David Lowell Lovelady

On the asymptotic classes of solutions of a superlinear differential equation

Czechoslovak Mathematical Journal, Vol. 27 (1977), No. 2, 242–245

Czechoslovak Mathematical Journal, 27 (102) 1977, Praha

ON THE ASYMPTOTIC CLASSES OF SOLUTIONS OF A SUPERLINEAR DIFFERENTIAL EQUATION

DAVID LOWELL LOVELADY, Tallahassee

(Received May 8, 1975)

Let n be an integer, $n \geqq 2$, let q be a positive continuous function on $[0, \infty)$, and let α be a real number, $\alpha > 1$. It is known (see I. LIČKO and M. ŠVEC [3] and G. H. RYDER and D. V. V. WEND [6]) that

$$\int_0^\infty t^{2n-1} q(t)\, dt < \infty \tag{1}$$

is a necessary and sufficient condition for the existence of a nonoscillatory solution of

$$u^{(2n)}(t) + q(t)\, |u(t)|^\alpha \operatorname{sgn}(u(t)) = 0\,. \tag{2}$$

The sufficiency of (1) is usually shown by showing the existence of a solution u of (2) with

$$\lim_{t \to \infty} u(t) \tag{3}$$

existing and not being zero. On the other hand, I. T. KIGURADZE [2] has shown that the nonoscillatory solutions of (2) fall into n distinct classes, and one of these classes contains all solutions u for which (3) exists. We shall obtain separate necessary and sufficient conditions for each of the n classes.

Suppose u is an eventually positive solution of (2). Now there is $c \geqq 0$ such that u is defined on $[c, \infty)$ and none of $u, u', \ldots, u^{(2n-1)}$ has a zero in $[c, \infty)$. Let j_u be the largest integer such that $u^{(i)} > 0$ on $[c, \infty)$ if $i \leqq j_u$ (where we write $u = u^{(0)}$). Now j_u is odd and $u^{(k)}u^{(k+1)} < 0$ on $[c, \infty)$ if $j_u \leqq k \leqq 2n - 1$. Since j_u is odd, we see that there are n possible values for the function described by $u \to j_u$, and that the eventually positive solutions of (2) fall into n classes. If u is an eventually negative solution of (2), then $-u$ is an eventually positive solution, so similar analyses apply. (See Kiguradze [2], Ryder and Wend [6], and the present author [4], [5] for details on the above construction).

Theorem. *Suppose k is an odd integer in $[0, 2n]$. Then*

$$\int_0^\infty t^{2n-k+\alpha(k-1)}\, q(t)\, dt < \infty \tag{4}$$

if and only if there is an eventually positive solution u of (2) with $k = j_u$.

Note that our Theorem is an improvement of [5, Corollary 2], in which it was shown that if $k > 1$, $\alpha > k/(k-1)$, and

$$\int_0^\infty t^{2n-k+\alpha(k-1)-1}\, q(t)\, dt = \infty$$

then (2) has no eventually positive solution u with $k = j_u$. If m is an odd integer in $[0, 2n]$ and $m < k$ then (4) implies

$$\int_0^\infty t^{2n-m+\alpha(m-1)}\, q(t)\, dt < \infty ,$$

since $\alpha > 1$. This gives the following result.

Corollary 1. *If m and k are odd integers in $[0, 2n]$, if $m < k$, and if there is an eventually positive solution u of (2) with $j_u = k$, then there is an eventually positive solution u of (2) with $j_u = m$.*

If u is an eventually positive solution of (2) and $k = j_u$, then $u^{(k)} > 0$ and $u^{(k+1)} < < 0$, eventually, so $\lim_{t\to\infty} u^{(k)}(t)$ exists. This and k applications of L'Hôpital's Rule say that $\lim_{t\to\infty} u(t)/t^k$ exists. Thus we have another corollary.

Corollary 2. *Suppose k is an odd integer in $[2, 2n]$. Then*

$$\int_0^\infty t^{2n-k+\alpha(k-1)}\, q(t)\, dt = \infty \tag{5}$$

if and only if

$$\lim_{t\to\infty} \frac{u(t)}{t^{k-2}} \tag{6}$$

exists and is finite whenever u is a nonoscillatory solution of (2).

Proof of the theorem. Suppose there is an eventually positive solution u of (2) with $k = j_u$. Find $c \geqq 0$ such that u is defined on $[c, \infty)$ and none of $u, u', \ldots, u^{(2n-1)}$ has a zero on $[c, \infty)$. Now

$$u^{(k)}(t) \geqq \frac{1}{(2n-k-1)!} \int_0^\infty (s-t)^{2n-k-1}\, q(s)\, u(s)^\alpha\, ds \tag{7}$$

if $t \geqq c$. Also, if $s \geqq t \geqq c$,

$$(8)\quad u(s) \geqq \frac{1}{(k-2)!} \int_c^s (s-\xi)^{k-2} u^{(k-1)}(\xi)\, d\xi \geqq \frac{1}{(k-2)!} \int_t^s (s-\xi)^{k-2} u^{(k-1)}(\xi)\, d\xi \geqq$$

$$\geqq \frac{u^{(k-1)}(t)}{(k-2)!} \int_t^s (s-\xi)^{k-2}\, d\xi = \frac{(s-t)^{k-1}}{(k-1)!} u^{(k-1)}(t),$$

since $u^{(k-1)}$ is increasing (recall $u^{(k)} > 0$). If $v = u^{(k-1)}$ and $\beta = ((2n-k-1)!)^{-1} \cdot ((k-1)!)^{-\alpha}$ then (7) and (8) say

