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GENERALIZED INTERVALS AND TOPOLOGY

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1. INTRODUCTION

The intervals of a totally ordered set generate a natural and useful topology, and much work has been done on the ramifications of transferring the formal definition of interval to the more general case of partially ordered sets (see, for example, [9], [12], and [13], among many others). However, the usual definition for a topology derived from these intervals does not yield a Hausdorff topology even for the plane ([2], p. 61).

In [19], we defined "generalized intervals" in an l-group by considering (for example) the set $\{(x, y) \mid -1 \leqq y \leqq 1\} \subseteq \mathbf{R} \times \mathbf{R}$ as an interval rather than the set $\{(0, y) \mid -1 \leqq y \leqq 1\}$. We then used these "generalized intervals" to define a group and lattice topology on an l-group, which was preserved by cardinal products. In this paper, we define generalized intervals in an arbitrary partially ordered set $(P, \leqq)$ (§ 2). Substituting generalized intervals for formal intervals in the definition of the interval topology defines a topology on P (§ 3) which is preserved by cardinal products of dually directed sets (§ 4). To ensure that intervals appear as generalized intervals, we may modify our definitions slightly to produce generalized star-intervals (§ 2). The topology generated by generalized star-intervals then contains the interval topology (§ 3) but may not be preserved by cardinal products of totally ordered sets (§ 5). Our definitions are based on certain kinds of semi-ideals (§ 2), and in § 3 we detail several, previously defined, ideal-like topologies. These topologies are in general incomparable with our generalized interval and star-interval topologies (§§ 3 and 5).

Notation. For terms left undefined, we refer the reader to [4], [6], and [14]. Our notation is standard with possibly the following exceptions: If $(P, \leqq)$ is a partially ordered set and if $A \subseteq P$, then we let

$$u(A) = \{p \in P \mid p \geqq a \text{ for all } a \in A\},$$

$$b(A) = \{p \in P \mid p \leqq a \text{ for all } a \in A\}.$$

If $x, y \in P$, then $u(x, y) = u(\{x, y\})$, $b(x, y) = b(\{x, y\})$, $(-\infty, x] = b(\{x\})$, and $[y, \infty) = u(\{y\})$. Also, if $x, y \in P$, then $x \wedge y$ $(x \vee y)$ denotes the greatest lower bound (least upper bound) of $\{x, y\}$ if it exists, and $x \wedge y = z$ $(x \vee y = z)$ means that $x \wedge y$ $(x \vee y)$ exists and equals z.

We write functions on the right, including inverse "functions" which denote the pre-image of a set in the codomain. We denote the empty set by $\square$. We use **N**, **Z**, and **R** for the natural numbers, the integers, and the real numbers, respectively. Unless otherwise noted, **N**, **Z**, and **R** have their usual orders.

A function f from a partially ordered set $(P, \leqq)$ to itself is an *o-automorphism* if f is bijective and $x \leqq y$ in P if and only if $xf \leqq yf$. If $\{P_\alpha \mid \alpha \in A\}$ is a collection of partially ordered sets, then the *cardinal product* of the P_α, denoted by $|\prod_A| P_\alpha$, is their cartesian product with pointwise order: $f \leqq g$ if and only if $\alpha f \leqq \alpha g$ for all $\alpha \in A$. If $\{G_\alpha \mid \alpha \in A\}$ is a collection of l-groups, then their *cardinal sum*, denoted by $|\sum_A| G_\alpha$, is the l-subgroup of $|\prod_A| G_\alpha$ consisting of those functions which are 0 for all but a finite number of α.

2. GENERALIZED INTERVALS

Let $(P, \leqq)$ be a partially ordered set. Sets of the form $(-\infty, p]$ for some $p \in P$ are called *initial segments*; sets of the form $[p, \infty)$ for some $p \in P$ are called *final segments*; and sets of the form $[r, t] = [r, \infty) \cap (-\infty, t]$ for some $r, t \in P$ with $r \leqq t$ are called (*closed*) *intervals*.

In [19], we used the polars of a lattice-ordered group G to define "generalized intervals" in G. Such a generalized interval about a point $x \in G$ was a set of the form

$$x + [-g, g] + g^\perp = [x - g, x + g] + g^\perp$$

for some $g \in G^+$, where

$$g^\perp = \{x \in G \mid |x| \wedge |g| = 0\}$$

was the *polar* of g in G (see [7], [22]). The intervals of a totally ordered set "span" the set (in the sense that there are no elements perpendicular to the intervals), and the introduction of polars into the idea of interval was an attempt to incorporate this spanning ability into non-totally ordered groups. In this section, we suggest two related generalizations of the notion of polar to arbitrary partially ordered sets. Using these extended "polars", we define generalized intervals in arbitrary partially ordered sets analogously to the definition given above for l-groups. Since arbitrary partially ordered sets lack the symmetry of l-groups, each extended notion of polar naturally has an upper and a lower component.

Let $r, s, t \in P$ be such that $r \leqq s \leqq t$. The *upper polar of t with respect to s* (or the *s, t-upper polar*) is the set

$$(s, t)^\perp = \{x \in P \mid x \wedge t = s\} ;$$

the *lower polar of r with respect to s* (or the *s, r-lower polar*) is the set

$$(s, r)_{\mathrm{T}} = \{x \in P \mid x \vee r = s\}\,.$$

Clearly, $(s, r)_{\mathrm{T}} \cap (s, t)^{\perp} = \{s\}$.

