_j, q_j) = \lim (m_i, q_j) = (\mathrm{pr}_1 x, q_j) \in \omega^* \times \omega\,;$$

and then

$$xy = \lim_j (\mathrm{pr}_1 x, q_j) = (\mathrm{pr}_1 x, \mathrm{pr}_2 y) \in \omega_P^* \times \omega^*.$$

Case (c) follows in the same way. Case (d) is like (a).

If we replace ω by $\tilde{\omega}$, the topological structure remains unchaged of course. We can therefore use figure 1 to represent $\beta(\tilde{\omega} \times \tilde{\omega})$, though we shall place tildas over all the constituent parts, and speak of, for example, $\tilde{\omega}_P^*$. The space $\alpha\tilde{\omega}$ is

$\{-\infty, \ldots, -3, -2, -1\}$. We shall again use Φ for the canonical homomorphism $\beta(\tilde{\omega} \times \tilde{\omega}) \to \alpha\tilde{\omega} \times \alpha\tilde{\omega}$, and $\mathrm{pr}_i : \beta(\tilde{\omega} \times \tilde{\omega}) \to \beta\tilde{\omega}$ $(i = 1, 2)$ for the natural projections. We shall also let $\mathrm{pr}_i : \alpha\tilde{\omega} \times \alpha\tilde{\omega} \to \alpha\tilde{\omega}$ be the projection onto the i^{th} coordinate $(i = 1, 2)$; this will have advantages and will not lead to confusion.

4.2. Theorem. *The multiplication in $\beta(\tilde{\omega} \times \tilde{\omega})$ is as follows:*

(i) *if* $\Phi(xy) \in \tilde{\omega} \times \tilde{\omega}$, *then*
$$xy = (\max\{\mathrm{pr}_1 \Phi(x), \mathrm{pr}_1 \Phi(y)\}, \max\{\mathrm{pr}_2 \Phi(x), \mathrm{pr}_2 \Phi(y)\})\,;$$
(ii) *if* $\Phi(xy) = (-n, -\infty)$ *where* $-n \in \tilde{\omega}$, *then*
$$xy = (\max\{\mathrm{pr}_1 \Phi(x), \mathrm{pr}_1 \Phi(y)\}, \mathrm{pr}_2 y) \in \tilde{\omega} \times \tilde{\omega}^*\,;$$
(iii) *if* $\Phi(xy) = (-\infty, -n)$, *the product is obtained by symmetry from* (ii);

(iv) *if* $\Phi(xy) = (-\infty, -\infty)$, *then* $xy = y$.

Proof. The proof follows the methods of the last theorem, and we will prove only (iv) as an illustration. If $(-m_i, -n_i) \to x$, $(-p_j, -q_j) \to y$, then as at the beginning of the proof of 4.1 we find
$$xy = \lim_j \lim_i (\max\{-m_i, -p_j\}, \max\{-n_i, -q_j\})\,.$$
Now since $\Phi(-m_i, -n_i) \to (-\infty, -\infty)$ we must have eventually $-m_i \leqq -p_j$, $-n_i \leqq -q_j$ for fixed j. Hence
$$xy = \lim_j (-p_j, -q_j) = y\,.$$

To discuss $\beta(\omega \times \tilde{\omega})$, we again use figure 1, but the constituent parts will now be labelled $\omega \times \tilde{\omega}$, $\omega^* \times \tilde{\omega}$, $\omega_P^* \times \tilde{\omega}^*$, $\omega \times \tilde{\omega}^*$, $\omega^* \times \tilde{\omega}_P^*$ and Ω. We again have a homomorphism $\Phi : \beta(\omega \times \tilde{\omega}) \to \alpha\omega \times \alpha\tilde{\omega}$, and projections $\mathrm{pr}_1 : \beta(\omega \times \tilde{\omega}) \to \beta\omega$, $\mathrm{pr}_2 : \beta(\omega \times \tilde{\omega}) \to \beta\tilde{\omega}$, $\mathrm{pr}_1 : \alpha\omega \times \alpha\tilde{\omega} \to \alpha\omega$, $\mathrm{pr}_2 : \alpha\omega \times \alpha\tilde{\omega} \to \alpha\tilde{\omega}$.