$$v'(t) \geqq \beta\, v(t)^\alpha \int_t^\infty (s-t)^{2n-k-1+\alpha(k-1)} q(s)\, ds,$$

$$v'(t)\, v(t)^{-\alpha} \geqq \beta \int_t^\infty (s-t)^{2n-k-1+\alpha(k-1)} q(s)\, ds,$$

$$\frac{1}{\alpha-1}\left(v(c)^{1-\alpha} - v(t)^{1-\alpha}\right) \geqq \beta \int_c^t \left(\int_s^\infty (\xi-s)^{2n-k-1+\alpha(k-1)} q(\xi)\, d\xi\right) ds$$

if $t \geqq c$. Since $\lim_{t\to\infty} v(t)^{1-\alpha}$ exists, because $\alpha > 1$, this says

$$(9)\qquad \int_c^\infty \left(\int_s^\infty (\xi-s)^{2n-k-1+\alpha(k-1)} q(\xi)\, d\xi\right) ds < \infty.$$

But (9) implies (4), so the first part of the proof is complete.

Now suppose (4) holds, and let

$$\gamma = \int_0^\infty t^{2n-k+\alpha(k-1)} q(t)\, dt.$$

Find positive numbers β and b such that

$$(10)\qquad \beta + \frac{b^\alpha \gamma}{(k-1)!\,(2n-k-1)!} \leqq b.$$

(Clearly there are such numbers, since $\alpha > 1$.) Let $\mathcal{F}$ be the set to which f belongs if and only if f is a continuous function from $[0, \infty)$ to $[0, \infty)$, and $f(t) \leqq bt^{k-1}$ if $t \geqq 0$. If f is in $\mathcal{F}$ then (4) says

$$\int_0^\infty t^{2n-k-1} q(t)\, f(t)^\alpha\, dt < \infty.$$

Let T be that function on $\mathcal{F}$, each value of which is a continuous function from $[0, \infty)$ to $[0, \infty)$, such that $g = T(f)$ if and only if

$$g(t) = \beta t^{k-1} + \frac{1}{(k-1)!\,(2n-k-1)!} \int_0^t (t-s)^{k-1} \left(\int_s^\infty (\xi-s)^{2n-k-1} q(\xi) f(\xi)^\alpha\, d\xi\right) ds$$

whenever $t \geqq 0$. Suppose f is in $\mathscr{F}$ and $g = (Tf)$. If $t \geqq 0$,

$$g(t) \leqq \beta t^{k-1} + \frac{b^\alpha}{(k-1)!\,(2n-k-1)!} \int_0^t t^{k-1} \left(\int_s^\infty \xi^{2n-k-1+\alpha(k-1)} q(\xi)\, \mathrm{d}\xi \right) \mathrm{d}s =$$

$$= t^{k-1} \left(\beta + \frac{b^\alpha}{(k-1)!\,(2n-k-1)!} \int_0^t \left(\int_s^\infty \xi^{2n-k-1+\alpha(k-1)} q(\xi)\, \mathrm{d}\xi \right) \mathrm{d}s \right) \leqq$$

$$\leqq \left(\beta + \frac{b^\alpha \gamma}{(k-1)!\,(2n-k-1)!} \right) t^{k-1} \leqq b t^{k-1} ,$$

from (10), of g is in $\mathscr{F}$, and T maps $\mathscr{F}$ into $\mathscr{F}$. Now routine computations show that T is continuous with respect to the topology of locally uniform convergence, and that the range of T is locally equicontinuous. Thus the fixed point theorem of J. Schauder [7] (see also W. A. Coppel [1, p. 9]) says that there is u in $\mathscr{F}$ with $u = T(u)$, i.e.,

$$u(t) = \beta t^{k-1} + \tag{11}$$

$$+ \frac{1}{(k-1)!\,(2n-k-1)!} \int_0^t (t-s)^{k-1} \left(\int_s^\infty (\xi - s)^{2n-k-1}\, q(\xi)\, u(\xi)^\alpha\, \mathrm{d}\xi \right) \mathrm{d}s$$

if $t \geqq 0$. Now (11) says $u(t) \geqq \beta t^{k-1}$ if $t \geqq 0$, so $u(t)$ is positive if $t \geqq 0$. Also, it is easily seen that u is a solution of (2), and $j_u = k$. The proof is complete.

Corollaries 1 and 2 are now obvious.

References

[1] *W. A. Coppel:* Stability and asymptotic behavior of differential equations, D. C. Heath & Co., Boston, 1965.

[2] *I. T. Kiguradze:* Oscillatory properties of certain ordinary differential equations, Soviet Math. Dokl., *3* (1962), 649—652 (Russian).

[3] *I. Ličko* and *M. Švec:* Le caractère oscillatoire des solutions de l'équation $y^{(n)} + f(x)\, y^\alpha = 0$, $n > 1$, Czechoslovak Math. J. *13*, (*88*), (1963), 481—491.

[4] *D. L. Lovelady:* Oscillation and even order linear differential equations, Rocky Mountain Math. J., *6* (1976), 299—304,

[5] *D. L. Lovelady:* Nonlinear even order oscillation criteria according to Kiguradze classes, (unpublished).

[6] *G. H. Ryder* and *D. V. V. Wend:* Oscillation of solutions of certain ordinary differential equations of n^{th} order, Proc. Amer. Math. Soc., *25* (1970), 463—469.

[7] *J. Schauder:* Der Fixpunktsatz in Funktionalräumen, Studia Math., *2* (1930), 171—180.

Author's address: Department of Mathematics, The Florida State University, Tallahassee, Florida 32306, U. S. A.