Jozef Kačur; Alojz Wawruch

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ON AN APPROXIMATE SOLUTION FOR QUASILINEAR PARABOLIC EQUATIONS

JOZEF KAČUR and ALOJZ WAWRUCH, Bratislava

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Introduction. In this paper we consider the first initial-boundary value problem for quasilinear parabolic equations

$$(1) \qquad \frac{\partial u}{\partial t} + \sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij}(x)\, D^j u) + a(t, x, Du) = 0$$

in the domain $Q = \Omega \times (0, T)$, where $t \in \langle 0, T \rangle$ $(T < \infty)$, Ω is a bounded domain $x \in \Omega \subset E^N$ (N-dimensional Euclidean space) with a Lipschitzian boundary $\partial\Omega$, $i = (i_1, \ldots, i_N)$ is a multi-index,

$$D^i \equiv \frac{\partial^{|i|}}{\partial x_1^{i_1} \ldots \partial x_N^{i_N}} \quad \text{with} \quad |i| = \sum_{p=1}^{N} i_p ,$$

and Du is the vector function $Du \equiv (D^i u, |i| \leq k)$.

The function $a(t, x, \xi)$, $\xi \in E^d$ $(d = \text{card}\{i, |i| \leq k\})$ is Lipschitz continuous in t and ξ.

Initial-boundary conditions are of the form

$$(2) \qquad u(x, 0) = u_0(x) , \quad D_\nu^l u|_{\partial\Omega \times (0,T)} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k - 1 ,$$

where D_ν^l is the outward normal derivative of order l and $u_0(x) \in \mathring{W}_2^k(\Omega)$ (Sobolev space).

An approximate solution $u^n(x, t)$ of the problem (1), (2) is constructed in terms of functions $u_s(x)$, $s = 1, \ldots, n$ which are obtained in the following way:

Let $\{t_s\}_{s=1}^n$ be the uniform partition of $\langle 0, T \rangle$, $h = T/n$ and $t_s = s \,.\, h$. Successively for $s = 1, \ldots, n$ we solve the linear Dirichlet boundary value problem

$$(1') \qquad \frac{u_s - u_{s-1}}{h} + \sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij} D^j u_s) + a(t_s, x, Du_{s-1}) = 0 ,$$

$$(2') \qquad D_\nu^l\, u_s(x)|_{\partial\Omega} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k - 1$$

where $u_0 = u_0(x)$ is taken from (2). Then we construct (Rothe's function)

$$u^n(x, t) = u_{s-1}(x) + (t - t_{s-1})\, h^{-1}(u_s(x) - u_{s-1}(x)) \quad \text{for}$$
$$t_{s-1} \leqq t \leqq t_s\,, \quad s = 1, \ldots, n\,.$$

In fact, this method is Rothe's method which is called also the method of lines.

Under certain assumptions on a_{ij} (see (3), (4) below) we prove that $u^n(x, t)$ converges for $n \to \infty$ to the unique weak solution $u(x, t)$ of (1), (2) (see Definition 3) in the norm of the space $C(\langle 0, T \rangle, L_2(\Omega))$. Moreover, we prove that $u^n(x, t) \to \to u(x, t)$ for $n \to \infty$ in the norm of the space $W_2^{2k-1}(\Omega') \cap W_2^k(\Omega)$ for all $t \in (0, T)$, where Ω' is an arbitrary subdomain of Ω with $\bar{\Omega}' \subset \Omega$. If (3) is satisfied for $l = k$, then our weak solution $u(x, t)$ satisfies (1) for a.e. $(x, t) \in Q$ in the classical sense.

Analogous results are obtained also in the case when $u^n(x, t)$ is constructed in terms of u_s $(s = 1, \ldots, n)$ which are obtained by the following predictor-corrector scheme:

Let v_s, u_s $s = 1, \ldots, n$ be the weak solutions of the linear Dirichlet boundary value problems $(u_0 = u_0(x))$

$$(1'') \qquad \frac{v_s - u_{s-1}}{h} + \sum_{|i|,|j| \leqq k} (-1)^{|i|}\, D^i(a_{ij} D^j v_s) + a(t_s, x, Du_{s-1}) = 0\,,$$

$$(2'') \qquad D_v^l v_s\big|_{\partial\Omega} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k - 1$$

and

$$(1''') \qquad \frac{u_s - u_{s-1}}{h} + \sum_{|i|,|j| \leqq k} (-1)^{|i|}\, D^i(a_{ij} D^j u_s) + a(t_s, x, Dv_s) = 0\,,$$

$$(2''') \qquad D_v^l u_s\big|_{\partial\Omega} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k - 1\,.$$

This is the special case of the predictor-corrector scheme of the Crank-Nicholson method (see [11]).

Rothe's method was introduced in [5] and later on has been used by many authors. The conception of our paper corresponds to the recent papers [1–4].

NOTATION AND DEFINITIONS

By $C^{0,1}(\bar{\Omega})$ we denote the space of Lipschitz continuous functions in $\bar{\Omega}$ and by $C^{p,1}(\bar{\Omega})$ the subset of all $v \in C^{0,1}(\bar{\Omega})$ such that $D^i v \in C^{0,1}(\bar{\Omega})$ for all i with $|i| = p$.

We shall assume

$$(3) \qquad a_{ij}(x) \in C^{p_{i,l},1}(\bar{\Omega}) \quad \text{for all} \quad |i|, |j| \leqq k\,,$$

where $p_{i,l} = \max\{0, |i| + l - k - 1\}$ and l is an integer satisfying $1 \leqq l \leqq k$.

Ellipticity is assumed in the form

$$(4) \qquad \sum_{|i|,|j|=k} a_{ij}(x)\, \xi_i \xi_j \geqq C_1 \sum_{|i|=k} \xi_i^2 \quad (C_1 > 0)\,.$$

$a(t, x, \xi)$ is continuous in all variables t, x, ξ and satisfies

$$(5) \qquad |a(t, x, \xi) - a(t', x, \xi')| \leqq C(1 + |t - t'| + |t - t'|\,|\xi| + |\xi - \xi'|)$$

where C is a positive constant.

Let us consider the Sobolev space

$$W_2^k(\Omega) \equiv W = \{u \in L_2(\Omega);\ D^i u \in L_2(\Omega) \text{ for all } |i| \leqq k\}$$

($D^i u$ are derivatives in the sense of distributions) with the norm $\|\cdot\|_{W^2{}_k} = \|\cdot\|_W = \sum_{|i|\leqq k} \|D^i u\|$, where $\|\cdot\|$ is the norm in $L_2(\Omega)$. The scalar product in $L_2(\Omega)$ is denoted $(.,.)$.

Let $C_0^\infty(\Omega)$ be the set of all infinitely differentiable functions with support in Ω. We denote $\mathring{W}_2^k(\Omega) = \overline{C_0^\infty(\Omega)}$, where the closure is taken in the norm of the space W_2^k.

By means of the bilinear form

$$[Au, v] = \int_\Omega \sum_{|i|,|j|\leqq k} a_{ij}(x)\, D^j u\, D^i v \,\mathrm{d}x \quad \text{for} \quad u, v \in \mathring{W}_2^k(\Omega)$$

we define a linear continuous operator A from $\mathring{W}_2^k(\Omega)$ into W_2^{-k}(W_2^{-k} is the dual space to $\mathring{W}_2^k(\Omega)$).

Let X be a Banach space with a norm $\|\cdot\|_X$ and let $v(t) : \langle 0, T\rangle \to X$ be an abstract function. By $\|v(t)\|_X$ we denote the norm $\|\cdot\|_X$ of the element $v(t) \in X$ at a fixed t.

Definition 1. We denote by $L_p(\langle 0, T\rangle, X)$ $(1 \leqq p \leqq \infty)$ the set of all measurable abstract functions $v(t)$ from $\langle 0, T\rangle$ into X (see [10]) such that

$$\|v\|^p_{L_p(\langle 0,T\rangle,X)} = \int_0^T \|v(t)\|_X^p \,\mathrm{d}t < \infty \quad \text{for} \quad 1 \leqq p < \infty$$

and

$$\|v\|_{L_\infty(\langle 0,T\rangle,X)} = \sup_{t\in\langle 0,T\rangle} \text{ess}\, \|v(t)\|_X < \infty \quad \text{for} \quad p = \infty\,.$$

Let $C(\langle 0, T\rangle, X)$ be the set of all continuous functions

$$v(t) : \langle 0, T\rangle \to X \quad \text{with} \quad \|v\|_{C(\langle 0,T\rangle,X)} = \max_{t\in\langle 0,T\rangle} \|v(t)\|_X < \infty\,.$$

The set of all abstract functions $v(t) : \langle 0, T\rangle \to X$ such that $x^*(v(t)) \in C(\langle 0, T\rangle)$ for all $x^* \in X^*$ (X^* is the dual space to X) is denoted by $C_w(\langle 0, T\rangle, X)$.