4.3. Theorem. *The multiplication in $\beta(\omega \times \tilde{\omega})$ is as follows:*

(i) *if* $\Phi(xy) \in \omega \times \tilde{\omega}$, *then*
$$xy = (\max\{\mathrm{pr}_1 x, \mathrm{pr}_1 y\}, \max\{\mathrm{pr}_2 \Phi(x), \mathrm{pr}_2 \Phi(y)\})$$
(*notice that here, necessarily* $\mathrm{pr}_1 x, \mathrm{pr}_1 y \in \omega$, *but one of* $\mathrm{pr}_2 \Phi(x)$, $\mathrm{pr}_2 \Phi(y)$ *could be* $-\infty$);

(ii) *if* $\Phi(xy) = (\infty, -n)$, *then if* $\mathrm{pr}_1 \Phi(x) = \infty$,
$$xy = (\mathrm{pr}_1 x, \max\{\mathrm{pr}_2 \Phi(x), \mathrm{pr}_2 \Phi(y)\})\,,$$
but if $\mathrm{pr}_1 \Phi(x) \in \omega$,
$$xy = (\mathrm{pr}_1 y, \max\{\mathrm{pr}_2 \Phi(x), \mathrm{pr}_2 \Phi(y)\})\,;$$

(iii) *if* $\Phi(xy) = (n, -\infty)$, *then*

$$xy = (\max\{\mathrm{pr}_1 x, \mathrm{pr}_1 y\}, \mathrm{pr}_2 y) ;$$

(iv) *if* $\Phi(xy) = (\infty, -\infty)$ *then* (a) *if* $\mathrm{pr}_1 x \in \omega$,

$$xy = y ;$$

but (b) *if* $\mathrm{pr}_1 \Phi(x) = \infty$, *then*

$$xy = (\mathrm{pr}_1 x, \mathrm{pr}_2 y) \in \omega_P^* \times \tilde{\omega}^* .$$

Proof. Once again the proofs are on similar lines, so we shall consider only (iv). The general formula for the product is

$$xy = \lim_j \lim_i (\max\{m_i, p_j\}, \max\{-n_i, -q_j\}) .$$

Now in case (a), (m_i) is eventually constant, so that for large j, $m_i \leqq p_j$. Also both $\Phi(-n_i) \to -\infty$ and $\Phi(-q_j) \to -\infty$, so that, for fixed j, eventually, $-n_i \leqq -q_j$. Hence, in this case, for large j,

$$\lim_i (\max\{m_i, p_j\}, \max\{-n_i, -q_j\}) = (p_j, -q_j) ,$$

and we conclude $xy = y$.

For case (b), we have $\mathrm{pr}_1 \Phi(m_i, -n_i) \to \infty$, so that, for fixed j, eventually $m_i \geqq p_j$ and $-n_i \leqq -q_j$. Hence

$$\lim_i (\max\{m_i, p_j\}, \max\{-n_i, -q_j\}) = \lim_i (m_i, -q_j) = (\mathrm{pr}_1 x, -q_j) \in \omega^* \times \tilde{\omega} ;$$

thus,

$$xy = \lim_j (\mathrm{pr}_1 x, -q_j) = (\mathrm{pr}_1 x, \mathrm{pr}_2 y) \in \omega_P^* \times \tilde{\omega} .$$

We could now go on to give complete descriptions of the multiplication in $\beta(\omega^r \times \tilde{\omega}^s)$ for any positive integers r and s. It is of course true that since the cardinals of ω^r and $\tilde{\omega}^s$ are $\aleph_0$, $\beta(\omega^r \times \tilde{\omega}^s)$ is homeomorphic with $\beta(\omega \times \omega)$, and indeed with βN. To give a complete account of the multiplication it would be necessary to use induction ($\beta(\omega^3) = \beta(\omega^2 \times \omega)$, etc.) to provide an analysis of the structure of $\beta(\omega^r \times \tilde{\omega}^s)$ on the lines of that in section 3. However, an adequate picture of the multiplication can be obtained simply by investigating the part of $\beta(\omega^r \times \tilde{\omega}^s)$ 'at the corner at infinity'.

