Czechoslovak Mathematical Journal

H. C. Gupta; Bhu Dev Sharma

On non-additive measures of inaccuracy

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 4, 584–595

Persistent URL: http://dml.cz/dmlcz/101429

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

ON NON-ADDITIVE MEASURES OF INACCURACY

H. C. GUPTA and BHU DEV SHARMA, Delhi

(Received January 6, 1975)

1. INTRODUCTION

Shannon's entropy

$$H_S(P) = -\sum_{i=1}^{n} p_i \log_2 p_i\,, \quad \sum_{i=1}^{n} p_i = 1 \tag{1.1}$$

of a discrete probability distribution $P = (p_1, p_2, \ldots, p_n)$ has played important role in Information Theory and has found applications in so many other branches. However from statistical point of view as also for some economic analysis Kerridge's [6] measure of inaccuracy given by

$$H(P; Q) = -\sum_{i=1}^{n} p_i \log_2 q_i = H_K(P; Q) \tag{1.2}$$

associated with a pair of distributions $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n)$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$ is more suitable. $H_K(P; Q)$ is a generalization of Shannon's entropy $H_S(P)$. Both of these are additive for independent distributions. This property for $H(P; Q)$ may be put as follows:

$$H(P{*}P'; Q{*}Q') = H(P; Q) + H(P'; Q') \tag{1.3}$$

where

$$P{*}P' = (p_1p'_1, \ldots, p_1p'_m, \ldots, p_np'_1, \ldots, p_np'_m)$$

etc., where $\sum_{i=1}^{n} p_i = 1$ and $\sum_{j=1}^{m} p'_j = 1$. $H_K(P; Q)$ in (1.2) is clearly arithmetic average of quantities $(-\log_2 q_i)$ with weights p_i. Considering inaccuracy as a generalized mean

$$\phi^{-1}\Big(\sum_i p_i \theta(-\log_2 q_i)\Big) \tag{1.4}$$

where ϕ is strictly monotonic and continuous function, Sharma [8] has obtained (refer Rényi for entropy [7]) inaccuracy of order α given by

$$(1.5)\qquad H_\alpha(P; Q) = \frac{1}{1-\alpha} \log_2 \sum_{i=1}^{n} p_i q_i^{\alpha-1}, \quad \alpha \neq 1, \quad \alpha > 0.$$

This is additive, satisfying (1.3).

On the other hand some recent researches ([2], [4], [11], [13]) have been made for non-additive measures. For inaccuracy taking non-additivity as

$$(1.6)\quad H(P{*}P'; Q{*}Q') = H(P; Q) + H(P'; Q') + \lambda\, H(P; Q)\, H(P'; Q'), \quad \lambda \neq 0,$$

under additional property

$$(1.7)\qquad H(P; Q) = \sum_{i=1}^{n} f(p_i; q_i),$$

a functional equation

$$(1.8)\qquad \sum_i \sum_j f(p_i p_j'; q_i q_j') = \sum_i f(p_i; q_i) + \sum_j f(p_j'; q_j') + \lambda \sum_i \sum_j f(p_i; q_i) f(p_j'; q_j')$$

is formed. Under the condition $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$, Sharma and Taneja [10] have obtained the most general solutions of (1.8) given by

$$f(p; q) = (2^{1-\beta} - 1)^{-1} (p q^{\beta-1} - 1), \quad \beta \neq 1$$

and then (1.7) gives type β inaccuracy, viz.

$$(1.9)\qquad H^\beta(P; Q) = (2^{1-\beta} - 1)^{-1} \left(\sum_i p_i q_i^{\beta-1} - 1\right), \quad \beta \neq 1.$$

Clearly (1.9) has arisen from (1.6) under the sum property (1.7) of inaccuracy. This satisfies the mean value representation also. It is naturally interesting to examine what are all possible mean value forms (1.4) which satisfy non-additivity (1.6). This is attempted in this paper. The two new classes of measures so characterized interestingly contain all the measures of inaccuracy viz. (1.2), (1.5) and (1.9) as particular cases. Properties, nature including convexity and bivariate studies of new measures have been included.

