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GENERALIZED WEINGARTEN SURFACES

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We are going to prove the following

Theorem. *Let G be a bounded domain in $\mathscr{R}^2$, ∂G its boundary and $M : G \cup \partial G \to \to E^3$ a surface such that $M(\partial G)$ consists of umbilical points. Let there exist functions $R_i : M \to \mathscr{R}$; $i = 1, 2, 3, 4$; such that*

$$R_1 \, \mathrm{d}H + R_2 \, \mathrm{d}K + R_3 * \mathrm{d}H + R_4 * \mathrm{d}K = 0 \,, \tag{1}$$

H and K being the mean and Gauss curvatures of $M(G)$ resp. Further, let

$$R_1^2 + R_3^2 + 4H(R_1 R_2 + R_3 R_4) + 4K(R_2^2 + R_4^2) > 0 \,. \tag{2}$$

Then $M(G \cup \partial G)$ is a part of a sphere.

Proof. (1) Consider a field of orthonormal moving frames $\{m, v_1, v_2, v_3\}$ associated to $M \equiv M(G \cup \partial G)$. Then

$$\begin{aligned} \mathrm{d}m &= \omega^1 v_1 + \omega^2 v_2 \,, \\ \mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 \,, \\ \mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 \,, \\ \mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 \end{aligned} \tag{3}$$

with the usual integrability conditions. From

$$\omega^3 = 0 \,, \tag{4}$$

we get

$$\omega_1^3 = a\omega^1 + b\omega^2 \,, \quad \omega_2^3 = b\omega^1 + c\omega^2 \tag{5}$$

and

$$\begin{aligned} \mathrm{d}a - 2b\omega_1^2 &= \alpha\omega^1 + \beta\omega^2 \,, \\ \mathrm{d}b + (a - c)\,\omega_1^2 &= \beta\omega^1 + \gamma\omega^2 \,, \\ \mathrm{d}c + 2b\omega_1^2 &= \gamma\omega^1 + \delta\omega^2 \,. \end{aligned} \tag{6}$$

The mean and Gauss curvature are given by

$$H = \tfrac{1}{2}(a + c)\,, \quad K = ac - b^2 \tag{7}$$

respectively. From this,

$$\begin{aligned} dH &= \tfrac{1}{2}(\alpha + \gamma)\,\omega^1 + \tfrac{1}{2}(\beta + \delta)\,\omega^2\,, \\ dK &= (a\gamma + c\alpha - 2b\beta)\,\omega^1 + (a\delta + c\beta - 2b\gamma)\,\omega^2\,. \end{aligned} \tag{8}$$

The $*$-operator is given (as usually) by

$$* : \tau = p\omega^1 + q\omega^2 \to *\tau = -q\omega^1 + p\omega^2\,. \tag{9}$$

Taking in regard another field $\{m; w_1, w_2, w_3\}$ of moving frames with

$$\begin{aligned} v_1 &= \varepsilon_1(\cos\varphi\,.\,\omega_1 - \sin\varphi\,.\,w_2)\,, \\ v_2 &= \sin\varphi\,.\,w_1 + \cos\varphi\,.\,w_2\,, \\ v_3 &= \varepsilon_2 w_3\,, \qquad \varepsilon_1^2 = \varepsilon_2^2 = 1\,, \end{aligned} \tag{10}$$

we get

$$dm = \Omega^1 w_1 + \Omega^2 w_2 \tag{11}$$

with

$$\Omega^1 = \varepsilon_1 \cos\varphi\,.\,\omega^1 + \sin\varphi\,.\,\omega^2\,, \quad \Omega^2 = -\varepsilon_1 \sin\varphi\,.\,\omega^1 + \cos\varphi\,.\,\omega^2$$

and, for $\tau = P\Omega^1 + Q\Omega^2$,

$$p = \varepsilon_1(P\cos\varphi - Q\sin\varphi)\,, \quad q = P\sin\varphi + Q\cos\varphi\,. \tag{12}$$

From this,

$$*\tau = \varepsilon_1(-Q\Omega^1 + P\Omega^2) \tag{13}$$

so that the $*$-operator depends just on the orientation of M. For further use, let us choose one of the orientations of M; the result is, of course, independent of the chosen orientation.

(2) We have

$$\begin{aligned} *dM &= -\tfrac{1}{2}(\beta + \delta)\,\omega^1 + \tfrac{1}{2}(\alpha + \gamma)\,\omega^2\,, \\ *dK &= -(a\delta + c\beta - 2b\gamma)\,\omega^1 + (a\gamma + c\alpha - 2b\beta)\,\omega^2 \end{aligned} \tag{14}$$

so that the equation (1) yields

$$\begin{aligned} &R_1(\alpha + \gamma) + 2R_2(a\gamma + c\alpha - 2b\beta) - \\ &- R_3(\beta + \delta) - 2R_4(a\delta + c\beta - 2b\gamma) = 0\,, \\ &R_1(\beta + \delta) + 2R_2(a\delta + c\beta - 2b\gamma) + \\ &+ R_3(\alpha + \gamma) + 2R_4(a\gamma + c\alpha - 2b\beta) = 0\,. \end{aligned} \tag{15}$$

On M, choose a coordinate system (u, v) such that

(16) $$I = r^2\, du^2 + s^2\, dv^2\,, \quad \text{i.e.,} \quad \omega^1 = r\, du\,, \quad \omega^2 = s\, dv\,, \quad rs \neq 0\,.$$

From $d\omega^1 = -\omega^2 \wedge \omega_1^2$, $d\omega^2 = \omega^1 \wedge \omega_1^2$, we get