To see what is meant here, consider the natural homomorphism

$$\Phi : \beta(\omega^r \times \tilde{\omega}^s) \to (\alpha\omega)^r \times (\alpha\tilde{\omega})^s .$$

The subset $\Phi^{-1}(\infty, \ldots, \infty, -\infty, \ldots, -\infty)$ of $\beta(\omega^r \times \tilde{\omega}^s)$ is a subsemigroup. Any element of $\beta(\omega^r \times \tilde{\omega}^s)$ will be in a subsemigroup of the form $\Phi^{-1}(x_1, \ldots, x_r, y_1, \ldots, y_s)$

where $x_i \in \omega \cup \{\infty\}$, $y_j \in \tilde{\omega} \cup \{-\infty\}$, and is therefore in a subsemigroup isomorphic to one of the kind $\Phi^{-1}(\infty, \ldots, \infty, -\infty, \ldots, -\infty)$ in $\beta(\omega^u \times \tilde{\omega}^v)$ where $0 \leqq u \leqq r$, $0 \leqq v \leqq s$. The only problem – and that complicated rather than difficult – is to see how distinct subsemigroups of this kind combine. We shall content ourselves with the following result.

4.4. Theorem. (i) *If* $\Phi(x) = \Phi(y) = (\infty, \ldots, \infty)$ *and* $x, y \in \beta(\omega^r)$, *then* $xy = x$.
(ii) *If* $\Phi(x) = \Phi(y) = (-\infty, -\infty, \ldots, -\infty)$ *and* $x, y \in \beta(\tilde{\omega}^s)$, *then* $xy = y$.
(iii) *If* $\Phi(x) = \Phi(y) = (\infty, \ldots, \infty, -\infty, \ldots, -\infty)$, $\Phi(z) = (1, 1, \ldots, 1, -\infty, \ldots$ $\ldots, -\infty)$ *for* $x, y, z \in \beta(\omega^r \times \tilde{\omega}^s)$, *then* $xy = (\mathrm{pr}_1 x, \mathrm{pr}_2 y) \in (\omega^r)^*_P \times (\tilde{\omega}^s)^*$, $yz = (\mathrm{pr}_1 y, \mathrm{pr}_2 z) \in (\omega^r)^*_P \times (\tilde{\omega}^s)^*$, *and* $zy = y$.

Proof. As we are again using the same method of proof, we shall prove only (iii). Let $x = \lim_i (n_i, m_i)$ $(n_i \in \omega^r, m_i \in \tilde{\omega}^s)$, $y = \lim_j (p_j, q_j)$ $(p_j \in \omega^r, q_j \in \tilde{\omega}^s)$. Then $\Phi(n_i), \Phi(p_j) \to (\infty, \ldots, \infty)$, $\Phi(m_i), \Phi(q_j) \to (-\infty, \ldots, -\infty)$. Thus, for fixed j, we have eventually $(n_i, m_i)(p_j, q_j) = (n_i, q_j)$, so that

$$xy = \lim_j \lim_i (n_i, q_j) = (\mathrm{pr}_1 x, \mathrm{pr}_2 y) \in (\omega^r)^*_P \times (\tilde{\omega}^s)^*$$

The product yz is obtained in the same way. To find zy, let $z = \lim_k ((1, \ldots, 1), u_k)$. Then $\Phi(u_k) \to (-\infty, \ldots, -\infty)$, and so eventually, for fixed j,

$$z(p_j, q_j) = \lim_k ((1, 1, \ldots, 1), u_k)(p_j, q_j) = \lim_k (p_j, q_j) = (p_j, q_j)$$

so that $zy = y$ as required.