2. CHARACTERIZATION

Let $H(p_i; q_i)$ be the elementary inaccuracy of the ith event x_i. Then we shall consider inaccuracy as mean of these elementary inaccuracies with weights as functions of p_i i.e.

$$(2.1)\qquad H(P; Q; \theta) = \phi^{-1}\left(\frac{\sum_i f(p_i)\, \phi(H(p_i; q_i))}{\sum_i f(p_i)}\right)$$

where ϕ is a strictly monotonic and continuous function and f is a weight function.

We first characterize $H(p; q)$ under the following postulates:

E 1. $H(p; q)$ is non-additive satisfying the relation

$$H(pp'; qq') = H(p; q) + H(p'; q') + \lambda H(p; q) H(p'; q') \tag{2.2}$$

where $p, p' \in [0, 1]$, $q, q' \in (0, 1]$.

E 2. $H(p; 1) = 0$.

E 3. $H(1; \frac{1}{2}) = 1$.

E 4. $H(p, q)$ is continuous for $q \in (0, 1]$.

Lemma. *The elementary inaccuracy $H(p; q)$ of an event which is found to occur with probability q while it was expected to occur with probability p, satisfying the above four postulates is given by*

$$H(p; q) = (2^{1-\beta} - 1)^{-1} (q^{\beta-1} - 1), \quad \beta \neq 1. \tag{2.3}$$

Proof. Since $\lambda \neq 0$, we may write (2.2) as

$$1 + \lambda H(pp'; qq') = [1 + \lambda H(p; q)] [1 + \lambda H(p'; q')] \tag{2.4}$$

or

$$F(pp'; qq') = F(p; q) F(p'; q') \tag{2.5}$$

where

$$F(p; q) = 1 + \lambda H(p; q). \tag{2.6}$$

Putting $q = 1$ in (2.6) and using postulate E 2, we get

$$F(p; 1) = 1 + \lambda H(p; 1) = 1. \tag{2.7}$$

Again putting, $p' = 1$, $q = 1$, in (2.5) and using (2.7) we get

$$F(p; q') = F(p; 1) F(1; q') = F(1; q'). \tag{2.8}$$

This shows that $F(p; q')$ is a function of q' alone.

Next for $p = 1$, $p' = 1$, (2.5) gives

$$F(1; qq') = F(1; q) F(1; q'). \tag{2.9}$$

The most general continuous solution of (2.9) (refer Aczel [1]) is given by

$$F(1; q) = q^C \tag{2.10}$$

where C is some constant.

From (2.8) and (2.10) we get

$$F(p; q) = q^C . \tag{2.11}$$

Hence

$$H(p; q) = \lambda^{-1}(q^C - 1) = \lambda^{-1}(q^{\beta-1} - 1) \tag{2.12}$$

where $C = \beta - 1$, $\beta \neq 1$, and $\lambda \neq 0$.

The result now follows as under postulate E 3, $\lambda = 2^{1-\beta} - 1$.

Next we characterize non-additive inaccuracy $H(P; Q)$ of a distribution Q with respect to P under the following postulates:

I-1. Non-additive. $H(P; Q)$ is non-additive satisfying (1.6).

I-2. Mean-value. $H(P; Q)$ is a generalized mean (2.1) of the elementary inaccuracies (2.12) of various events with weights p_i i.e. $f(x) = x$.

Theorem 1. *The non-additive inaccuracy $H(P; Q)$ of a distribution $Q = (q_1, \ldots, q_n)$ with respect to $P = (p_1, \ldots, p_n)$ satisfying the postulates* I-1 *and* I-2 *can be only of one of the following two forms*

$$H(P; Q; 1, \beta) = (2^{1-\beta} - 1)^{-1} \left[\exp_2 \{(\beta - 1) \sum_{i=1}^{n} p_i \log_2 q_i\} - 1\right], \quad \beta \neq 1 \tag{2.13}$$

and

$$H(P; Q; \alpha, \beta) = (2^{1-\beta} - 1)^{-1} \left[\left(\sum_{i=1}^{n} p_i q_i^{\alpha-1}\right)^{(\beta-1)/(\alpha-1)} - 1\right], \tag{2.14}$$