Definition 2. $C^1_w((0, T), L_2(\Omega))$ is the set of all $v \in C(\langle 0, T\rangle, L_2(\Omega))$ such that

$$\frac{d}{dt}(v(t), w)) \in C((0, T)) \cap L_\infty(\langle 0, T\rangle) \quad \text{for all} \quad w \in L_2(\Omega)$$

and

$$\left|\frac{d}{dt}(v(t), w)\right| \leqq C\|w\| .$$

In this case there exists

$$g(t) \in L_\infty(\langle 0, T\rangle, L_2(\Omega)) \cap C_w((0, T), L_2(\Omega))$$

(uniquely determined) such that

$$\frac{d}{dt}(v(t), w) = (g(t), w) \quad \text{for all} \quad w \in L_2(\Omega)$$

and we denote by $d\, v(t)/dt = g(t)$ the weak derivative of $v(t)$.

By $F(t)\, u = a(t, x, Du)$ we denote the operator from $\langle 0, T\rangle \times W^k_2(\Omega)$ into $L_2(\Omega)$ (see (5)).

Definition 3. $u(t) \in L_\infty(\langle 0, T\rangle, \mathring{W}^k_2(\Omega))$ is a weak solution of the problem (1), (2), if

$$u(t) \in C^1_w((0, T), L_2(\Omega)), \quad u(0) = u_0$$

and

$$\left(\frac{d\, u(t)}{dt}, v\right) + [A\, u(t), v] + (F(t)\, u(t), v) = 0$$

holds for all $v \in \mathring{W}^k_2(\Omega)$ and $t \in (0, T)$.

We shall assume the following additional regularity property of u_0 from (2) and A:

$$Au_0 \in L_2(\Omega)\,. \tag{6}$$

Remark 1. If $u_0(x) \in W^{2k}_2(\Omega)$ and (3) holds for $l = k$, then (6) is satisfied.

The strong convergence is denoted by $\to$ while $\rightharpoonup$ stands for the weak convergence. Positive constants are denoted by C and the fact that C depends on a parameter ε is indicated by writing $C(\varepsilon)$. Symbols C or $C(\varepsilon)$ can denote also different constants in the same discussion.

1. A PRIORI ESTIMATES

$u_s \in \mathring{W}^2_k$ $(s = 1, \ldots, n)$ is a solution of (1'), (2'), if

$$\left(\frac{u_s - u_{s-1}}{h}, v\right) + [Au_s, \; v] + (F(t_s)\, u_{s-1}, v) = 0 \tag{7}$$

holds for all $v \in \mathring{W}^k_2(\Omega)$.

By Ω' we denote an arbitrary subdomain of Ω with $\bar{\Omega}' \subset \Omega$.

Lemma 1. *If* (3)–(5) *are satisfied, then there exists a unique solution* $u_s \in \in \mathring{W}_2^k(\Omega) \cap W_2^{k+l}(\Omega')$ $(s = 1, \ldots, n)$ of (1′), (2′).

Proof. From (3), (4) and due to a lemma of J. L. Lions (see [7] Chap. I, Lemma 5.1) we obtain easily

(8) $$[Au, u] \geqq C_1 \|u\|_W^2 - C_4 \|u\|^2$$

by virtue of the theorem on equivalent norms in $\mathring{W}_k^2(\Omega)$ (see [7]). Thus, the operator $Au + h^{-1}u$ is $\mathring{W}_2^k$ elliptic (see [7]) for all $h \leqq h_0 \leqq C_4^{-1}$.

If $u_{s-1} \in W_2^k(\Omega)$, then $F(t_s)\, u_{s-1} \in L_2(\Omega)$ because of (5). From the results on linear elliptic equations [7] (Theorem 3.1, Chap. 1) we conclude that there exists a unique solution $u_s \in \mathring{W}_2^k$ of (1′), (2′) for $h \leqq h_0$. Since $u_0 \in W_2^k(\Omega)$, we obtain successively $u_s \in \mathring{W}_2^k$ for $s = 1, \ldots, n$.

On the other hand, $u_s \in \mathring{W}_2^k(\Omega)$ is a solution of the equation

$$Au = \frac{u_s - u_{s-1}}{h} + F(t_s)\, u_{s-1} \equiv f_{h,s}$$

where $f_{h,s} \in L_2(\Omega)$. We prove that $D^\alpha f_{h,s} \in W_2^{(-k+1)}$ for $|\alpha| \leqq l - 1$ ($W_2^{(-k+1)}$ is the dual space to $\mathring{W}_2^{k-1}(\Omega)$). Indeed, we have

$$\sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ \|\varphi\| W_2^{k-1} \leqq 1}} |(D^\alpha f_{h,s}, \varphi)| = \sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ \|\varphi\| W_2^{k-1} \leqq 1}} |(f_{h,s}, D^\alpha \varphi)| =$$

$$= \sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ \|\varphi\| W_2^{k-1} \leqq 1}} \left| \int_\Omega f_{h,s}(x) \,.\, D^\alpha \varphi(x)\, dx \right| \leqq \|f_{h,s}\| \,.$$

(Here $(D^\alpha f_{h,s}, \varphi)$ denotes the value of the distribution $D^\alpha f_{h,s}$ at the point φ). Thus, from [7] (Theorem 1.2, Exercise 1.2, Chap. 4) we deduce that $u_s \in W_2^{k+l}(\Omega')$ and the estimate

(9) $$\|u_s\|_{W_2^{k+l}(\Omega')} \leqq C(\Omega') \left(\|u_s\|_W + \|f_{h,s}\| \right)$$

holds for all $h \leqq h_0$ and $s = 1, \ldots, n$.

In the sequel we shall assume that (3)–(6) are satisfied.

Lemma 2. *There exists* C *and* $h_0 > 0$ *such that the estimates*

$$\|u_s\| \leqq C\,, \quad \sum_{s=1}^n h \|u_s\|_W^2 \leqq C$$

take place for all $h \leqq h_0$, $s = 1, \ldots, n$.

Proof. Let us put $v = u_s$ into (7) and then sum up for $s = 1, \ldots, p$ where $1 \leqq p \leqq n$. We obtain

$$(10) \qquad \sum_{s=1}^{p} (u_s - u_{s-1}, u_s) + h \sum_{s=1}^{p} [Au_s, u_s] + \sum_{s=1}^{p} h(F(t_s)\, u_{s-1}, u_s) = 0 .$$

From the identity

$$2 . \sum_{s=1}^{p} (u_s - u_{s-1}, u_s) = \sum_{s=1}^{p} \|u_s - u_{s-1}\|^2 + \|u_p\|^2 - \|u_0\|^2$$

and from (8) we deduce

$$(11) \qquad \|u_p\|^2 + C_1 \sum_{s=1}^{p} h \|u_s\|_W^2 \leqq \|u_0\|^2 + C_2 \sum_{s=1}^{p} h \|u_s\|^2 + \sum_{s=1}^{p} h(F(t_s)\, u_{s-1}, u_s) .$$

Applying Young's inequality

$$(12) \qquad |ab| \leqq \frac{\varepsilon^2 a^2}{2} + \frac{b^2}{2\varepsilon^2}$$

we estimate

$$(13) \qquad \sum_{s=1}^{p} h |(F(t_s)\, u_{s-1}, u_s)| \leqq \sum_{s=1}^{p} h \|F(t_s)\, u_{s-1}\| \, \|u_s\| \leqq$$

$$\leqq \sum_{s=1}^{p} \frac{\varepsilon h}{2} \|F(t_s)\, u_{s-1}\|^2 + \sum_{s=1}^{p} \frac{h}{2\varepsilon} \|u_s\|^2 .$$

Owing to (5) we have

$$(14) \qquad \|F(t_s)\, u_{s-1}\|^2 \leqq C_3 + C_4 \|u_{s-1}\|_W^2$$

and hence, due to (11), (13) and (14), we conclude

$$\|u_p\|^2 + (C_1 - \varepsilon C_4) \sum_{s=1}^{p} h \|u_s\|_W^2 \leqq C(u_0) + \left(C_2 + \frac{1}{2\varepsilon}\right) \sum_{s=1}^{p} h \|u_s\|^2 .$$