5. ALMOST PERIODIC COMPACTIFICATIONS

Let $T_1, T_2, \ldots, T_n$ be totally ordered separately continuous Hausdorff topological semigroups. For each i, αT_i denotes the compact semigroup constructed from T_i by the method of section 2 (see just before Proposition 2.6). Our aim is to use the properties of the Stone-Čech compactification to prove the following result.

5.1. Theorem. *Both the weakly almost periodic compactification* $W(T_1 \times \ldots \times T_n)$ *and the almost periodic compactification* $A(T_1 \times \ldots \times T_n)$ *of the product* $T_1 \times \ldots \times T_n$ *coincide with* $\alpha T_1 \times \ldots \times \alpha T_n$.

Recall that $W(T_1 \times \ldots \times T_n)$ resp. $A(T_1 \times \ldots \times T_n)$ is the solution for $T_1 \times \ldots \times T_n$ to the universal mapping problem into compact separately resp. jointly continuous semigroups [1]. Together with the universal properties of $\beta(T_1 \times \ldots \times T_n)$, this yields canonical surjective maps

$$\beta(T_1 \times \ldots \times T_n) \to W(T_1 \times \ldots \times T_n) \to A(T_1 \times \ldots \times T_n) \to \alpha T_1 \times \ldots \times \alpha T_n .$$

We have seen that $\beta(T_1 \times \ldots \times T_n)$ may not be a semigroup, but when it is, all these maps are homomorphisms.

We must find some way of circumventing the problem that $\beta(T_1 \times \ldots \times T_n)$ is not a semigroup in general. This is, in fact, simply done. Let $|T_i|$ denote T_i with the discrete topology; then (Proposition 1.1), $\beta(|T_1| \times \ldots \times |T_n|)$ is always a semigroup. The continuity of the identity map $|T_1| \times \ldots \times |T_n| \to T_1 \times \ldots \times T_n$ yields a continuous surjective homomorphism $W(|T_1| \times \ldots \times |T_n|) \to W(T_1 \times \ldots \times T_n)$. From all the maps at our disposal, we shall only need to consider

$$\beta(|T_1| \times \ldots \times |T_n|) \xrightarrow{\nu} W(|T_1| \times \ldots \times |T_n|) \xrightarrow{\mu}$$
$$\xrightarrow{\mu} W(T_1 \times \ldots \times T_n) \xrightarrow{\lambda} \alpha T_1 \times \ldots \times \alpha T_n ,$$

and these are all continuous surjective homomorphisms. As all the spaces are compact, to establish Theorem 5.1 it will be enough to show that λ is injective; for it is then a homeomorphism, so an isomorphism, and we have observed that $A(T_1 \times \ldots \times T_n)$ is sandwiched between $W(T_1 \times \ldots \times T_n)$ and $\alpha T_1 \times \ldots \times \alpha T_n$.

To this end, let $(x_1, \ldots, x_n) \in \alpha T_1 \times \ldots \times \alpha T_n$. The construction of αT_i means that elements of αT_i are of three kinds (see section 1) and by permuting the indices if necessary, we can suppose $(x_1, \ldots, x_n)$ is of the form

$$(s(U_1), \ldots, s(U_r),\ i(V_1), \ldots, i(V_s),\ a_1, \ldots, a_t)$$

where $U_1, \ldots, U_r, V_1, \ldots, V_s$ are segments, $0 \leqq r, s, t$, and $r + s + t = n$. By the construction of αT_i, each U_i is clopen and each $T_i \setminus V_i$ is clopen in the relevant T_i, while the closures in αT_i, $\overline{U}_i =]-\infty, s(U_i)]$ and $\overline{T \setminus V_i} = [i(V_i), \infty[$ are also clopen in αT_i.