We may then define generalized segments and (closed) intervals as follows: If $r, s, t \in P$ are such that $r \leqq s \leqq t$, we let

$$[r, s, \infty) = \{x \in P \mid \Box \neq b(a, r) \subseteq (-\infty, x] \text{ for some } a \in (s, r)_{\mathrm{T}}\}\,,$$

$$(-\infty, s, t] = \{x \in P \mid \Box \neq u(c, t) \subseteq [x, \infty) \text{ for some } c \in (s, t)^{\perp}\}\,,$$

$$[r, s, t] = [r, s, \infty) \cap (-\infty, s, t]\,.$$

A *generalized final segment* of P is a set of the form $[r, s, \infty)$ for some $r \leqq s$; a *generalized initial segment* of P is a set of the form $(-\infty, s, t]$ for some $s \leqq t$; and a *generalized (closed) interval* of P is a set of the form $[r, s, t]$ for some $r \leqq \leqq s \leqq t$.

An *ideal* of a lattice L is universally defined as a subset I of L such that if $x, y \in I$ and $z \in L$ are such that $z \leqq x$, then $x \vee y \in I$ and $z \in I$. The definition of an ideal in an arbitrary partially ordered set, however, may vary from author to author (e.g. [8], [10]). In this paper, we will be concerned with the following kinds of ideal-like subsets: Let $(P, \leqq)$ be a partially ordered set. LAWSON's *semi-ideal* [15] of P (called a "partie à gauche" by GUILLAUME [11]) is a set $S \subseteq P$ such that

$$S = \{x \in P \mid x \leqq s \text{ for some } s \in S\}\,.$$

An **F**-*ideal* of P (FRINK's definition of ideal in [10] – see also [17]) is a semi-ideal $J \subseteq P$ such that for any finite set $F \subseteq J$, $b(u(F)) \subseteq T$. An *ideal* of P is a semi-ideal $I \subseteq P$ such that for any finite set $F \subseteq I$ such that $u(F) \neq \Box$, $b(u(F)) \subseteq I$. (This definition of ideal is meant to approximate closely our definition of generalized initial segments.) *Dual semi-ideals*, *dual* **F**-*ideals*, and *dual ideals* are defined in the obvious (dual) fashion. Clearly, for any lattice, the notions of **F**-ideal, ideal as defined above for partially ordered sets, and ideal in the usual sense, are equivalent.

If L is the lattice in the illustration, then $(-\infty, s, t] = L \setminus \{x\}$ is not an ideal. However, we do have Propositions 2.1 and 2.3 below, whose proofs are straightforward.

Proposition 2.1. *For any partially ordered set $(P, \leqq)$, the generalized initial segments are semi-ideals and the generalized final segments are dual semi-ideals.*

Corollary 2.2. *Let $(P, \leqq)$ be a partially ordered set. For $r, s, t \in P$ such that $r \leqq s \leqq t$,*

(i) $(-\infty, t) \subseteq (-\infty, s, t]$; (ii) $[r, \infty) \subseteq [r, s, \infty)$; (iii) $[r, t] \subseteq [r, s, t]$.

Proposition 2.3. *For any distributive lattice* $(L, \leqq)$ *the generalized initial segment* $(-\infty, s, t]^{\perp}$ *is the ideal generated by* $\{t\} \cup (s, t)^{\perp}$ *and the generalized final segment* $[r, s, \infty)$ *is the dual ideal generated by* $\{r\} \cup (s, r)_{\top}$.

We noted above that generalized intervals are meant to extend definitions originally given for l-groups. The next result says that such an extension does indeed take place.

Proposition 2.4. *Let* $(G, \leqq)$ *be an l-group. Let* $x \in G$, $g \in G^{+}$. *Then* $x + [-g, g] + g^{\perp} = [x - g, x, x + g]$.

Proof. Let $y \in x + [-g, g] + g^{\perp}$. By Lemma 2.12 of [19], $[-g, g] + g^{\perp}$ is a convex sublattice of G, and hence both $(-x + y)^{+}$ and $(-x + y)^{-}$ are members of it, i.e., $(-x + y)^{+} = a + b$ and $(-x + y)^{-} = c + d$ for $a, c \in [-g, g]$ and $b, d \in g^{\perp}$. By use of Lemma 2.3 of [19], it is not difficult to show that

$$(-g) \wedge d \leqq c \wedge d \leqq -x + y \leqq a \vee b \leqq g \vee b\,.$$

It is also straightforward to see that

$$x + (g^{\perp})^{+} = (x, x + g)^{\perp} \quad \text{and} \quad x - (g^{\perp})^{+} = (x, x - g)_{\top}\,, \tag{1}$$

and therefore, $y \in [x - g, x, x + g]$. Conversely, let $y \in [x - g, x, x + g]$. Then there exist $a \in (x, x - g)_{\top}$ and $c \in (x, x + g)^{\perp}$ such that

$$(-g) \wedge (-x + a) \leqq -x + y \leqq (g) \vee (-x + b)\,.$$

Using Lemmas 2.2 and 2.3 of [19] and the relations (1) above, one may easily show that $(-g) \wedge (-x + a) \in [-g, g] + g^{\perp}$ and $(g) \vee (-x + b) \in [-g, g] + g^{\perp}$. Since $[-g, g] + g^{\perp}$ is convex, $y \in x + [-g, g] + g^{\perp}$.

Generalized intervals and segments are also meant to extend the ordinary intervals and segments of a totally ordered set. The next result shows that in non-degenerate cases this extension also occurs.

Proposition 2.5. *If* $(T, \leqq)$ *is a totally ordered set, then for all* $r, s, t \in T$ *with* $r < s < t$, $(-\infty, s, t] = (-\infty, t]$, $[r, s, \infty) = [r, \infty)$, *and* $[r, s, t] = [r, t]$.

Proof. By Corollary 2.2, $(-\infty, s, t] \supseteq (-\infty, t]$. Conversely, if $x \in (-\infty, s, t]$, then there exists $a \in (s, t)^{\perp}$ such that $x \leqq a \vee t$. Then $a \wedge t = s$, and since $t \neq s$ and $(T, \leqq)$ is totally ordered, $a = s$. Thus, $x \leqq s \vee t = t$, i.e., $x \in (-\infty, t]$. Similarly, $[r, s, \infty) = [r, \infty)$, and therefore, $[r, s, t] = [r, t]$.