Let us choose $\varepsilon > 0$ so that $C_1 - \varepsilon C_4 = \frac{1}{2} C_1$. Then we obtain

$$(15) \qquad \|u_p\|^2 + \tfrac{1}{2} C_1 \sum_{s=1}^{p} h \|u_s\|_W^2 \leqq C_5 + C_6 \sum_{s=1}^{p} h \|u_s\|^2$$

and, in particular,

$$\|u_p\|^2 \leqq C_5 + C_6 \sum_{s=1}^{p} h \|u_s\|^2 \quad \text{for all} \quad p = 1, \ldots, n ,$$

which $(h \leqq h_0 < C_6^{-1})$ implies successively

$$\|u_1\|^2 \leqq C_5 (1 - C_6 h)^{-1} , \quad \|u_2\|^2 \leqq C_5 (1 - C_6 h)^{-1} (1 + C_5 h (1 - C_6 h)^{-1})$$

and

$$(16)\qquad \|u_s\|^2 \leqq C_5(1 - C_6h)^{-1} . (1 + C_5h(1 - C_6h)^{-1})^{s-1}$$

for $s = 1, \ldots, n$. There exists C such that

$$(1 + C_5h(1 - C_6h)^{-1})^{s-1} \leqq C \quad \text{for all} \quad h \leqq h_0 < C_6^{-1}$$

and $s = 1, \ldots, n$. Thus, (16) implies the first part of Lemma 2. The rest of the proof follows from (11).

Lemma 3. *There exist C and $h_0 > 0$ such that the estimates*

$$\left\|\frac{u_s - u_{s-1}}{h}\right\|^2 \leqq C\,, \quad h^{-1}\|u_s - u_{s-1}\|_W^2 \leqq C$$

take place for all $h \leqq h_0$ and $s = 1, \ldots, n$.

Proof. Let us consider (7) for $s = i$ and $s = i - 1$, putting $v = u_i - u_{i-1}$. Subtracting these equalities we obtain

$$\left(\frac{u_i - u_{i-1}}{h}, u_i - u_{i-1}\right) + [A(u_i - u_{i-1}), u_i - u_{i-1}] +$$

$$+ (F(t_i)\, u_{i-1} - F(t_{i-1})\, u_{i-2}, u_i - u_{i-1}) = \left(\frac{u_{i-1} - u_{i-2}}{h}, u_i - u_{i-1}\right)$$

from where, due to (8), we deduce

$$(17)\qquad \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + C_1h^{-1}\|u_i - u_{i-1}\|_W^2 \leqq$$

$$\leqq \left\|\frac{u_{i-1} - u_{i-2}}{h}\right\| \left\|\frac{u_i - u_{i-1}}{h}\right\| + \left\|\frac{u_i - u_{i-1}}{h}\right\| \|F(t_i)\, u_{i-1} -$$

$$- F(t_{i-1})\, u_{i-2}\| + C_2 \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 . h .$$

By virtue of (5) we estimate

$$(18)\qquad \|F(t_i) - u_{i-1} - F(t_{i-1})\, u_{i-2}\| \leqq$$

$$\leqq C . (h + h\|u_{i-1}\|_W + \|u_{i-1} - u_{i-2}\|_W) .$$

Applying (12) we estimate

$$\left\|\frac{u_{i-1} - u_{i-2}}{h}\right\| \left\|\frac{u_i - u_{i-1}}{h}\right\| \leqq 2^{-1} \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + 2^{-1} \left\|\frac{u_{i-1} - u_{i-2}}{h}\right\|^2$$

and, owing to (18)

$$\left\|\frac{u_i - u_{i-1}}{h}\right\| \|F(t_i)\, u_{i-1} - F(t_{i-1})\, u_{i-2}\| \leqq C_3 \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 . h + $$
$$+ C_4 h + C_5 h \|u_{i-1}\|_W^2 + C_1 (2h)^{-1} \|u_{i-1} - u_{i-2}\|_W^2 .$$

From these estimates and from (17) we obtain

$$\text{(19)} \qquad \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 (1 - C_6 h) + C_1 h^{-1} \|u_i - u_{i-1}\|_W^2 \leqq$$
$$\leqq \left\|\frac{u_{i-1} - u_{i-2}}{h}\right\|^2 + C_1 h^{-1} \|u_{i-1} - u_{i-2}\|_W^2 + C_5 \|u_{i-1}\|_W^2 + C_4 h .$$

The estimate (19) is recurrent and enables us to obtain successively ($h \leqq h_0 < C_6^{-1}$)

$$\text{(20)} \qquad \left(\left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + C_1 h^{-1} \|u_i - u_{i-1}\|_W^2\right) (1 - C_6 h)^{i-1} \leqq$$
$$\leqq \left\|\frac{u_1 - u_0}{h}\right\|^2 + C_1 h^{-1} \|u_1 - u_0\|_W^2 + \sum_{j=2}^{i} (1 - C_6 h)^{i-j}\, C_5 h \|u_{j-1}\|_W^2 +$$
$$+ \sum_{j=1}^{i-1} (1 - C_6 h)^{j-1}\, C_4 h$$

where $2 \leqq i \leqq n$. Since $1 \geqq (1 - C_6 h)^{i-1} \geqq e^{-C_6 T}$ for all $h \leqq h_0$ and $i = 1, \ldots, n$, (20) implies

$$\text{(21)} \qquad \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + C_1 h^{-1} \|u_i - u_{i-1}\|_W^2 \leqq C . \left(\left\|\frac{u_1 - u_0}{h}\right\|^2 + \right.$$
$$\left. + C_1 h^{-1} \|u_1 - u_0\|_W^2 + \sum_{j=1}^{i} h \|u_{j-1}\|_W^2 + 1\right).$$

Now, we estimate the right hand side in (21). Putting $v = u_1 - u_0$ we have from (7)

$$\left(\frac{u_1 - u_0}{h}, u_1 - u_0\right) + [Au_1, u_1 - u_0] + (F(t_1)\, u_0, u_1 - u_0) = 0 .$$

Hence we deduce

(22)

$$\left\|\frac{u_1 - u_0}{h}\right\|^2 + h^{-1} [A(u_1 - u_0), u_1 - u_0] \leqq C(u_0) \left\|\frac{u_1 - u_0}{h}\right\| - \left[Au_0, \frac{u_1 - u_0}{h}\right].$$

Owing to (6) we estimate

$$\left\| \left[Au_0, \frac{u_1 - u_0}{h} \right] \right\| \leqq \|Au_0\| \left\| \frac{u_1 - u_0}{h} \right\|$$

and hence applying (12) we obtain from (22)

$$\left\| \frac{u_1 - u_0}{h} \right\|^2 (1 - \varepsilon C) + h^{-1}[A(u_1 - u_0), u_1 - u_0] \leqq C(u_0, \varepsilon) .$$

Thus, due to (8) we have

$$(23) \qquad \left\| \frac{u_1 - u_0}{h} \right\|^2 (1 - \varepsilon C - C_2 h) + C_1 h^{-1} \|u_1 - u_0\|_W^2 \leqq C(u_0, \varepsilon) ;$$

since $h \leqq h_0 < C_2^{-1}$ we can choose $\varepsilon > 0$ so that $(1 - \varepsilon C - C_2 h) > \alpha > 0$ for all $h \leqq h_0$, where α is a suitable constant. The proof of Lemma 3 follows from (23), (21) and Lemma 2.

Lemma 4. *There exists C and $h_0 > 0$ such that*

$$(24) \qquad \|u_i\|_W \leqq C \quad \textit{for all} \quad h \leqq h_0 \quad \textit{and} \quad i = 1, \ldots, n .$$

Proof. From (8) and (7) with $v = u_s$ we obtain

$$C_1 \|u_s\|_W^2 \leqq \left\| \frac{u_s - u_{s-1}}{h} \right\| \|u_s\| + \|F(t_s)\, u_{s-1}\| \, \|u_s\| + C_2 \|u_s\|^2 .$$

Owing to (5) the estimate

$$(25) \qquad \|F(t_s)\, u_{s-1}\| \leqq C \, . \, (1 + \|u_{s-1}\|_W)$$

takes place and hence Lemma 2, Lemma 3 and (24) imply

$$C_1 \|u_s\|_W^2 \leqq C_3 + C_4 \|u_{s-1}\|_W .$$

Due to Lemma 3 we have

$$\|u_{s-1}\|_W \leqq \|u_s\|_W + \|u_s - u_{s-1}\|_W \leqq \|u_s\|_W + C_5$$

and thus the estimate

$$C_1 \|u_s\|_W^2 \leqq C_6 + C_4 \|u_s\|_W$$

takes place for all $h \leqq h_0$ and $s = 1, \ldots, n$. Applying (13) to the last inequality we obtain the result required.