(17) $$\omega_1^2 = -s^{-1} r_v\, du + r^{-1} s_u\, dv\,.$$

We have, from (6),

(18) $$\begin{aligned} d(a - c) &= \quad 4b\omega_1^2 + (\alpha - \gamma)\,\omega^1 + (\beta - \delta)\,\omega^2\,, \\ db &= -(a - c)\,\omega_1^2 + \beta\omega^1 + \gamma\omega^2\,, \end{aligned}$$

i.e.,

(19) $$(a - c)_u + 4b\frac{r_v}{s} = (\alpha - \gamma)\, r\,, \quad b_u - (a - c)\frac{r_v}{s} = \beta r\,,$$

$$(a - c)_v - 4b\frac{s_u}{r} = (\beta - \delta)\, s\,, \quad b_v + (a - c)\frac{s_u}{r} = \gamma s\,.$$

Finally,

(20) $$\begin{aligned} \alpha rs &= \quad s(a - c)_u + rb_v + (\cdot)(a - c) + (\cdot)\, b\,, \\ \beta rs &= \quad sb_u + (\cdot)(a - c) + (\cdot)\, b\,, \\ \gamma rs &= \quad rb_v + (\cdot)(a - c) + (\cdot)\, b\,, \\ \delta rs &= -r(a - c)_v + sb_u + (\cdot)(a - c) + (\cdot)\, b\,. \end{aligned}$$

The system (15) becomes

(21) $$\begin{aligned} a_{11}(a - c)_u + a_{12}(a - c)_v + b_{11}b_u + b_{12}b_v &= c_{11}(a - c) + c_{12}b\,, \\ a_{21}(a - c)_u + a_{22}(a - c)_v + b_{21}b_u + b_{22}b_v &= c_{21}(a - c) + c_{22}b \end{aligned}$$

with

(22) $$\begin{aligned} a_{11} &= \quad s(R_1 + 2cR_2)\,, \\ a_{12} &= \quad r(R_3 + 2aR_4)\,, \\ b_{11} &= -2s(2bR_2 + R_3 + 4HR_4)\,, \\ b_{12} &= \quad 2r(R_1 + 2HR_2 + 2bR_4)\,, \\ a_{21} &= \quad s(R_3 + 2cR_4)\,, \\ a_{22} &= -r(R_1 + 2aR_2)\,, \\ b_{21} &= \quad 2s(R_1 + 2HR_2 - 2bR_4)\,, \\ b_{22} &= \quad 2r(-2bR_2 + R_3 + 2HR_4)\,. \end{aligned}$$

Recall that the system (21) is called elliptic if the form

$$\Phi = (a_{12}b_{22} - a_{22}b_{12})\,\mu^2 - (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})\,\mu\nu + (a_{11}b_{21} - a_{21}b_{11})\,\nu^2 \tag{23}$$

is definite. In our case,

$$a_{12}b_{22} - a_{22}b_{12} = 2r^2[2(H + a)(R_1R_2 + R_3R_4) + 2b(R_1R_4 - R_2R_3) + 4Ha(R_2^2 + R_4^2) + R_1^2 + R_3^2],$$

$$a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11} = 4rs[-2b(R_1R_2 + R_3R_4) + (a - c)(R_1R_4 - R_2R_3) - 4bH(R_2^2 + R_4^2)],$$

$$a_{11}b_{21} - a_{21}b_{11} = 2s^2[R_1^2 + R_2^2 + 2(H + c)(R_1R_2 + R_3R_4) + 2b(R_2R_3 - R_1R_4) + 4cH(R_2^2 + R_4^2)].$$

Denoting by Δ the discriminant of Φ, we get

$$\frac{\Delta}{6r^2s^2} = [(R_1 + 2HR_2)^2 + (R_3 + 2HR_4)^2] \times [R_1^2 + R_3^2 + 4H(R_1R_2 + R_3R_4) + 4K(R_2^2 + R_4^2)]. \tag{24}$$

The first term of the product cannot be equal to zero; indeed, let us suppose, on the contrary, $R_1 + 2HR_2 = R_3 + 2HR_4 = 0$. Then the second term would be $-4(H^2 - K)(R_2^2 + R_2^2) \leqq 0$, which is a contradiction to (2). This means that (2) induces the system (21) to be elliptic. On the boundary ∂G, $a - c = b = 0$ according to the supposition. From this, $a - c = b = 0$ on G, i.e., $4(H^2 - K) = (a - c)^2 + 4b^2 = 0$ on G, and M is a part of a sphere. QED.

From our Theorem, we get immediately the following

Corollary. *Let G be a bounded domain in $\mathscr{R}^2$, ∂G its boundary and $M : G \cup \partial G \to E^3$ be a surface such that $M(\partial G)$ consists of umbilical points and there exists a function $f(x, y)$ on G such that*

$$f(H, K) = 0\,, \quad f_H^2 = 4Hf_Hf_K + 4kf_K^2 > 0 \tag{25}$$

on G. Then M is a part of a sphere.

The proof is trivial, because $f(H, K)$ implies $f_H\,\mathrm{d}H + f_K\,\mathrm{d}K = 0$, and we are in the situation of our Theorem for $R_1 = f_H$, $R_2 = f_K$, $R_3 = R_4 = 0$. This Corollary has been proved by A. Švec, for ovaloids, in his paper [1].

Bibliography

[1] *A. Švec:* Several new characterizations of the sphere. Czech. Math. J., *25 (100)* 1975, 645—652.

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