For degenerate cases, however, Proposition 2.5 may fail to hold. For example, if $(T, \leqq)$ is a totally ordered set and if $r < t$ in T, then $(-\infty, t, t] = T = [r, r, \infty)$ and hence $[r, r, t] = (-\infty, t]$ and $[r, t, t] = [r, \infty)$. Furthermore, for any lattice $(L, \leqq)$, $[r, r, r] = L$ for all $r \in L$.

To avoid this pathology, we introduce our second set of definitions by slightly modifying our previous set as follows: Let $(P, \leqq)$ be a partially ordered set; let $r, s, t \in$

$\in P$ be such that $r \leqq s \leqq t$. The *upper star-polar of t with respect to s* is the set $*(s, t)^{\perp} = (s, t)^{\perp}$ if $s < t$ and the set $*(s, t)^{\perp} = \{s\}$ if $s = t$; similarly, the *lower star-polar of r with respect to s* is the set $*(s, r)_{\mathrm{T}} = (s, r)_{\mathrm{T}}$ if $r < s$ and the set $*(s, r)_{\mathrm{T}} = \{s\}$ if $r = s$. We let

$$*[r, s, \infty) \quad = \{x \in P \mid \square \neq b(a, r) \subseteq (-\infty, x] \text{ for some } a \in *(s, r)_{\mathrm{T}}\},$$

$$*(-\infty, s, t] = \{x \in P \mid \square \neq u(c, t) \subseteq [x, \infty) \text{ for some } c \in *(s, t)^{\perp}\},$$

$$*[r, s, t] \quad = *[r, s, \infty) \cap *(-\infty, s, t],$$

and say that *generalized final star-segments* are sets of the form $*[r, s, \infty)$, *generalized initial star-segments* are sets of the form $*(-\infty, s, t]$, and *generalized (closed) star-intervals* are sets of the form $*[r, s, t]$.

Clearly, in any partially ordered set $(P, \leqq)$, $*(-\infty, t, t] = (-\infty, t]$, $*[r, r, \infty) = [r, \infty)$, and $*[r, r, r] = \{r\}$ for all $r, t \in P$. Thus, we have the following unrestricted analogue of Proposition 2.4.

Proposition 2.6. *If $(T, \leqq)$ is a totally ordered set, then for all $r, s, t \in T$ with $r \leqq s \leqq t$, $*(-\infty, s, t] = (-\infty, t]$, $*[r, s, \infty) = [r, \infty)$, and $*[r, s, t] = [r, t]$.*

Furthermore, the following analogues of Propositions 2.1, 2.3, and 2.4 follow immediately from the previous result (Proposition 2.9) or from the definitions (Propositions 2.7 and 2.8).

Proposition 2.7. *For any partially ordered set $(P, \leqq)$, the generalized initial star-segments are semi-ideals and the generalized final star-segments are dual semi-ideals.*

Proposition 2.8. *For any distributive lattice $(L, \leqq)$, the generalized initial star-segment $*(-\infty, s, t]$ is the ideal generated by $\{t\} \cup *(s, t)^{\perp}$, and the generalized final star-segment $*[r, s, \infty)$ is the dual ideal generated by $\{r\} \cup *(s, r)_{\mathrm{T}}$.*

Proposition 2.9. *Let $(G, \leqq)$ be an l-group. Let $x \in G$, $g \in G^{+} \setminus \{0\}$. Then $x + [-g, g] + g^{\perp} = *[x - g, x, x + g]$.*

3. GENERALIZED INTERVAL TOPOLOGIES

Let $(P, \leqq)$ be a partially ordered set. The *interval topology* [9] on P, denoted by $\mathscr{I}(P)$, is the topology which takes the final segments and the initial segments as a subbase for its closed sets.

Analogously, we define the *generalized interval topology* (*generalized star-interval topology*) on P, denoted by $\mathscr{G}(P)$ ($\mathscr{G}^*(P)$), to be the topology which takes as a subbase for its closed sets all the generalized final segments (generalized final star-segments) and all the generalized initial segments (generalized initial star-segments), together with $\square$.

By Propositions 2.1 and 2.7, both $\mathscr{G}(P)$ and $\mathscr{G}^*(P)$ are topologies that are generated by certain semi-ideals and dual semi-ideals of P. In this section (and in part of § 5), we wish to compare $\mathscr{G}(P)$ and $\mathscr{G}^*(P)$ with other topologies whose definitions are related to intervals or ideal-like segments.

In [3], BIRKHOFF suggests an alternative to $\mathscr{I}(P)$, viz., the *new interval topology*, denoted by $\mathscr{N}(P)$. A set B is closed with respect to $\mathscr{N}(P)$ if and only if the intersection of B with the intersection of finite unions of closed intervals of P is itself an intersection of finite unions of closed intervals. In [20] and [21], RENNIE uses intervals in a different way to define his *L-topology*, denoted by $\mathscr{L}(P)$. A set B is a basic open set of $\mathscr{L}(P)$ in case B is convex and the intersection of B with any maximal chain C of P is open with respect to $\mathscr{I}(C)$.