Lemma 5. *There exist* $C(\Omega')$ *and* $h_0 > 0$ *such that* $\|u_s\|_{W_2^{k+1}(\Omega')} \leqq C(\Omega')$ *for all* $h \leqq h_0$ *and* $s = 1, \ldots, n$.

Proof. From Lemma 3, Lemma 4 and (25) we deduce that there exists C such that

$$(26) \qquad \|f_{h,s}\| \leqq C \text{ for all } \quad h \leqq h_0 \quad \text{and} \quad s = 1, \ldots, n$$

where $f_{h,s} = -(u_s - u_{s-1})/h + F(t_s)\, u_{s-1}$. Thus, the result follows from (26), Lemma 4 and (9).

In the sequel we present some consequences from the a priori estimates just obtained.

We denote by $u^n(t)$ Rothe's function

$$u^n(t) = u_{s-1} + (t - t_{s-1})\, h^{-1}(u_s - u_{s-1}) \quad \text{for} \quad t_{s-1} \leqq t \leqq t_s\,, \quad s = 1, \ldots, n\,.$$

Lemma 6. *There exists* $u(t) \in C(\langle 0, T\rangle, L_2(\Omega))$ *and a subsequence* $\{u^{n_k}(t)\}$ *from* $\{u^n(t)\}$ *such that* $u^{n_k}(t) \to u(t)$ *in the norm of the space* $C(\langle 0, T\rangle, L_2(\Omega))$.

Proof. Lemma 4 implies the estimate

$$(27) \qquad \|u^n(t)\|_{W_2^k(\Omega)} \leqq C \quad \text{for all } n \text{ and } \quad t \in \langle 0, T\rangle$$

(C is independent of n). From the compactness of the imbedding $W_2^k(\Omega) \to L_2(\Omega)$ we conclude that for fixed $t \in \langle 0, T\rangle$ it is possible to choose a subsequence of $\{u^n(t)\}$ convergent in the norm of the space $L_2(\Omega)$. By the diagonal method we choose a subsequence $\{u^{n_k}(t)\}$ such that $u^{n_k}(t)$ is convergent in $L_2(\Omega)$ for each rational point $t \in \langle 0, T\rangle$. Owing to Lemma 3 we prove that $u^{n_k}(t)$ is convergent for all $t \in \langle 0, T\rangle$. From Lemma 3 and the triangle inequality we deduce

$$(28) \qquad \|u^n(t) - u^n(t')\| \leqq C|t - t'| \quad \text{for all } n \text{ and } \quad t, t' \in \langle 0, T\rangle\,.$$

Let $t' \in \langle 0, T\rangle$ be an irrational point and $t \in \langle 0, T\rangle$ a rational one. Thus, the inequality

$$(29) \qquad \|u^{n_k}(t') - u^{n_r}(t')\| \leqq \|u^{n_k}(t') - u^{n_k}(t)\| + \|u^{n_k}(t) - u^{n_r}(t)\| + \\ + \|u^{n_r}(t) - u^{n_r}(t')\|$$

together with (28) implies that $u^{n_k}(t)$ is convergent in $L_2(\Omega)$ for all $t \in \langle 0, T\rangle$. There exists $u(t) : \langle 0, T\rangle \to L_2(\Omega)$ such that $u^{n_k}(t) \to u(t)$ in $L_2(\Omega)$ for all $t \in \langle 0, T\rangle$. Regarding (28) we have $u(t) \in C(\langle 0, T\rangle, L_2(\Omega))$. From (29), passing to the limit for $r \to \infty$, we conclude that $u^{n_k}(t) \to u(t)$ locally uniformly, i.e., if $\varepsilon > 0$ there exist $K > 0$ and $\delta_t(\varepsilon) > 0$ such that $\|u^{n_k}(t') - u(t')\| < \varepsilon$ for all t' satisfying $|t' - t| < \delta_t(\varepsilon)$ and $k \geqq K$. Thus, the rest of the proof follows from the Borel covering theorem.

For a moment, denote the sequence $\{u^{n_k}(t)\}$ from Lemma 6 by $\{u^n(t)\}$.

Lemma 7. *Let $u(t) \in C(\langle 0, T\rangle, L_2(\Omega))$ be the same as in Lemma 6. The following assertions hold*:

a) *$u(t)$ is Lipschitz continuous from $\langle 0, T\rangle$ into $L_2(\Omega)$, i.e., $\|u(t) - u(t')\| \leqq \leqq C|t - t'|$ for all $t, t' \in \langle 0, T\rangle$;*

b) *$u(t) \in L_\infty(\langle \delta, T\rangle, \mathring{W}_2^k(\Omega) \cap W_2^{k+l}(\Omega'))$ for all $0 < \delta < T$. If $u_0 \in W_2^{k+l}(\Omega)$, then $\delta = 0$.*

c) *$u^n(t) \to u(t)$ in the norm of the space $W_2^{k+l-1}(\Omega') \cap W_2^{k-1}(\Omega)$ for all $t \in (0, T)$.*

d) *$u(t) \in C(\langle \delta, T\rangle, W_2^{k+l-1}(\Omega') \cap \mathring{W}_2^{k-1}(\Omega))$. If $u_0 \in W_2^{k+l}(\Omega)$, then $\delta = 0$.*

Proof. Assertion a) follows from Lemma 6 and (28). b) The space $H \equiv \mathring{W}_2^k(\Omega) \cap \cap W_2^{k+l}(\Omega')$ is a separable Hilbert space (with respect to the scalar product

$$(.,.)_H = (.,.)_{W_2^k(\Omega)} + (.,.)_{W_2^{k+l}(\Omega')}) .$$

Since $L_\infty(\langle 0, T\rangle, H)$ is the dual space to the separable Banach space $L_1(\langle 0, T\rangle, H')$ (see [8]), where H' is the dual space to H, bounded sets in $L_\infty(\langle 0, T\rangle, H)$ are compact with respect to the weak* topology (see [9], [10]). From Lemma 4 and Lemma 5 we deduce that

$$\max_{t \in \langle T/n, T\rangle} \|u^n(t)\|_H \leqq C \quad \text{for all } n \text{ (} C \text{ is independent of } n\text{)} . \tag{30}$$

Thus, if $0 < \delta < T$ then there exists $w \in L_\infty(\langle \delta, T\rangle, H)$ and a subsequence $\{u^{n_k}\}$ of $\{u^n\}$ such that $u^{n_k} \to_{w*} w$ in $L_\infty(\langle \delta, T\rangle, H)$ (weak* convergence). From this fact it follows also that $u^{n_k} \rightharpoonup w$ in $L_2(\langle \delta, T\rangle, L_2(\Omega)) \supset L_\infty(\langle \delta, T\rangle, H)$ (weak convergence) and hence due to Lemma 6, we have $u(t) \equiv w(t)$. If $u_0 \in W_2^{k+l}(\Omega)$ then we can put $\delta = 0$. Moreover, owing to (30) and Assertion a) we deduce easily that $u(t) \in H$ for all $t \in \langle T/n, T\rangle$ and the estimate

$$\sup_{t \in \langle \delta, T\rangle} \|u(t)\|_H \leqq C \quad \text{for all} \quad \delta > 0 , \quad \text{where } C \text{ is from (27)} . \tag{31}$$

(If $u_0 \in W_2^{k+l}(\Omega)$, then $\delta = 0$.)

c) Assertion c) follows from the compactness of the imbeddings $W_2^k(\Omega) \to W_2^{k-1}(\Omega)$ and $W_2^{k+l}(\Omega') \to W_2^{k+l-1}(\Omega')$, from the estimate (30) and Lemma 6.

d) Assertion d) follows from (31), from the compactness of the imbeddings

$$W_2^k(\Omega) \to W_2^{k-1}(\Omega) , \quad W_2^{k+l}(\Omega') \to W_2^{k+l-1}(\Omega')$$

and Assertion a).