$$\alpha \neq 1, \quad \beta \neq 1 .$$

Proof. Taking $P' = \{p\}$, $Q' = \{q\}$, $0 < p, q < 1$ and using postulates I-1 and I-2 (c.f. 2.12), we get

$$\phi^{-1}\left(\sum_i p_i \phi\left(\frac{q_i^{\beta-1} q^{\beta-1} - 1}{2^{1-\beta} - 1}\right)\right) = \frac{q^{\beta-1} - 1}{2^{1-\beta} - 1} + \tag{2.15}$$

$$+ q^{\beta-1} \phi^{-1}\left(\sum_i p_i \phi\left(\frac{q_i^{\beta-1} - 1}{2^{1-\beta} - 1}\right)\right), \quad \beta \neq 1$$

or

$$\psi_q^{-1}\left(\sum_i p_i \psi_q\left(\frac{q_i^{\beta-1} - 1}{2^{1-\beta} - 1}\right)\right) = \phi^{-1}\left(\sum_i p_i \phi\left(\frac{q_i^{\beta-1} - 1}{2^{1-\beta} - 1}\right)\right), \quad \beta \neq 1 \tag{2.16}$$

where

$$\phi\left(\frac{q_i^{\beta-1}\, q^{\beta-1} - 1}{2^{1-\beta} - 1}\right) = \psi_q\left(\frac{q_i^{\beta-1} - 1}{2^{1-\beta} - 1}\right), \quad \beta \neq 1\,. \tag{2.17}$$

(2.16) shows that there exists a linear relation (Theorem 83 in [5]) between ψ_q and ϕ i.e.

$$\phi\left(\frac{q_i^{\beta-1} q^{\beta-1} - 1}{2^{1-\beta} - 1}\right) = A(q)\,\phi\left(\frac{q_i^{\beta-1} - 1}{2^{1-\beta} - 1}\right) + B(q)\,, \quad \beta \neq 1 \tag{2.18}$$

where $A(q)$ and $B(q)$ are constants depending upon q and $A(q) \neq 0$. Let

$$\phi[(2^{1-\beta} - 1)^{-1}\,(q_i^{\beta-1} - 1)] = G(q_i)\,, \quad \beta \neq 1\,. \tag{2.19}$$

Then (2.18) becomes

$$G(q_i q) = A(q)\, G(q_i) + B(q)\,. \tag{2.20}$$

The most general solution of (2.20) (Theorem 84 in [5]) is

$$G(q_i) = k \log q_i \quad \text{or} \quad G(q_i) = \frac{q_i^{\alpha-1} - 1}{k} \tag{2.21}$$

where $k \neq 0$ and $\alpha \neq 1$ are constants, i.e.

$$\phi[(2^{1-\beta} - 1)^{-1}\,(q_i^{\beta-1} - 1)] = k \log q_i \quad \text{or} \quad \frac{q_i^{\alpha-1} - 1}{k}\,, \quad \alpha \neq 1\,, \quad \beta \neq 1\,, \quad k \neq 0\,,$$

so that

$$\phi^{-1}(x) = (2^{1-\beta} - 1)^{-1}\,[\exp_2 (\beta-1)\, x/k - 1]\,, \quad \beta \neq 1\,, \quad k \neq 0$$

$$\text{or} = (2^{1-\beta} - 1)^{-1}\,[(1 + kx)^{(\beta-1)/(\alpha-1)} - 1]\,, \quad \alpha \neq 1\,, \quad \beta \neq 1\,, \quad k \neq 0\,.$$

For these values of ϕ, (2.1) gives (2.13) and (2.14) respectively.