Frink's *ideal topology* [10] (cf. [17]), denoted by $\mathscr{Y}(P)$, takes as a subbase for its open sets all **F**-ideals and dual **F**-ideals which are completely irreducible in the following sense: A proper (dual) **F**-ideal is *completely irreducible* if it is not the intersection of a collection of (dual) **F**-ideals distinct from it. A set $B \subseteq P$ is **L**-*closed* if, for all $X \subseteq B$, $\bigvee X \in P$ implies $\bigvee X \in B$ and $\bigwedge X \in P$ implies $\bigwedge X \in B$. NAITO's *CP-ideal topology* [18], denoted by $c\mathscr{P}(P)$, takes as a subbase for its closed sets all **L**-closed, prime ideals and dual ideals, where a (dual) ideal $I \subseteq P$ is *prime* if for all $x, y \in P$ such that $\square \neq b(x, y) \subseteq I$ $(\square \neq u(x, y) \subseteq I)$, either $x \in I$ or $y \in I$. Guillaume's "*topologie gauche*" ("*topologie droite*") [11], denoted by $\mathscr{T}g(P)$ $(\mathscr{T}d(P))$, takes for its open sets the (dual) semi-ideals of P. A set $B \subseteq P$ is *Dedekind-closed* if whenever $\{x_\beta\} \subseteq B$ is a net ascending (i.e., $\alpha \geqq \beta$ implies $x_\beta \leqq x_\alpha$) to $\bigvee x_\beta \in P$, then $\bigvee x_\beta \in B$ and whenever $\{y_\delta\} \subseteq B$ is a net descending (i.e., $\alpha \geqq \delta$ implies $y_\delta \geqq y_\alpha$) to $\bigwedge y_\delta \in P$, then $\bigwedge y_\delta \in B$. Lawson's *semi-ideal topology* [15], denoted by $\mathscr{S}(P)$, takes as a subbase for its closed sets all the Dedekind-closed semi-ideals and dual semi-ideals. Finally, a set $B \subseteq P$ is closed with respect to Lawson's *chain topology* [15] (Guillaume calls it the "topologie longitudinal" in [11]), denoted by $\mathscr{T}\mathscr{L}(P)$, if, whenever $\square \neq T \subseteq B$ is totally ordered, then $\bigwedge T \in P$ implies $\bigwedge T \in B$ and $\bigvee T \in P$ implies $\bigvee T \in B$.

A topology on $(P, \leqq)$ is *intrinsic* [4] if every o-automorphism of P is continuous.

Proposition 3.1. *For any partially ordered set* $(P, \leqq)$, *all the topologies defined above are intrinsic.*

Proof. The result follows easily from the definitions and the fact that o-automorphisms preserve arbitrary meets and joins when they exist.

The interrelations noted in the next result will considerably simplify the comparisons of $\mathscr{G}(P)$ and $\mathscr{G}^*(P)$ with the other topologies defined above.

Proposition 3.2. *Let $(P, \leqq)$ be a partially ordered set. Then*

(1) $\mathscr{I}(P) \subseteq \mathscr{N}(P) \subseteq \mathscr{T}\mathscr{L}(P)$;

(2) $\mathscr{I}(P) \subseteq \mathscr{S}(P) \subseteq \mathscr{L}(P) \subseteq \mathscr{T}\mathscr{L}(P)$;

(3) $\mathscr{I}(P) \subseteq \mathscr{Y}(P)$;

(4) $c\mathscr{P}(P) \subseteq \mathscr{S}(P)$.

Proof. (1) Clearly $\mathscr{I}(P) \subseteq \mathscr{N}(P)$. Suppose that C is closed with respect to $\mathscr{N}(P)$, and let $T \subseteq C$ be a totally ordered set such that $\bigvee T \in P$. Let $t \in T$, and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in P$ such that $C \cap [t, \bigvee T] \subseteq \bigcup [a_i, b_i]$. Let $M = \{i \mid a_i \leqq s$ for some $s \in C \cap [t, \bigvee T]\}$. Since T is totally ordered, there exists $j \in M$ such that $s \leqq b_j$ for all $s \in C \cap [t, \bigvee T]$, and then

$$a_j \leqq \bigvee T = \bigvee (C \cap [t, \bigvee T]) \leqq b_j,$$

i.e., $\bigvee T \in \bigcup [a_i, b_i]$. Since C is closed with respect to $\mathscr{N}(P)$, we conclude that $\bigvee T \in C$. Similarly, if $T \subseteq C$ is totally ordered and $\bigwedge T \in P$, then $\bigwedge T \in C$, and hence C is closed with respect to $\mathscr{T}\mathscr{L}(P)$. For (2), it is clear that $\mathscr{I}(P) \subseteq \mathscr{S}(P)$. Suppose that S is a Dedekind-closed dual semi-ideal of P, and let C be a maximal chain of P. Suppose that $e \in C \cap (P \setminus S)$. If there exists $t \in C \cap (P \setminus S)$ such that $e < t$, then $e \in C \setminus [t, \infty) \in \mathscr{I}(C)$ and $C \setminus [t, \infty) \subseteq P \setminus S$. Suppose that such a t does not exist. If $e = \bigwedge (C \cap S)$, then $e \in S$ since S is Dedekind-closed. This contradicts our choice of e, and hence there exists $r \in C \cap S$ such that $C \cap [e, r] = \{e, r\}$. Then $e \in C \setminus [r, \infty) \in \mathscr{I}(C)$ and $C \setminus [r, \infty) \subseteq P \setminus S$, and therefore, $P \setminus S \in \mathscr{L}(P)$. Similarly, complements of Dedekind-closed semi-ideals are elements of $\mathscr{L}(P)$, and thus $\mathscr{S}(P) \subseteq \mathscr{L}(P)$. A similar type of argument shows that $\mathscr{L}(P) \subseteq \mathscr{T}\mathscr{L}(P)$, and hence (2) holds. Finally, (3) is proven in Theorem 4 of [23], and (4) is clear.