Remark 2. In virtue of (31) and Lemma 7 (a)) we prove easily that $u(t) \in \in C_w(\langle \delta, T\rangle, H)$ for all $0 < \delta < T$. If $u_0 \in W_2^{k+l}(\Omega)$, then $\delta = 0$.

Indeed, if $t_n \to t_0$ ($t_n, t_0 \in \langle \delta, T\rangle$) then $u(t_{n_k}) \rightharpoonup w$ in the reflexive space H, because of (31). But, owing to Lemma 7 (a)) we have $w = u(t_0)$. Thus, $u(t_n) \rightharpoonup u(t_0)$ in H from which the required result follows.

2. EXISTENCE, UNIQUENESS, REGULARITY AND CONVERGENCE OF THE METHOD

In this section we prove that $u(t)$ from Lemma 6 is the unique solution of (1), (2).

Let us define the step function $w^n(t) : (0, T\rangle \to H$ where $H \equiv W_2^{k+l}(\Omega') \cap \mathring{W}_2^k(\Omega)$ by

$$w^n(t) = u_s \quad \text{for} \quad t_{s-1} < t \leqq t_s, \quad s = 1, \dots, n$$

and $w^n(t) = u_0$ for $t \in \langle -T/n, 0\rangle$.

Similarly, we define $F^n(t) : W_2^k \to L_2(\Omega)$ by $F^n(t)\,u = F(t_s)\,u$ for all $u \in W_2^k(\Omega)$, $t_{s-1} < t \leqq t_s$, $s = 1, \dots, n$ and $F^n(0)\,u = F(0)\,u$.

The identity (7) can be rewritten in the form

$$(32) \qquad \left(\frac{\mathrm{d}^- u^n(t)}{\mathrm{d}t}, v\right) + [A\,w^n(t), v] + \left(F^n(t)\,w^n\left(t - \frac{T}{n}\right), v\right) = 0$$

for all $v \in \mathring{W}_2^k(\Omega)$ and $t \in (0, T)$, where $\mathrm{d}^-/\mathrm{d}t$ is the lefthand derivative. Integrating (32) over $\langle 0, t)$ we obtain

$$(33) \qquad (u^n(t), v) + \int_0^t [A\,w^n(\tau), v]\,\mathrm{d}\tau + \int_0^t \left(F^n(\tau)\,w^n\left(\tau - \frac{T}{n}\right), v\right)\mathrm{d}\tau - (u_0, v) = 0 \quad \text{for all} \quad v \in \mathring{W}_2^k(\Omega)\,.$$

Before we pass to the limit $n \to \infty$ in (33) we prove some auxiliary assertions.

Lemma 3 and (30) imply

$$(34) \qquad \|w^n(t) - u^n(t)\| \leqq Cn^{-1} \quad \text{for all} \quad t \in \langle 0, T\rangle$$

and

$$(35) \qquad \|w^n(t)\|_H + \left\|w^n\left(t - \frac{T}{n}\right)\right\|_H \leqq C \quad \text{for all } n \text{ and } \frac{2T}{n} < t \leqq T\,.$$

From (34), (35) we deduce easily that

$$(36) \qquad w^n(t) \to u(t) \quad \text{and} \quad w^n\left(t - \frac{T}{n}\right) \to u(t)$$

in the norm of the space $W_2^{k+l-1}(\Omega') \cap W_2^{k-1}(\Omega)$ for all $t \in (0, T)$.

Lemma 8. a) If $v \in \mathring{W}_2^k(\Omega)$, then

$$\left(F^n(t)\,w^n\left(t - \frac{T}{n}\right), v\right) \to (F(t)\,u(t), v) \quad \text{for} \quad n \to \infty\,;$$

b) $(F(t)\,u(t), v) \in C(\langle 0, T\rangle) \cap L_\infty(\langle 0, T\rangle)$ *and the estimate*

$$|(F(t)\,u(t), v)| \leqq C\|v\|$$

takes place for all $t \in (0, T\rangle$ *and* $v \in L_2(\Omega)$.

Proof. It suffices to prove a), b) for $v \in C_0^\infty(\Omega)$, because of the density. To $v \in C_0^\infty(\Omega)$ there exists Ω' with $\bar{\Omega}' \subset \Omega$ such that the support of v is a subset of Ω'. From (36) and (5) we obtain

$$F^n(t)\,w^n\left(t - \frac{T}{n}\right) \to F(t)\,u(t) \quad \text{with} \quad n \to \infty \quad \text{in} \quad L_2(\Omega')$$

from which a) follows.

b) Similarly as in Assertion a) from (5) and Lemma 7 (c)) we deduce $(F(t)\,u(t), v) \in C(\langle 0, T\rangle)$. From the estimate (31) and from (5) we obtain

$$|(F(t)\,u(t), v)| \leqq \|F(t)\,u(t)\|\,\|v\| \leqq C\|v\|$$

for all $t \in \langle 0, T\rangle$, $v \in L_2(\Omega)$ and the proof is complete.

Lemma 9. *Let* $u(t)$ *be from Lemma* 6. *Then*

a) $A\,u(t) \in L_2(\Omega)$ *for all* $t \in \langle 0, T\rangle$ *with*

$$\|A\,u(t)\| \leqq C \quad \textit{for all} \quad t \in \langle 0, T\rangle\,.$$

b) $[A\,u(t), v] \in C(\langle 0, T\rangle)$ *for all* $v \in \mathring{W}_2^k(\Omega)$.

Proof. From Lemma 3 we have $\|\mathrm{d}^-\,u^n(t)/\mathrm{d}t\| \leqq C$ for all $t \in (0, T\rangle$. From the definition of $w^n(t)$, Lemma 4 and (5) we deduce easily the estimate

$$\left\|F^n(t)\,w^n\left(t - \frac{T}{n}\right)\right\| \leqq C \quad \text{for all} \quad t \in \langle 0, T\rangle$$

and hence (33) implies the estimate

(37) $$|[A\,w^n(t), v]| \leqq C\|v\| \quad \text{for all} \quad t \in \langle 0, T\rangle \quad \text{and} \quad v \in C_0^\infty(\Omega)\,.$$

Since

(38) $$[A\,w^n(t), v] \to [A\,u(t), v]$$

(see (36)), we obtain from (37) that

(39) $$|[A\,u(t), v]| \leqq C\|v\| \quad \text{for all} \quad t \in \langle 0, T\rangle \quad \text{and} \quad v \in C_0^\infty(\Omega)\,.$$

The density $C_0^\infty(\Omega)$ in $L_2(\Omega)$ and (39) then implies Assertion a).

b) It suffices to prove Assertion b) for $v \in C_0^\infty(\Omega)$. In this case the result required follows from the continuity of the operator $A : \mathring{W}_2^k \to W_2^{-k}$ and from Lemma 7 (d)).

Remark 3. Lemma 9 implies

$$A\,u(t) \in L_\infty(\langle 0, T\rangle, L_2(\Omega)) \cap C_w(\langle 0, T\rangle, L_2(\Omega))\,.$$

Indeed, it follows from Lemma 9 (a), b)), (39) and the density of $\mathring{W}_2^k(\Omega)$ in $L_2(\Omega)$ that

$$[A\,u(t), v] \in C(\langle 0, T)) \quad \text{for all} \quad v \in L_2(\Omega)$$

and hence (see [10]) $A\,u(t)$ is a measurable abstract function from $\langle 0, T\rangle \to L_2(\Omega)$. The rest of the proof follows from Lemma 9.

Our main result is

Theorem 1. *There exists a solution $u(t)$ of the problem* (1), (2) *with the following properties*:

a) $u(t) : \langle 0, T\rangle \to L_2(\Omega)$ *is Lipschitz continuous*;

b) $u(t) \in C_w^1(\langle 0, T), L_2(\Omega))$ *and there exists $u'(t)$ (strong derivative) for a.e. $t \in (0, T)$ with $u'(t) = \mathrm{d}\,u(t)/\mathrm{d}t \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$*;

c) $u(t) \in L_\infty(\langle 0, T\rangle, W_2^{k+l}(\Omega') \cap \mathring{W}_2^k(\Omega)) \cap C_w(\langle 0, T\rangle, W_2^{k+l}(\Omega') \cap \mathring{W}_2^k(\Omega))$. If $u_0 \in W_2^{k+l}(\Omega)$ *then we can put $\langle 0, T\rangle$ instead of $(0, T\rangle$.*

d) $u(t) \in C(\langle \delta, T\rangle, W_2^{k+l-1}(\Omega') \cap \mathring{W}_2^{k-1}(\Omega)$ *for all* $0 < \delta < T$. If $u_0 \in W_2^{k+l}(\Omega)$ *then* $\delta = 0$.

e) $A\,u(t) \in L_\infty(\langle 0, T\rangle, L_2(\Omega)) \cap C_w(\langle 0, T\rangle, L_2(\Omega))$.