REMARKS. GENERALIZED WEIGHTS

If we take $f(x) = x^\gamma$ in (2.1), then proceeding as in the above theorem, it can be easily shown that non-additive inaccuracy $H(P; Q)$ of a distribution Q with respect to P satisfying the postulates I-1 and I-2 can be only of one of the following two forms

$$H(P; Q; \beta, \gamma) = (2^{1-\beta} - 1)^{-1}\,[\exp_2\,((\beta - 1)\,(\textstyle\sum_i p_i^\gamma \log_2 q_i)/\sum_i p_i^\gamma) - 1]\,, \quad \beta \neq 1$$

and

$$H(P; Q; \alpha, \beta, \gamma) = (2^{1-\beta} - 1)^{-1} \left[\left(\left(\sum_i p_i^\gamma (q_i^{\alpha-1} - 1)\right)/\sum_i p_i^\gamma\right)^{(\beta-1)/(\alpha-1)} - 1\right],$$

$$\alpha \neq 1, \quad \beta \neq 1,$$

of which (2.13) and (2.14) respectively are particular cases.

3. LIMITING AND PARTICULAR CASES

I. $\lim_{\alpha \to 1} H(P; Q; \alpha, \beta) = H(P; Q; 1, \beta)$.

This follows from the following:

$$\lim_{\alpha \to 1} \left[\log_2 \left((2^{1-\beta} - 1) H(P; Q; \alpha, \beta) + 1\right)\right] = \lim_{\alpha \to 1} \frac{(\beta - 1) \log_2 \left(\sum_i p_i q_i^{\beta-1}\right)}{\alpha - 1} =$$

$$= (\beta - 1) \sum_i p_i \log_2 q_i .$$

II. $\lim_{\beta \to 1} H(P; Q; 1, \beta) = -\sum_i p_i \log_2 q_i = H_K(P; Q)$.

III. $H(P; Q; \beta, \beta) = (2^{1-\beta} - 1)^{-1} \left(\sum_i p_i q_i^{\beta-1} - 1\right)$, $\beta \neq 1$, type β inaccuracy.

This inaccuracy function was studied by Sharma and Ram Autar [11].

Next taking $Q \equiv P$, we get

$$H(P; P; \beta, \beta) = (2^{1-\beta} - 1)^{-1} \left(\sum_i p_i^\beta - 1\right), \quad \beta \neq 1$$

which is information function of type β, first studied by J. Havrda and F. Charvát [4] and later on by I. Vajda [13] and Daroćzy [2]. Further it would be noted that

$$\lim_{\beta \to 1} H(P; Q; \beta, \beta) = H_k(P; Q)$$

and

$$\lim_{\beta \to 1} H(P; P; \beta, \beta) = - \sum_i p_i \log_2 p_i .$$

Thus $\lim_{\beta \to 1} H(P; P; \beta, \beta)$ is Shannon's entropy [12] of the distribution P.

IV. $\lim_{\beta \to 1} H(P; Q; \alpha, \beta) = \lim_{\beta \to 1} \left[(2^{1-\beta} - 1)^{-1} \left(\left(\sum_i p_i q_i^{\alpha-1}\right)^{(\beta-1)/(\alpha-1)} - 1\right]\right.$, $\alpha \neq 1$, $\beta \neq 1$

$$= \frac{1}{1 - \alpha} \log_2 \left(\sum_i p_i q_i^{\alpha-1}\right), \quad \alpha \neq 1 .$$

This is inaccuracy function studied by Sharma [8] such that if $Q \equiv P$,

$$\lim_{\beta \to 1} H(P; P; \alpha, \beta) = \frac{1}{1-\alpha} \log_2 \left(\sum_i p_i^\alpha\right), \quad \alpha \neq 1$$

which is Rényi's entropy [7].

The above four are limiting and particular cases. In general, we note the following relations:

V.

$$(3.1) \qquad H(P; Q; 1, \beta) = (2^{1-\beta} - 1)^{-1} \left[\exp_2 \{(1-\beta) H_K(P; Q)\} - 1\right], \quad \beta \neq 1$$

where $H_K(P; Q)$ is Kerridge's inaccuracy;
and

$$(3.2) \qquad H(P; Q; \alpha, \beta) = (2^{1-\beta} - 1)^{-1} \left[\exp_2 \{(1-\beta) H_\alpha(P; Q)\} - 1\right], \quad \alpha \neq 1, \quad \beta \neq 1.$$

Remark. $P \equiv Q$ reduces the two non-additive inaccuracies to non-additive entropies studied by Sharma and Mittal [9].