Proposition 3.3. *If $(T, \leqq)$ is a totally ordered set, then*

$$\mathscr{I}(T) = \mathscr{N}(T) = \mathscr{S}(T) = \mathscr{L}(T) = \mathscr{T}\mathscr{L}(T) = \mathscr{Y}(T) = c\mathscr{P}(T).$$

Proposition 3.4. *For any partially ordered set $(P, \leqq)$, $\mathscr{I}(P) \subseteq \mathscr{G}^*(P)$, and thus $\mathscr{G}^*(P)$ is T_1. Furthermore, for any totally ordered set $(T, \leqq)$, $\mathscr{I}(T) = \mathscr{G}^*(T)$.*

Proof. That $\mathscr{I}(P) \subseteq \mathscr{G}^*(P)$ follows from the definitions; that $\mathscr{I}(T) = \mathscr{G}^*(T)$ follows from Proposition 2.5.

A partially ordered set $(P, \leqq)$ *has trivial polars* if for all $p \in P$, there exists $r, t \in P$ such that $r < p < t$, $(r, p)^{\perp} = \{r\}$, and $(t, p)_{\mathrm{T}} = \{t\}$. If, for all $p \in P$, there exist $r, t \in P$ such that $r < p < t$, then P is *unbounded*. Not every unbounded set has trivial polars, e.g., $|\sum_{\mathbf{N}}| \mathbf{R}$.

Proposition 3.5. *If $(P, \leqq)$ has trivial polars, then $\mathscr{I}(P) \subseteq \mathscr{G}(P)$ and hence $\mathscr{G}(P)$ is T_1. Furthermore, if $(T, \leqq)$ is totally ordered and unbounded, then $\mathscr{I}(T) = \mathscr{G}(T)$.*

Proof. Since $(P, \leqq)$ has trivial polars, then for all $p \in P$ there exist $r, t \in P$ such that $r < p < t$, $(-\infty, r, p] = (-\infty, p]$, and $[p, t, \infty) = [p, \infty)$. Thus $\mathscr{I}(P) \subseteq \mathscr{G}(P)$. Since $(T, \leqq)$ is unbounded as well as totally ordered, it has trivial polars and hence $\mathscr{I}(T) \subseteq \mathscr{G}(T)$. That $\mathscr{I}(T) \supseteq \mathscr{G}(T)$ follows from Proposition 2.4.

If $\mathscr{A}$ and $\mathscr{B}$ are topologies on a set X, then $\tau(\mathscr{A}, \mathscr{B})$ denotes the topology on X generated by $\mathscr{A} \cup \mathscr{B}$.

Proposition 3.6. *If $(P, \leqq)$ is dually directed, then $\mathscr{G}^*(P) = \tau(\mathscr{G}(P), \mathscr{I}(P))$, and if $(P, \leqq)$ also has trivial polars, then $\mathscr{G}(P) = \mathscr{G}^*(P)$.*

Proof. The first part of the Proposition is straightforward; the second part then follows from Proposition 3.5.

It is easy to see that in $\mathbf{R}$, $(-\infty, 2] \in \mathscr{T}g(\mathbf{R}) \setminus \mathscr{G}(\mathbf{R})$ and $\mathbf{R} \setminus (-\infty, 2] \in \mathscr{G}(\mathbf{R}) \setminus \mathscr{T}g(\mathbf{R})$. Thus $\mathscr{G}(\mathbf{R}) \nsubseteq \mathscr{T}g(\mathbf{R})$ and $\mathscr{G}(\mathbf{R}) \nsupseteq \mathscr{T}g(\mathbf{R})$. By Proposition 3.6, $\mathscr{G}(\mathbf{R}) = \mathscr{G}^*(\mathbf{R})$, and hence $\mathscr{G}^*(\mathbf{R})$ is also not comparable to $\mathscr{T}g(\mathbf{R})$. Dually, neither $\mathscr{G}(\mathbf{R})$ nor $\mathscr{G}^*(\mathbf{R})$ is comparable to $\mathscr{T}d(\mathbf{R})$.

Example 3.7. We construct a partially ordered set $(P, \leqq)$ such that $\mathscr{G}(P) \nsubseteq \mathscr{G}^*(P)$.

Let $D = \mathbf{R} \times \mathbf{R}$ be partially ordered by: $(r, s) \preccurlyeq (x, y)$ if and only if $r = x$ and $s = y$ or $r < x$ and $s < y$. (This is a "tight Riesz order" in the sense of LOY and MILLER [16].) Clearly $d = a \wedge b$ in D if and only if $d = a$ or $d = b$. Thus, if $d \preccurlyeq b$ in D, then $*(d, b)^{\perp} = \{d\}$, and hence $*(-\infty, d, b] = (-\infty, b]$. Similarly $*[a, d, \infty) = [a, \infty)$ for all $a \preccurlyeq d$, and therefore, if $\{*X_i \mid 1 \leqq i \leqq n\}$ is a (finite) set of generalized star-segments, then $D \nsubseteq \bigcup(*X_i)$.

Let $P = D \times \{1, 2\}$ be partially ordered by: $(d, \alpha) \leqq (b, \beta)$ if and only if $d \preccurlyeq b$ and $\alpha = \beta$. By the argument above, $D \times \{1\}$ is not contained in a finite union of generalized star-segments of P, and hence $D \times \{1\}$ is not closed with respect to $\mathscr{G}^*(P)$. However, $D \times \{1\} = (-\infty, (1, 1, 1), (1, 1, 1)]$ is closed with respect to $\mathscr{G}(P)$, and therefore, $\mathscr{G}(P) \nsubseteq \mathscr{G}^*(P)$.

Example 3.8. We construct a distributive lattice $(L, \leqq)$ such that neither $\mathscr{G}(L)$ nor $\mathscr{G}^*(L)$ is contained in any of $\mathscr{I}(L)$, $\mathscr{N}(L)$, $\mathscr{S}(L)$, $\mathscr{L}(L)$, $c\mathscr{P}(L)$, $\mathscr{T}\mathscr{L}(L)$, and $\mathscr{Y}(L)$.