Proof. We prove that $u(t)$ from Lemma 6 is a solution of (1), (2). Let $v \in C_0^\infty(\Omega)$ in (33). Owing to Lemma 6, Lemma 8, (37), (38) and Lebesque's theorem, the limiting process $n \to \infty$ in (33) enables us to deduce

$$(40) \quad (u(t), v) + \int_0^t [A\,u(\tau), v]\,\mathrm{d}\tau + \int_0^t (F(\tau)\,u(\tau), v)\,\mathrm{d}\tau - (u_0, v) = 0 \quad \text{for all}$$

$$t \in (0, T\rangle \quad \text{and} \quad v \in C_0^\infty(\Omega) \quad \text{and hence also for} \quad v \in \mathring{W}_2^k(\Omega)\,.$$

From (40), with regard to Lemma 8, Lemma 9 and (39), we conclude $u(t) \in C_w^1(\langle 0, T), L_2(\Omega))$. Differentiating in (40) we obtain that $u(t)$ is a solution of (1), (2). The identity $(\mathrm{d}/\mathrm{d}t)\,(u(t), w) = (\mathrm{d}\,u(t)/\mathrm{d}t, w)$ for all $t \in (0, T)$ and $w \in L_2(\Omega)$ implies the identity

$$\int_0^T (u(t), \psi'(t)\,w)\,\mathrm{d}t = -\int_0^T \left(\frac{\mathrm{d}\,u(t)}{\mathrm{d}t}, \psi(t)\,w\right)\mathrm{d}t$$

for all $w \in L_2(\Omega)$ and $\psi(t) \in C_0^\infty(\langle 0, T\rangle)$. Thus, $u(t) \in W_2^1(\langle 0, T\rangle, L_2(\Omega))$ – see [2] (Definition 3) – and hence, owing to [2] (Lemma 1) there exists the strong derivative

$u'(t)$ for a.e. $t \in (0, T)$ and the equality $u'(t) = \mathrm{d}\, u(t)/\mathrm{d}t$ is true. From (40), owing to Lemma 7 and Lemma 8, we deduce

$$\left|\left(\frac{\mathrm{d}\, u(t)}{\mathrm{d}t}, v\right)\right| \leqq C\|v\| \quad \text{for all} \quad t \in (0, T) \quad \text{and} \quad v \in L_2(\Omega)\,.$$

The continuity of $(\mathrm{d}\, u(t)/\mathrm{d}t, w)$ implies the measurability of the abstract function $\mathrm{d}\, u(t)/\mathrm{d}t$. Thus, Assertion b) is proved. The other assertions are proved in the previous lemmas and remarks. Assertion c) follows from Lemma 7 and the estimate (31), where C is independent of δ.

The idea of the proof of uniqueness is due to [6].

Theorem 2. *The solution of* (1), (2) *is unique.*

Proof. Let $u_1(t)$ and $u_2(t)$ be two solutions of (1), (2). Then $u(t) = u_1(t) - u_2(t)$ satisfies

$$\left(\frac{\mathrm{d}\, u(t)}{\mathrm{d}t}, v\right) + [A\, u(t), v] + (F(t)\, u_1(t) - F(t)\, u_2(t), v) = 0 \tag{41}$$

for all $v \in \mathring{W}_2^k(\Omega)$. Let us put $v = e^{-\lambda t}\, u(t)$ into (41). From the properties of $u(t)$ (Theorem 1c)) and from (41) we deduce that $[A\, u(t), u(t)]$ is a continuous function in $t \in (0, T)$. Thus, integrating (41) over the interval $\langle 0, t_0)$ $(0 < t_0 \leqq T)$ we have

$$\int_0^{t_0} e^{-\lambda t}\left(\frac{\mathrm{d}\, u(t)}{\mathrm{d}t}, u(t)\right) \mathrm{d}t + \int_0^{t_0} e^{-\lambda t}\{[A\, u(t), u(t)] + (F(t)\, u_1(t) - F(t)\, u_2(t), u(t))\}\, \mathrm{d}t = 0\,. \tag{42}$$

Since $(\mathrm{d}/\mathrm{d}t)\, \|u(t)\|^2 = 2(\mathrm{d}\, u(t)/\mathrm{d}t, u(t))$ and

$$e^{-\lambda t} \frac{\mathrm{d}}{\mathrm{d}t}\, \|u(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\,(\|u(t)\|^2\, e^{-\lambda t}) + \lambda\,.\, \|u(t)\|^2\, e^{-\lambda t}$$

due to $u(0) = 0$, we obtain from (42) that

$$2^{-1}\,.\, \|u(t_0)\|^2\, e^{-\lambda t_0} + \int_0^{t_0} e^{-\lambda t}(\lambda\,.\,2^{-1}\|u(t)\|^2 + [A\, u(t), u(t)] + (F(t)\, u_1(t) - F(t)\, u_2(t), u(t)))\, \mathrm{d}t = 0\,. \tag{43}$$

Owing to (5) and Schwartz's inequality we estimate

$$|(F(t)\, u_1(t) - F(t)\, u_2(t), u(t))| \leqq C\|u(t)\|_W\,.\, \|u(t)\| \leqq \leqq \varepsilon^2\,.\, 2^{-1}\|u(t)\|_W^2 + C^2\,.\,2^{-1}\varepsilon^{-2}\|u(t)\|^2\,. \tag{44}$$

From (8) and (44) (putting $\varepsilon = \sqrt{C_3}$) we obtain

$$[A\, u(t), u(t)] + (F(t)\, u_1(t) - F(t)\, u_2(t), u(t)) \geqq -\, C\|u(t)\|^2$$

for all $t \in (0, T)$. If we take $\lambda > 2C$, then (43) implies $\|u(t_0)\| \leqq 0$ for all $0 < t_0 \leqq T$, which yields the result required.

In Lemma 6 and Lemma 7 (c) we have proved that there exists a subsequence $\{u^{n_k}(t)\}$ of the sequence $\{u^n(t)\}$ (sequence of Rothe's functions) which converges to the solution $u(t)$ of (1), (2) in the corresponding norms. As a consequence of the Uniqueness Theorem 2 we obtain

Theorem 3. *The sequence of Rothe's functions converges to the solution $u(t)$ of (1), (2) in the following norms:*

a) $u^n(t) \to u(t)$ *in* $C(\langle 0, T\rangle, L_2(\Omega))$,

b) $u^n(t) \to u(t)$ *in the norm of the space* $W_2^{k-1}(\Omega) \cap W_2^{k+l-1}(\Omega')$ *for each* $t \in (0, T)$.

Theorem 4. *If the assumption (3) is satisfied for $l = k$, then the solution $u(t) = u(x, t)$ of (1), (2) satisfies (1) in the classical sense for a.e. $(x, t) \in Q \equiv \Omega \times (0, T)$.*

Proof. Owing to Theorem 1 (c) with $l = k$) it suffices to prove that there exists the distribution derivative $\partial u(x, t)/\partial t \in L_2(Q)$ (see [7], Theorem 2.2 Chap. 2 and Remark 1.2 Chap. 4). Let $\psi(t) \in C_0^\infty(\langle 0, T\rangle)$ and $\varphi(x) \in C_0^\infty(\Omega)$. Then, we have

$$\int_0^T \int_\Omega u(x, t)\, \psi'(t)\, \varphi(x)\, \mathrm{d}x\, \mathrm{d}t = - \int_0^T \int_\Omega g(x, t)\, \psi(t)\, \varphi(x)\, \mathrm{d}x\, \mathrm{d}t$$

where $g(x, t) \in L_2(Q)$ is generated by the abstract function $\mathrm{d}\, u(t)/\mathrm{d}t \in L_\infty(\langle 0, T\rangle, L_2(\Omega)) \subset L_2(Q)$ – see the proof of Theorem 1. Since linear combinations of all $\psi(t)\, \varphi(x)$ are dense in $C_0^\infty(Q)$, Theorem 4 is proved.