4. PROPERTIES OF GENERALIZED INACCURACY FUNCTIONS

We first list some of the simple properties:

1. $H(P; Q; 1, \beta)$ is non-negative for $\beta \neq 1$.
It follows from the fact that $-\sum_i p_i \log_2 q_i$ is non-negative.

2. $H(P; P; 1, \beta) \leqq H(P; Q; 1, \beta)$, $\beta \neq 1$ with equality iff $P \equiv Q$.
It follows from the well known inequality

$$-\sum_i p_i \log_2 p_i \leqq -\sum_i p_i \log_2 q_i, \quad \sum_i p_i = \sum_i q_i = 1$$

where the equality holds iff $P \equiv Q$.

3.

$$(4.1) \qquad H(P; Q; \alpha, \beta) \gtrless (2^{1-\beta} - 1)^{-1} \left[\exp_2 \{(\beta - 1) \log_2 (\sum_i p_i q_i)\} - 1\right], \quad \beta \neq 1$$

according as $\alpha \lessgtr 2$.

It follows from the inequality (page 523 in [3])

$$(\sum_k a_k x_k)^r \gtrless \sum_k a_k x_k^r \tag{4.2}$$

according as $r \lesseqgtr 1$, $\sum_k a_k = 1$, by taking

$$x_k = q_k^{\alpha-1}, \quad a_k = p_k, \quad r = \frac{1}{\alpha - 1}.$$

From this it is further easy to see that

$$H(P; Q; \alpha, \beta) > 0, \quad \text{for} \quad \alpha < 2, \quad \beta \neq 1.$$

4. Convexity. It is basically a useful property and we enunciate it in the following theorems.

Theorem 2. *$H(P; Q; 1, \beta)$ is convex $\cap$ or $\cup$ function of P according as $\beta \gtrless 1$.*

The result follows immediately from the fact that e^{ax} is a convex $\cup$ function of x and $(2^{1-\beta} - 1) \gtrless 0$ according as $\beta \lesseqgtr 1$.

Theorem 3. *$H(P; Q; \alpha, \beta)$ is convex $\cap$ or $\cup$ function of distribution P according as*

(i) $\alpha < \beta < 1$; $1 < \alpha < \beta$; $\alpha > 1, \beta < 1$

or

(ii) $1 < \beta < \alpha$; $\beta < \alpha < 1$; $\alpha < 1, \beta > 1$.

Proof. Let $P_0(X) = \{p_0(x_1), \ldots, p_0(x_n)\}$ be the distribution of X such that $p_0(x_i) = \sum_{j=1}^{r} a_j p_j(x_i)$, where a_j's are non-negative numbers which sum to unity and $\{p_j(x_i)\}$, $j = 1, 2, \ldots, r$, are some probability distributions of X.

We then have

$$\Delta = \sum_{j=1}^{r} a_j H(P_j; Q; \alpha, \beta) - H(P_0; Q; \alpha, \beta), \quad \alpha \neq 1, \quad \beta \neq 1,$$

$$= \sum_{j=1}^{r} a_j [(\sum_{i=1}^{n} p_j(x_i)\, q_i^{\alpha-1})^{(\beta-1)/(\alpha-1)} - 1] (2^{1-\beta} - 1)^{-1} -$$

$$- [(\sum_{i=1}^{n} p_0(x_i)\, q_i^{\alpha-1})^{(\beta-1)/(\alpha-1)} - 1] (2^{1-\beta} - 1)^{-1}, \quad \alpha \neq 1, \quad \beta \neq 1,$$

$$= [\sum_{j=1}^{r} a_j (\sum_{i=1}^{n} p_j(x_i)\, q_i^{\alpha-1})^{(\beta-1)/(\alpha-1)} -$$

$$- (\sum_{i=1}^{n} \sum_{j=1}^{r} a_j\, p_j(x_i)\, q_i^{\alpha-1})^{(\beta-1)/(\alpha-1)}] (2^{1-\beta} - 1)^{-1}, \quad \alpha \neq 1, \quad \beta \neq 1,$$

$$= [\sum_{j=1}^{r} a_j (\sum_{i=1}^{n} p_j(x_i)\, q_i^{\alpha-1})^{(\beta-1)/(\alpha-1)} -$$

$$- (\sum_{j=1}^{r} a_j (\sum_{i=1}^{n} p_j(x_i)\, q_i^{\alpha-1}))^{(\beta-1)/(\alpha-1)}] (2^{1-\beta} - 1)^{-1}, \quad \alpha \neq 1, \quad \beta \neq 1.$$