Let $L = ([0, 2] \times]0, 2[) \cup \{(0, 0), (2, 2)\}$ be partially ordered as a subset of $\mathbf{R} \mid \times \mid \mathbf{R}$. Clearly, $(L, \leqq)$ is a distributive lattice. Furthermore, $[(0, 1), (1, 1), \infty) = L \setminus \{(0, 0)\}$, and hence $\{(0, 0)\} \in \mathscr{G}(L)$. However, $T = \{(1/n, 1/n) \mid n \in \mathbf{N}\}$ is a totally ordered subset of $L \setminus \{(0, 0)\}$ such that $\bigwedge T = (0, 0)$, and hence $\{(0, 0)\} \notin \mathscr{T}\mathscr{L}(L)$. Thus, $\mathscr{G}(L) \nsubseteq \mathscr{T}\mathscr{L}(L)$, and hence by Proposition 3.2, $\mathscr{G}(L)$ is contained in none of $\mathscr{I}(L)$, $\mathscr{N}(L)$, $\mathscr{S}(L)$, $\mathscr{L}(L)$, or $c\mathscr{P}(L)$. (In fact, as noted in Example 1, p. 239, of [18], $c\mathscr{P}(L)$ is indiscrete.) Since L is a lattice, Proposition 3.6 then implies that $\mathscr{G}^*(L)$ is also not contained in any of these topologies.

It remains to show that $\mathscr{G}(L) \nsubseteq \mathscr{Y}(L)$. If $\{(0, 0)\} \subseteq J$, a completely irreducible dual $\mathbf{F}$-ideal of L, then $J = L$. Thus, it suffices to show that $\{(0, 0)\}$ is not the intersection of a finite number of completely irreducible $\mathbf{F}$-ideals of L. By Corollary 1 on p. 305 of [5], any such **F**-ideal is maximal with respect to not containing some element of L. Thus, if $\{I_1, \ldots, I_n\}$ is a collection of completely irreducible **F**-ideals of L, each of which contains $\{(0, 0)\}$, then there exist $a_1, \ldots, a_n \in [0, 2]$ and $b_1, \ldots$ $\ldots, b_n \in [0, 2]$ such that I_i is maximal without (a_i, b_i). If $0 < s < \bigwedge b_i$, then clearly $(0, s) \in \bigcap I_i$, and hence $\{(0, 0)\} \neq I_i$. Thus $\{(0, 0)\} \notin \mathscr{Y}(L)$, and hence neither $\mathscr{G}(L)$ nor $\mathscr{G}^*(L)$ is contained in $\mathscr{Y}(L)$.

4. DUALLY DIRECTED CARDINAL PRODUCTS

In this section, we will show that the generalized interval topologies on certain dually directed subsets of a cardinal product are the topologies that the subsets inherit from the product of the generalized interval topologies on the factors.

Let $\{P_\delta \mid \delta \in \Delta\}$ be a collection of partially ordered sets. Let $P = |\prod_\Delta| P_\delta$, and denote the δth projection function by p_δ. If $s \in P_\beta$ and $y \in P$, then $s^y \in P$ is defined by

$$\delta s^y = \begin{cases} s & \text{if } \delta = \beta \\ \delta y & \text{if } \delta \neq \beta . \end{cases}$$

If $x, y, z \in P$ and $b_\beta \in P_\beta$ for each $\beta \in \Delta$ such that $\beta x \neq \beta z$, then $(b_\beta, x, z)^y \in P$ is defined by

$$\delta(b_\beta, x, z)^y = \begin{cases} b_\delta & \text{if } \delta x \neq \delta z \\ \delta y & \text{if } \delta x = \delta z . \end{cases}$$

A subset $Y \subseteq P$ *has single complements* if, whenever $p \in P_\delta$ for some $\delta \in \Delta$, then $p^y \in Y$ for all $y \in Y$; Y *has supported complements* if, whenever $x, z \in Y$ with $x \leqq z$ and we have a $b_\beta \in P_\beta$ for each $\beta \in \Delta$ such that $\beta x \neq \beta z$, then $(b_\beta, x, z)^y \in Y$ for all $y \in Y$.

Let $\mathscr{P}(\mathscr{P}^*)$ be the product of the $\mathscr{G}(P_\delta)$ $(\mathscr{G}^*(P_\delta))$. Then the sets of the form $(C)\, p_\delta^{-1}$, where $C \subseteq P_\delta$ is a generalized segment (star-segment) of P_δ, comprise a subbase for the closed sets of $\mathscr{P}(\mathscr{P}^*)$. We denote the topology that a subset $Y \subseteq P$ inherits from $\mathscr{P}(\mathscr{P}^*)$ by $\mathscr{P}_Y(\mathscr{P}_Y^*)$. Similarly, the segments, the generalized segments and star-segments, the upper and lower polars and star-polars, and the sets of upper and lower bounds are given a subscript Y when they refer to the subset $Y \subseteq P$ and left with no subscript when they refer to P.