Remark 4. If we consider a nonhomogeneous problem (1), (2), i.e., if (2) is of the form

$$u(x, 0) = u_0(x, 0)\,, \quad D_\nu^l\, u(x, t)|_{\partial\Omega \times (0,T)} = D_\nu^l\, u_0(x, t)|_{\partial\Omega \times (0,T)}$$

for $l = 0, 1, \ldots, k - 1$, where $u_0(x, t)$ is a sufficiently smooth function in Q, then we solve the homogeneous problem

$$\frac{\partial z}{\partial t} + \sum_{|i|,|j| \leq k} (-1)^{|i|}\, D^i(a_{ij}(x)\, D^j z) + a^*(t, x, Dz) = 0$$

$$z(x, 0) = 0\,, \quad D_\nu^l\, z(x, t)|_{\partial\Omega \times (0,T)} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k - 1$$

where

$$a^*(t, x, Dz) = a(t, x, Du_0 + Dz) -$$

$$- \sum_{|i|,|j| \leqq k} (-1)^{|i|} D^i(a_{ij}(x) D^j u_0) - \frac{\partial u_0(x, t)}{\partial t}.$$

Then the solution is of the form $u(x, t) = z(x, t) + u_0(x, t)$.

Remark 4. The assumption that $\{t_s\}_{s=1}^n$ is a uniform partition of the interval $\langle 0, T\rangle$ is not essential in this paper. We can consider an arbitrary partition $\{t_j\}_{j=1}^n$ of $\langle 0, T\rangle$, the norm of which converges to zero with $n \to \infty$.

Now, we shall be concerned with the dependence of the solution $u(t)$ on u_0 from (2) and on the operator $F(t)$.

Let $u_i(t)$ $(i = 1, 2)$ be the solution of (1), (2) corresponding to $u_0 = u_{0i}$ and $F(t) u = F_i(t) u = a_i(t, x, Du)$.

Theorem 5. *If*

(45) $$\|F_1(t) u - F_2(t) u\| \leqq a(t) + b(t) \|u\|_W \quad \text{for all} \quad u \in W_2^k(\Omega),$$

where $a(t)$, $b(t)$ are continuous nonnegative functions in $\langle 0, T\rangle$, then the estimate

(46)
$$\|u_1(t) - u_2(t)\|^2 \leqq e^{Kt} \left(\|u_{01} - u_{02}\|^2 + \max_{t \in \langle 0,T\rangle} \|u_1(t)\|_W^2 \cdot \int_0^t a^2(\tau) \, d\tau + \int_0^t b^2(\tau) \, d\tau \right)$$

takes place for all $t \in \langle 0, T\rangle$. (The constant $K > 0$ depends only on C_3, C_4 from (8) and C from (5).)

Proof. From Definition 3 we deduce

$$\left(\frac{d(u_1(t) - u_2(t))}{dt}, u_1(t) - u_2(t)\right) + [A(u_1(t) - u_2(t)), u_1(t) - u_2(t)] +$$
$$+ (F_1(t) u_1(t) - F_2(t) u_2(t), u_1(t) - u_2(t)) = 0$$

for all $t \in (0, T)$. Hence, integrating this equality over $\langle 0, t\rangle$ and using (8) we obtain

(47) $$\|u_1(t) - u_2(t)\|^2 + C_3 \int_0^t \|u_1(\tau) - u_2(\tau)\|_W^2 \, d\tau \leqq$$
$$\leqq \|u_{01} - u_{02}\|^2 + \int_0^t \|F_1(\tau) u_1(\tau) - F_2(\tau) u_2(\tau)\| \, \|u_1(\tau) - u_2(\tau)\| \, d\tau.$$

Owing to (5) and (45) we conclude

$$\|F_1(t)\,u_1(t) - F_2(t)\,u_2(t)\| \leqq \|F_1(t)\,u_1(t) - F_2(t)\,u_1(t)\| + \\ + \|F_2(t)\,u_1(t) - F_2(t)\,u_2(t)\| \leqq a(t)\,\|u_1(t)\|_W + C\|u_1(t) - u_2(t)\|_W + b(t)$$

and hence if (12) is applied suitably, then (47) yields

$$(48)\qquad \|u_1(t) - u_2(t)\|^2 + C_5 \int_0^t \|u_1(\tau) - u_2(\tau)\|_W^2\,d\tau \leqq \|u_{01} - u_{02}\|^2 + \\ + \max_{t\in\langle 0,T\rangle} \|u_1(t)\|_W^2 \,.\int_0^t a^2(\tau)\,d\tau + \int_0^t b^2(\tau)\,d\tau + K\int_0^t \|u_1(\tau) - u_2(\tau)\|^2\,d\tau\,.$$

Thus, (46) is a consequence of Gronwall's lemma (see [12]).

Let $u_n(t)$ $(n = 1, \ldots)$ be the solution of (1), (2) corresponding to u_{0n} (from (2)) and to the operator $F_n(t)\,v = a_n(t, x, Dv)$. We shall assume that

$$|a_n(t, x, \xi) - a_n(t', x, \xi')| \leqq C\,.\,(|t - t'| + |t - t'|\,|\xi| + |\xi - \xi'|)$$

holds for all $t, t' \in \langle 0, T\rangle$, $\xi', \xi \in E^d$ and $n = 1, \ldots$ and

$$\|F_n(t)\,u - F(t)\,u\| \leqq a_n(t)\,\|u\|_W + b_n(t)\,.$$

As a consequence of Theorem 5 we obtain

Theorem 6. *If*

$$\int_0^T a_n^2(\tau)\,d\tau \to 0\,, \quad \int_0^T b_n^2(\tau)\,d\tau \to 0\,, \quad \|u_{0n} - u_0\| \to 0$$

for $n \to \infty$, *then* $u_n(t) \to u(t)$ *in the norm of the space* $C(\langle 0, T\rangle, L_2(\Omega))$.

3.

In this section we shall be concerned with the approximate solution $u^n(t)$ (Rothe's function) which we construct by means of the predictor-corrector scheme – see the problems (1″), (2″) and (1‴), (2‴) in the introduction. First of all we prove a certain a priori estimate for $u^n(t)$ from which, similarly as in § 1, § 2, we deduce that $u^n(t)$ converges to the solution $u(t)$ of (1), (2). A priori estimates are obtained by similar techniques as in § 1 and thus we do not go into details. Assumption (5) will be considered (for simplicity) in the more special form

$$(5^*)\qquad |a(t, x, \xi) - a(t', x, \xi')| \leqq C\,.\,(|t - t'| + |\xi - \xi'|)\,.$$

$v_s \in \mathring{W}_2^k(\Omega)$ $(s = 1, \ldots, n)$ is a weak solution of (1″), (2″) (u_{s-1} being given) if

$$(49)\qquad \left(\frac{v_s - u_{s-1}}{h}, w\right) + [Av_s, w] + (F(t_s)\,u_{s-1}, w) = 0$$

holds for all $w \in \mathring{W}_2^k(\Omega)$.

$u_s \in \mathring{W}_2^k(\Omega)$ $(s = 1, \ldots, n)$ is a weak solution of (1'''), (2''') if

$$(50) \qquad \left(\frac{u_s - u_{s-1}}{h}, w\right) + [Au_s, w] + (F(t_s)\, v_s, w) = 0$$

holds for all $w \in \mathring{W}_2^k(\Omega)$.

Existence, uniqueness and regularity of u_s, v_s $(s = 1, \ldots, n)$ are guaranteed by Lemma 1.