Now using (4.2), we get

$$\left(\sum_{j=1}^{r} a_j\left(\sum_{i=1}^{n} p_j(x_i)\, q_i^{\alpha-1}\right)\right)^{(\beta-1)/(\alpha-1)} \gtrless \sum_{j=1}^{r} a_j\left(\sum_{i=1}^{n} p_j(x_i)\, q_i^{\alpha-1}\right)^{(\beta-1)/(\alpha-1)},$$

$$\alpha \neq 1, \quad \beta \neq 1$$

according as $(\beta - 1)/(\alpha - 1) \lesseqgtr 1$.

Convexity $\cap$ or $\cup$ follows for $\Delta \lesseqgtr 0$ taing into account the nature $(2^{1-\beta} - 1)^{-1} \gtrless \gtrless 0$ according as $\beta \lesseqgtr 1$.

In the next theorem we prove convexity of $H(P; Q; \alpha, \beta)$ with respect to Q.

Theorem 4. *$H(P; Q; \alpha, \beta)$ is convex $\cap$ function of distribution Q whenever $\alpha < \beta < 1$ or $\beta < 1$, $\alpha > 2$ or $2 < \alpha < \beta$ and convex $\cup$ function of Q whenever* $1 < \beta < \alpha < 2$.

Proof. As in theorem 3, $q_0(x_i) = \sum_{j=1}^{r} a_j\, q_j(x_i)$, where $\{q_j(x_i)\}$, $j = 1, 2, \ldots, r$ are some probability distributions of X.

We then have

$$\begin{aligned}
\Delta &= \sum_{j=1}^{r} a_j\, H(P; Q_j; \alpha, \beta) - H(P; Q_0; \alpha, \beta), \quad \alpha \neq 1, \quad \beta \neq 1, \\
&= \sum_{j=1}^{r} a_j\left(\left(\sum_{i=1}^{n} p_i\, q_j^{\alpha-1}(x_i)\right)^{(\beta-1)/(\alpha-1)} - 1\right)(2^{1-\beta} - 1)^{-1} \\
&\quad - \left(\left(\sum_{i=1}^{n} p_i\, q_0^{\alpha-1}(x_i)\right)^{(\beta-1)/(\alpha-1)} - 1\right)(2^{1-\beta} - 1)^{-1}, \quad \alpha \neq 1, \quad \beta \neq 1, \\
&= \Big[\sum_{j=1}^{r} a_j\left(\sum_{i=1}^{n} p_i\, q_j^{\alpha-1}(x_i)\right)^{(\beta-1)/(\alpha-1)} - \\
&\quad - \left(\sum_{i=1}^{n} p_i\left(\sum_{j=1}^{r} a_j\, q_j(x_i)\right)^{\alpha-1}\right)^{(\beta-1)/(\alpha-1)}\Big](2^{1-\beta} - 1)^{-1}, \quad \alpha \neq 1, \quad \beta \neq 1.
\end{aligned}$$

Now using (4.2) we have

$$\left(\sum_{j=1}^{r} a_j\, q_j(x_i)\right)^{\alpha-1} \gtrless \sum_{j=1}^{r} a_j\, q_j^{\alpha-1}(x_i), \quad \alpha \neq 1$$

according as $\alpha \lesseqgtr 2$, and therefore

$$\begin{aligned}
\lambda &= \left(\sum_{i=1}^{n} p_i\left(\sum_{j=1}^{r} a_j\, q_j(x_i)\right)^{\alpha-1}\right)^{(\beta-1)/(\alpha-1)} \gtrless \\
&\gtrless \left(\sum_{i=1}^{n} p_i \sum_{j=1}^{r} a_j\, q_j^{\alpha-1}(x_i)\right)^{(\beta-1)/(\alpha-1)}, \quad \alpha \neq 1, \quad \beta \neq 1
\end{aligned} \tag{4.3}$$

according as $\alpha \lesseqgtr 2$ if $(\beta - 1)/(\alpha - 1) > 0$ (the inequalities in (4.3) are reversed if $(\beta - 1)/(\alpha - 1) < 0$).