Theorem 4.1. *Let $Y \subseteq P$ be dually directed and have single complements. Then $\mathscr{G}(Y) \supseteq \mathscr{P}_Y$. If each P_δ also has trivial polars, then $\mathscr{G}^*(Y) \supseteq \mathscr{P}_Y^*$.*

Proof. To prove the first part of the theorem, it clearly suffices to show that for all $\delta \in \Delta$, for all $s, t \in P_\delta$ with $s \leqq t$, and for all $y \in Y$,

(i) $$((-\infty, s, t]\, p_\delta^{-1}) \cap Y = (-\infty, s^y, t^y]_Y ,$$

and dually. We will prove (i); the dual case follows from the dual argument. Since Y is dually directed, $u_Y(a, c) \neq \square$ for all $a, c \in Y$. Let $x \in ((-\infty, s, t]\, p_\delta^{-1}) \cap Y$. There exists $b \in (s, t)^\perp$ such that $\square \neq u(b, t) \subseteq [\delta x, \infty)$. Since Y has single complements, $b^x, s^y, t^y \in Y$, and if $z \in u_Y(b^x, y)$, then $b^z \in Y$. Suppose that $w \in u_Y(b^z, t^y)$. If $\beta \neq \delta$, then $\beta w \geqq \beta b^z = \beta z \geqq \beta b^x = \beta x$; since $\delta w \geqq \delta b^z = b$ and $\delta w \geqq \delta t^y = t$, then $\delta w \in u(b, t) \subseteq [\delta x, \infty)$. Thus $w \in [x, \infty)$, and hence $u_Y(b^z, t^y) \subseteq [x, \infty)_Y$. Furthermore, $s = (\delta b^z) \wedge (\delta t^y)$, and for $\beta \neq \delta$, $\beta y = (\beta b^z) \wedge (\beta t^y)$. Thus, $b^z \in (s^y, t^y)_Y^\perp$, and hence $x \in (-\infty, s^y, t^y]_Y$. Conversely, suppose that $z \in (-\infty, s^y, t^y]_Y$. Then $\square \neq u_Y(t^y, a) \subseteq [z, \infty)_Y$ for some $a \in (s^y, t^y)_Y^\perp$. Clearly, $\square \neq u(t, \delta a) \subseteq [\delta z, \infty)$, and $\delta a \in (s, t)^\perp$. Therefore, $z \in ((-\infty, s, t]\, p_\delta^{-1}) \cap Y$, and this proves (i).

The second part of the Theorem will follow if we show that for all $\delta \in \Delta$, for all $s, t \in P_\delta$ with $s \leqq t$, there exists $w \in P_\delta$ such that for all $y \in Y$,

$$\text{(ii)} \qquad (*(-\infty, s, t]\, p_\delta^{-1}) \cap Y = *(-\infty, w^y, t^y]_Y\,,$$

and dually. As above, it suffices to prove (ii). If $s < t$ in (ii), then (ii) follows from (i) with $w = s$. If $s = t$, let $w \in P_\delta$ be such that $w < t$ and $(w, t)^\perp = \{w\}$. Then $*(-\infty, s, t] = (-\infty, w, t]$. Since clearly $(-\infty, w^y, t^y]_Y = *(-\infty, w^y, t^y]_Y$, (ii) again follows from (i).

We note that we cannot use Proposition 3.6 to prove the second part of Theorem 4.1 since each P_δ can have trivial polars without Y having trivial polars. For example, $\boldsymbol{Z}$ has trivial polars, but $|\sum_{\boldsymbol{N}}|\, \boldsymbol{Z}$ is dually directed, has single complements, and does not have trivial polars.

Theorem 4.2. *(a) $\mathscr{G}(P) \subseteq \mathscr{P}$ and $\mathscr{G}^*(P) \subseteq \mathscr{P}^*$. (b) If $Y \subseteq P$ is dually directed and has supported complements, then $\mathscr{G}(Y) \subseteq \mathscr{P}_Y$ and $\mathscr{G}^*(Y) \subseteq \mathscr{P}_Y^*$.*

Proof. (a) It clearly suffices to show that for all $s, t \in P$ with $s \leqq t$,

$$\text{(iii)} \qquad (-\infty, s, t] = \textstyle\bigcap_\Delta((-\infty, \delta s, \delta t]\, p_\delta^{-1})\,,$$

$$\text{(iv)} \qquad (-\infty, t] = \textstyle\bigcap_\Delta((-\infty, \delta t]\, p_\delta^{-1})\,,$$

and dually. As above, it suffices to prove (iii) and (iv); (iv) is clear. To see that (iii) holds, let $x \in (-\infty, s, t]$. Then $\square \neq u(b, t) \subseteq [x, \infty)$ for some $b \in (s, t)^\perp$, and hence, for all $\delta \in \Delta$, $\square \neq u(\delta b, \delta t) \subseteq [\delta x, \infty)$ and $\delta b \in (\delta s, \delta t)^\perp$, i.e., $\delta x \in (-\infty, \delta s, \delta t]$. Conversely, if $x \in \bigcap_\Delta((-\infty, \delta s, \delta t]\, p_\delta^{-1})$, then for all $\delta \in \Delta$ there exists $b_\delta \in (\delta s, \delta t)^\perp$ such that $\square \neq u(\delta t, b_\delta) \subseteq [\delta x, \infty)$. Let $b \in P$ be such that $\delta b = b_\delta$ for all $\delta \in \Delta$. Then clearly $b \in (s, t)^\perp$ and $\square \neq u(t, b) \subseteq [x, \infty)$, i.e., $x \in (-\infty, s, t]$.

(b) For $a, b \in P$, let $S(a, b) = \{\delta \in \Delta \mid \delta a \neq \delta b\}$. It suffices to show that for $r, t \in P$ with $r < t$,

$$\text{(v)} \qquad (-\infty, r, t]_Y = \textstyle\bigcap_{S(r,t)}((-\infty, \delta r, \delta t]\, p_\delta^{-1} \cap Y)\,,$$

$$\text{(vi)} \qquad (-\infty, t]_Y = \textstyle\bigcap_\Delta((-\infty, \delta t]\, p_\delta^{-1} \cap Y)\,,$$