Now take $x, y \in \beta(|T_1| \times \ldots \times |T_n|)$ with

$$\lambda \circ \mu \circ \nu(x) = \lambda \circ \mu \circ \nu(y) = (s(U_1), \ldots, s(U_r),\ i(V_1), \ldots, i(V_s),\ a_1, \ldots, a_t) .$$

Then

$$x, y \in (\lambda \circ \mu \circ \nu)^{-1} (\overline{U}_1 \times \ldots \times \overline{U}_r \times \overline{T_{r+1} \setminus V_1} \times \ldots \times \overline{T_{r+s} \setminus V_s} \times$$
$$\times \{a_1\} \times \ldots \times \{a_t\}) .$$

This is a clopen subsemigroup of $\beta(T_1 \times \ldots \times T_{r+s} \times \{a_1\} \times \ldots \times \{a_t\})$ and is therefore seen to be the closure of

$$U_1 \times \ldots \times U_r \times (T_{r+1} \setminus V_1) \times \ldots \times (T_{r+s} \setminus V_s) \times \{a_1\} \times \ldots \times \{a_t\} .$$

The proof now follows that of Theorem 4.4. There are three cases: $r = 0$, $s > 0$; $r > 0$, $s = 0$; and $r, s > 0$. The last is the most difficult, and that is the one we shall consider. For brevity, we write $a = (a_1, \ldots, a_t)$.

Take nets $(m_i, n_i, a) \to x$, $(p_j, q_j, a) \to y$ where $m_i, p_j \in U_1 \times \ldots \times U_r$, $n_i, q_j \in \in (T_{r+1} \setminus V_1) \times \ldots \times (T_{r+s} \setminus V_s)$. By considering images under $\lambda \circ \mu \circ \nu$, we see that eventually for fixed j, $m_i \geqq p_j$ and $n_i \leqq q_j$. Hence

$$xy = \lim_j \lim_i (m_i, n_i, a)(p_j, q_j, a) = \lim_j \lim_i (m_i, q_j, a) .$$

Next, take a fixed $u \in U_1 \times \ldots \times U_r$, and write $z = \lim_j (u, q_j, a)$. (It is easy to see the limit exists by considering the projection onto $\beta(T_{r+1} \times \ldots \times T_{r+s})$.) Then, as above, we find

$$xz = \lim_j \lim_i (m_i, q_j, a) = xy .$$

By the same method, we also find

$$zx = \lim_i \lim_j (u, q_j, a)(m_i, n_i, a) = \lim_i \lim_j (m_i, n_i, a) = x .$$

Now observe that $W(|T_1| \times \ldots \times |T_n|)$ is a separately continuous semigroup with a dense commutative subsemigroup, so that it is itself commutative. We see that

$$v(xy) = v(xz) = v(x)\, v(z) = v(z)\, v(x) = v(zx) = v(x) .$$

Since x and y were arbitrarily chosen, we conclude that $v(x) = v(y)$. Therefore

$$\lambda^{-1}(s(U_1), \ldots, s(U_r), i(V_1), \ldots, i(V_s), a_1, \ldots, a_t) = \mu(v((\lambda \circ \mu \circ v)^{-1} (s(U_1), \ldots a_t)))$$

contains only one point. Hence λ is injective, and our theorem follows.

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References

[1] *J. F. Berglund* and *K. H. Hofmann:* Compact Semitopological Semigroups and Weakly Almost Periodic Functions, Lecture Notes in Mathematics, 42, Springer-Verlag, Berlin (1967).
[2] *G. Birkhoff:* Lattice Theory, American Math. Soc. Colloquium Publications, *25* (1948).
[3] *R. Ellis:* Lectures on Topological Dynamics, Benjamin, New York (1969).
[4] *P. Milnes:* Compactifications of topological semigroups, J. Australian Math. Soc. *15* (1973), 488—503.
[5] *A. K. Misra:* A topological view of *P*-spaces, General Topology and Applics. *2* (1972), 349—362.
[6] *L. Nachbin:* Topology and Order, Van Nostrand, Princeton, 1965.
[7] *J. S. Pym* and *H. L. Vasudeva:* The algebra of finitely additive measures on a partially ordered semigroup, Studia Math. *61* (1977), 7—19.
[8] *W. Ruppert:* Rechtstopologische Halbgruppen, J. Reine Angew. Math. *261* (1973), 123—133.
[9] *R. C. Walker:* The Stone-Čech Compactification, Ergebnisse Math. Vol. *83*, Springer-Verlag, Berlin, 1974.

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