Further

$$(4.4) \qquad (\sum_{j=1}^{r} a_j \sum_{i=1}^{n} p_i\, q_j^{\alpha-1}(x_i))^{(\beta-1)/(\alpha-1)} \gtrless$$

$$\gtrless \sum_{j=1}^{r} a_j (\sum_{i=1}^{n} p_i\, q_j^{\alpha-1}(x_i))^{(\beta-1)/(\alpha-1)} = \mu\,, \quad \alpha \neq 1\,, \quad \beta \neq 1$$

according as $(\beta - 1)/(\alpha - 1) \lesseqgtr 1$.

Combining (4.3) and (4.4) we get

$$\lambda > \mu \quad \text{for} \quad 1 < \beta < \alpha < 2 \quad \text{or} \quad \alpha < \beta < 1 \quad \text{or} \quad \beta < 1\,, \quad \alpha > 2$$

and

$$\lambda < \mu \quad \text{for} \quad 2 < \alpha < \beta\,.$$

Now taking into account the nature of $(2^{1-\beta} - 1)$ the result follows.

5. CASE OF TWO VARIABLES

Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_m)$ be two discrete random variables and $p(x_i, y_j) = P(X = x_i$ and $Y = y_j)$ and $q(x_i, y_j) = Q(X = x_i$ and $Y = y_j)$ be the two joint probability density functions. Then we define the non-additive joint inaccuracies of the distribution Q with respect to distribution P of XY as follows:

$$H(X; Y; P; Q; 1, \beta) =$$

$$= (2^{1-\beta} - 1)^{-1} \left[\exp_2 \{(\beta - 1) \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log_2 q(x_i, y_j)\} - 1\right], \quad \beta \neq 1\,,$$

$$H(X; Y; P; Q; \alpha, \beta) =$$

$$= (2^{1-\beta} - 1)^{-1} \left[(\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j)\, q^{\alpha-1}(x_i, y_j))^{(\beta-1)/(\alpha-1)} - 1\right], \quad \alpha \neq 1\,, \quad \beta \neq 1\,.$$

The following relations are rather immediate.

Theorem 5. *If X and Y are discrete statistically independent variables, then*

$$H(X; Y; P; Q; 1, \beta) = H(X; P; Q; 1, \beta) + H(Y; P; Q; 1, \beta) +$$

$$+ (2^{1-\beta} - 1)\, H(X; P; Q; 1, \beta)\, H(Y; P; Q; 1, \beta)\,, \quad \beta \neq 1\,,$$

and

$$H(X; Y; P; Q; \alpha, \beta) = H(X; P; Q; \alpha, \beta) + H(Y; P; Q; \alpha, \beta) +$$

$$+ (2^{1-\beta} - 1)\, H(X; P; Q; \alpha, \beta)\, H(Y; P; Q; \alpha, \beta)\,, \quad \alpha \neq 1\,, \quad \beta \neq 1\,.$$

Next we define the generalized conditional inaccuracy of X given Y as follows:

$$H(X/Y; P; Q; 1, \beta) =$$

$$= (2^{1-\beta} - 1)^{-1} \left[\exp_2 \left\{(\beta - 1) \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log_2 q(x_i/y_j)\right\} - 1\right], \quad \beta \neq 1,$$

$$H(X/Y; P; Q; \alpha, \beta) =$$

$$= (2^{1-\beta} - 1)^{-1} \left[\left(\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j)\, q^{\alpha-1}(x_i/y_j)\right)^{(\beta-1)/(\alpha-1)} - 1\right], \quad \alpha \neq 1, \quad \beta \neq 1.$$