Lemma 10. *There exist C and $h_0 > 0$ such that the estimate $\|(u_s - u_{s-1})/h\|^2 + h^{-1}\|u_s - u_{s-1}\|_W^2 \leqq C$ holds for all $s = 1, \ldots, n$ and $h \leqq h_0$.*

Proof. Subtracting (49) and (50), where $w = v_s - u_s$, we obtain

$$\left(\frac{v_s - u_s}{h}, v_s - u_s\right) + [A(v_s - u_s), v_s - u_s] + (F(t_s)\, u_{s-1} - F(t_s)\, v_s, v_s - u_s) = 0$$

and hence by applying (12) and (8) we deduce

$$\left\|\frac{v_s - u_s}{h}\right\|^2 (1 - C_3 h) + C_1 h^{-1}\|v_s - u_s\|_W^2 \leqq C_1 (4h)^{-1}\, \|u_s - v_s\|_W^2 .$$

Since

$$\|u_{s-1} - v_s\|_W^2 \leqq 2\|u_s - u_{s-1}\|_W^2 + 2\|u_s - v_s\|_W^2$$

we have the estimate

$$(51) \quad \left\|\frac{v_s - u_s}{h}\right\|^2 (1 - C_3 h) + C_1 (2h)^{-1}\, \|v_s - u_s\|_W^2 \leqq C_1 (2h)^{-1}\, \|u_s - u_{s-1}\|_W^2 .$$

Let us consider (50) for $s = i$ and $s = i - 1$ with $w = u_i - u_{i-1}$. Subtracting these equalities we obtain

$$\left(\frac{u_i - u_{i-1}}{h}, u_i - u_{i-1}\right) + [A(u_i - u_{i-1}), u_i - u_{i-1}] =$$
$$= -(F(t_i)\, v_i - F(t_{i-1})\, v_{i-1}, u_i - u_{i-1})$$

and hence applying again (12) and (8) we conclude that

$$(52) \qquad \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 (1 - C_4 h) + C_1 h^{-1}\|u_i - u_{i-1}\|_W^2 \leqq$$
$$\leqq \left\|\frac{u_{i-1} - u_{i-2}}{h}\right\|^2 + C_1 (12h)^{-1}\, \|v_i - v_{i-1}\|_W^2 + C_5 h .$$

The estimates

$$\|v_i - v_{i-1}\|_W^2 \leqq 3\|u_i - v_i\|_W^2 + 3\|u_i - u_{i-1}\|_W^2 + 3\|u_{i-1} - v_{i-1}\|_W^2 ,$$

(51) and (52) yield

$$(53) \qquad \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 (1 - C_4 h) + C_1 (2h)^{-1} \|u_i - u_{i-1}\|_W^2 \leqq$$

$$\leqq \left\|\frac{u_{i-1} - u_{i-2}}{h}\right\|^2 + C_1 (2h)^{-1} \|u_{i-1} - u_{i-2}\|_W^2 + C_5 h$$

where $i = 2, \ldots, n$ and $h \leqq h_0$ (h_0 is sufficiently small). From (53) we deduce successively

$$(54) \qquad \left(\left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + C_1 (2h)^{-1} \|u_i - u_{i-1}\|_W^2\right)(1 - C_4 h)^{i-2} \leqq$$

$$\leqq \left\|\frac{u_1 - u_0}{h}\right\|^2 + C_1 (2h)^{-1} \|u_1 - u_0\|_W^2 + CT \quad (j = 2, \ldots, n).$$

It remains to estimate the right hand side in (54). From (49) for $s = 1$ and $w = v_1 - u_0$ we have

$$\left(\frac{v_1 - u_0}{h}, v_1 - u_0\right) + [A(v_1 - u_0), v_1 - u_0] = -(F(t_1) u_0, v_1 - u_0) - [Au_0, v_1 - u_0].$$

Hence, by applying once more (12), (8) and (6) we conclude

$$(55) \qquad \left\|\frac{v_1 - u_0}{h}\right\|^2 (1 - C_2 h) + C_1 h^{-1} \|v_1 - u_0\|_W^2 \leqq C.$$

Similarly, (50) yields

$$(56) \qquad \left\|\frac{u_1 - u_0}{h}\right\|^2 (1 - C_2 h) + C_1 h^{-1} \|u_1 - u_0\|_W^2 \leqq C_3 + C_4 \|v_1\|_W.$$

But, (55) implies $\|v_1 - u_0\|_W \leqq C$ and hence the result required follows from (56) and (54).

Lemma 11. *There exists C and $h_0 > 0$ such that the estimate*

$$\left\|\frac{v_s - u_s}{h}\right\|^2 + h^{-1} \|v_s - u_s\|_W^2 \leqq C$$

takes place for all $s = 1, \ldots, n$ and $h \leqq h_0$.

Lemma 11 is a consequence of Lemma 10 and (51).

Lemma 12. *There exist C and $h_0 > 0$ such that the estimate $\|u_s\|_W + \|v_s\|_W \leqq C$ holds for all $s = 1, \ldots, n$ and $h \leqq h_0$.*

Proof. From Lemma 10 and the triangle inequality we deduce

(57) $$\|u_s\| \leqq C \quad \text{for all} \quad s = 1, \ldots, n \quad \text{and} \quad h \leqq h_0 .$$

From (50) for $w = u_s$ we have

$$\left(\frac{u_s - u_{s-1}}{h}, u_s\right) + [Au_s, u_s] = -(F(t_s)\, v_s, u_s) .$$

Thus, owing to (8), (57) and Lemma 10, we deduce

(58) $$\|u_s\|_W^2 \leqq C_1 + C_2\|v_s\|_W \quad \text{for all} \quad s = 1, \ldots, n \quad \text{and} \quad h \leqq h_0 .$$

Due to Lemma 11 we have the estimate

$$\|v_s\|_W \leqq C_3 + \|u_s\|_W$$

and hence, using (12), we conclude from (58) that

$$\|u_s\|_W \leqq C \quad \text{for all} \quad s = 1, \ldots, n .$$

The rest of the proof follows from Lemma 11.

By means of $u_s, v_s, s = 1, \ldots, n$ we define Rothe's functions $u^n(t), v^n(t)$. Lemma 11 implies

(59) $$\|u^n(t) - v^n(t)\|_W^2 \leqq Cn^{-1}$$

and Lemma 10 and Lemma 11 yield

(60) $$\left\|\frac{v_s - v_{s-1}}{h}\right\| \leqq C \quad \text{for all} \quad s = 1, \ldots, n \quad \text{and} \quad h \leqq h_0 \quad \text{where} \quad v_0 \equiv u_0 .$$

By virtue of the a priori estimates in Lemma 10, Lemma 11 and (59) we prove by the same method as in § 1 and § 2

Theorem 7. *Let $u^n(t)$ be (Rothe's function) of the form: $u^n(t) = u_{s-1} + (t - t_{s-1})\,.\,h^{-1}(u_s - u_{s-1})$ for $t_{s-1} \leqq t \leqq t_s$, $s = 1, \ldots, n$ ($u_0 = u_0(x)$ from (2)), where u_s ($s = 1, \ldots, n$) are solutions of (50). Then $u^n(t)$ converges to the unique solution $u(t)$ of (1), (2) in the following norms:*

a) *$u^n(t) \to u(t)$ in $C(\langle 0, T\rangle, L_2(\Omega))$,*
b) *$u^n(t) \to u(t)$ in the norm of the space $W_2^{k-1}(\Omega) \cap W_2^{k+l-1}(\Omega')$ for all $t \in (0, T)$.*

Remark 6. Rothe's function $v^n(t)$: $v^n(t) = v_{s-1} + (t - t_{s-1}) h^{-1}(v_s - v_{s-1})$ for $t_{s-1} \leqq t \leqq t_s$, $s = 1, \ldots, n$ $(v_0 \equiv u_0)$ where v_s $(s = 1, \ldots, n)$ are solutions of (49) also converges to the solution $u(t)$ of (1), (2). Theorem 7 holds true for $v^n(t)$ instead of $u^n(t)$, since the same a priori estimates have been proved for v_s $(s = 1, \ldots, n)$ as for u_s $(s = 1, \ldots, n)$.

Remark 7. Using the results on regularity for linear elliptic equations in the interior of the domain Ω we have proved regularity of the solution $u(t)$ of (1), (2):

$$u(t) \in L_\infty(\langle 0, T\rangle, W_2^{k+l}(\Omega')) \cap C_w((0,T), W_2^{k+l}(\Omega')) \cap C(\langle \delta, T\rangle, W_2^{k+l-1}(\Omega'))$$

(see Theorem 1). However, if $\partial\Omega$ is sufficiently smooth $(\partial\Omega \in C^{2k+l,1})$ then Lemma 1 and (9) hold true for $\Omega' \equiv \Omega$ – see [7] (Theorem 2.2, Chap. 4). Hence, by the same techniques we can prove regularity of $u(t)$ in Ω (we can put Ω instead of Ω' in (61)).

The results similar to those presented in this paper can be obtained also for more general boundary value problems than the Dirichlet problem by using the corresponding result for more general boundary value problems of linear elliptic equations – see [7] (Chap. I. 2.6 and Chap. IV. 2.2, 2.8).

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Adresa autorov: 816 31 Bratislava, Mlynská dolina, Matematický pavilón, ČSSR (Prírodovedecká fakulta UK).