These are actually generalized means of $H(X/Y = y_j; P; Q; .,.)$ with weights $p(y_j)$. Similarly $H(Y/X; P; Q; 1, \beta)$ and $H(Y/X; P; Q; \alpha, \beta)$ can be defined. We then have the following interesting relations:

Theorem 6.

$$\begin{aligned} H(X; Y; P; Q; 1, \beta) &= H(X; P; Q; 1, \beta) + H(Y/X; P; Q; 1, \beta) + \\ &\quad + (2^{1-\beta} - 1)\, H(X; P; Q; 1, \beta)\, H(Y/X; P; Q; 1, \beta), \quad \beta \neq 1, \\ &= H(Y; P; Q; 1, \beta) + H(X/Y; P; Q; 1, \beta) + \\ &\quad + (2^{1-\beta} - 1)\, H(Y; P; Q; 1, \beta)\, H(X/Y; P; Q; 1, \beta), \quad \beta \neq 1, \end{aligned}$$

and

$$\begin{aligned} H(X; Y; P; Q; \alpha, \beta) &= H(X; P; Q; \alpha, \beta) + H(Y/X; P; Q; \alpha, \beta) + \\ &\quad + (2^{1-\beta} - 1)\, H(X; P; Q; \alpha, \beta)\, H(Y/X; P; Q; \alpha, \beta), \quad \alpha \neq 1, \quad \beta \neq 1, \\ &= H(Y; P; Q; \alpha, \beta) + H(X/Y; P; Q; \alpha, \beta) + \\ &\quad + (2^{1-\beta} - 1)\, H(Y; P; Q; \alpha, \beta)\, H(X/Y; P; Q; \alpha, \beta), \quad \alpha \neq 1, \quad \beta \neq 1. \end{aligned}$$

CONCLUDING REMARKS

These studies were confined to the discrete case. It is proposed to report their continuous analogs in a subsequent paper.

References

[1] *Aczel, J.:* Lectures on Functional Equations and Their Applications. Academic Press, 1966.

[2] *Daróczy, Z.:* Generalized Information Functions. Information and Control, *16* (1970), 36—51.

[3] *Gallager, R. G.:* Information Theory and Reliable Communications. John Wiley and Sons, Inc. New York, 1968, pp. 523.

[4] *Havrda, J., F. Charvát:* Quantification Method of Classification Process. The concept of structural a — entropy. Kybernetika *3* (1967), 30—35.

[5] *Hardy, G. H., J. E. Littlewood, G. Pólya:* Inequalities. Cambridge University Press, 1964.

[6] *Kerridge, D. F.:* Inaccuracy and Inference. J. R. Statist. Soc., series B *23* (1961), 184—194.

[7] *Rényi, A.:* On Measures of Entropy and Information. Proc. Fourth Berkeley Symp. on Math. Stat. and Probability, University of California Press, (1960), 547—561.

[8] *Sharma, B. D.:* A Mean Value Study of Quantities in Information Theory. Ph. D. Thesis, Delhi University, 1970.

[9] *Sharma, B. D., D. P. Mittal:* New Non-additive Measures of Entropy for Discrete Probability Distributions. Journal of Mathematical Sciences, Vol. *10* (1975), 28—40.

[10] *Sharma, B. D., I. J. Taneja:* Entropy of Type (α, β) and Other Generalized Measures in Information Theory. METRIKA, Vol. *22* (1975), 205—215.

[11] *Sharma, B. D., Ram Autar:* On Characterization of a Generalized Inaccuracy Measure in Information Theory. Journal of Applied Probability, Vol. *10* (1973), 464—468.

[12] *Shannon, C. E.:* A Mathematical Theory of Communication, Bell System Tech. Journal, *27* (1948), 379—423.

[13] *Vajda, I.:* Axioms for a — entropy of a Generalized Probability Scheme. Kybernetika *2* (1968), 105—112.

Authors' addresses: Bhu Dev Sharma, Faculty of Mathematics, University of Delhi, India; H. C. Gupta, Department of Mathematics, Rajdhani College, (University of Delhi), Raja Garden, New Delhi, India.