and dually. As above, it suffices to prove (v) and (vi); and (vi) is clear. By (iii), $(-\infty, r, t]_Y \subseteq \bigcap_{S(r,t)}((-\infty, \delta r, \delta t]\, p_\delta^{-1} \cap Y)$. Conversely, let $x \in (-\infty, \delta r, \delta t]\,.\,. p_\delta^{-1} \cap Y$ for all $\delta \in S(r, t)$. If $\delta \in S(r, t)$, then there exists $b_\delta \in (\delta r, \delta t)^\perp$ such that $\Box \mathrel{+} u(b_\delta, \delta t) \subseteq [\delta x, \infty)$. Let $z \in u_Y(x, t)$, and let $z^* = (b_\delta, r, t)^z$. Since Y has supported complements, $z^* \in Y$. If $\delta \in S(r, t)$, then $\delta r = (\delta z^*) \wedge (\delta t)$, and if $\delta \notin S(r, t)$, then $\delta r = \delta t = (\delta z^*) \wedge (\delta t)$. Thus, $z^* \in (r, t)_Y^\perp$. Furthermore, if $w \in u_Y(z^*, t)$, then for $\delta \in S(r, t)$, $\delta w \in u(b_\delta\ \delta t) \subseteq [\delta x, \infty)$ and for $\delta \notin S(r, t)$, $\delta w \geqq \delta z \geqq \delta x$. Thus, $w \in [x, \infty)_Y$, and hence $\Box \mathrel{+} u_Y(z^*, t) \subseteq [x, \infty)_Y$. Therefore, $x \in (-\infty, r, t]_Y$ and (v) follows.

Corollary 4.3. *If P is the cardinal product of the dually directed sets $\{P_\alpha \mid \alpha \in A\}$, then $\mathscr{G}(P)$ is the product of the $\mathscr{G}(P_\alpha)$. If, in addition each P_α has trivial polars, then $\mathscr{G}^*(P)$ is the product of the $\mathscr{G}^*(P_\alpha)$ and $\mathscr{G}^*(P) = \mathscr{G}(P)$.*

Thus, in ensuring that the generalized star-interval topology is cardinally productive we have assumed enough to force the generalized star-interval topologies on the factors and on the product to be precisely the corresponding generalized interval topologies.

Corollary 4.4. *Let $\{G_\alpha \mid \alpha \in A\}$ be a collection of lattice-ordered groups. Then $\mathscr{G}(|\sum_A| G_\alpha)$ is the topology that $|\sum_A| G_\alpha$ inherits from the product of the $\mathscr{G}(G_\alpha)$. If each G_α has trivial polars, then $\mathscr{G}^*(|\sum_A| G_\alpha) = \mathscr{G}(|\sum_A| G_\alpha)$.*

We note that since clearly $|\sum_{\boldsymbol{N}}| \boldsymbol{Z}$ does not have trivial polars but by Corollary 4.4 $\mathscr{G}^*(|\sum_{\boldsymbol{N}}| \boldsymbol{Z}) = \mathscr{G}(|\sum_{\boldsymbol{N}}| \boldsymbol{Z})$, then the converse of the second part of Proposition 3.6 fails to hold.

5. EXAMPLES AND TRIVIALLY ORDERED SETS

We will show in this section that without the hypothesis of Corollary 3.3 that the factors be dually directed the product of the generalized interval topologies may not be the generalized interval topology on the cardinal product. Constructing this example will involve characterizing the generalized interval (and star-interval) topology on a trivially ordered set as the cofinite topology.

We will first finish the comparisons begun in § 3 and show that even for distributive lattices the generalized star-interval topology may not be cardinally productive.

Example 5.1. It is straightforward to show that the closure of $\{1\}$ with respect to $\mathscr{G}(\boldsymbol{N})$ is $\{1, 2\}$. It is also clear that $\mathscr{I}(\boldsymbol{N})$, and hence (by Propositions 3.3 and 3.4) $\mathscr{S}(\boldsymbol{N})$, $\mathscr{L}(\boldsymbol{N})$, $\mathscr{N}(\boldsymbol{N})$, $\mathscr{T}\mathscr{L}(\boldsymbol{N})$, $\mathscr{Y}(\boldsymbol{N})$, $c\mathscr{P}(\boldsymbol{N})$, and $\mathscr{G}^*(\boldsymbol{N})$, are all equivalent to the discrete topology on $\boldsymbol{N}$. Thus, $\mathscr{G}(\boldsymbol{N})$ contains none of these topologies.

Example 5.2. Let $P = |\prod_{\mathbf{N}}| \mathbf{N}$. We will show that $\mathscr{G}^*(P)$ is strictly contained in $\mathscr{P}^*$, the product of the $\mathscr{G}^*(\mathbf{N})$. Thus, $\mathscr{G}^*(P)$ contains none of $\mathscr{Y}(P)$, $c\mathscr{P}(P)$, $\mathscr{S}(P)$, $\mathscr{L}(P)$, or $\mathscr{T}\mathscr{L}(P)$.

Let $h \in P$ be defined by

$$nh = \begin{cases} 2 & \text{if} \quad n = 1 \\ 1 & \text{if} \quad n \neq 1 . \end{cases}$$

Suppose that the finite sets I, $J \subseteq N$ and $\{(a_i, b_i) \mid i \in I\}$. $\{(r_j, s_j) \mid j \in J\} \subseteq P \times P$, are such that

$$(2) \qquad (*(-\infty, 1, 1])\, p_1^{-1} = (\{1\})\, p_1^{-1} \subseteq (\textstyle\bigcup_I^* [a_i, b_i, \infty)) \cup (\bigcup_J^* (-\infty, r_j, s_j]) .$$

Suppose further that $h \notin \bigcup_I^* [a_i, b_i, \infty)$ and $h \notin \bigcup_J^* (-\infty, r_j, s_j]$. Then for each $i \in I$, there exists $k_i \in \mathbf{N}$ such that $1 < k_i a_i$ and either $na_i = nb_i$ for all $n \in \mathbf{N}$ or $k_i a_i < k_i b_i$; furthermore, for each $j \in J$, $nr_j = ns_j$ for all $n \in \mathbf{N}$ and $1r_j = 1$. Let $q \in \mathbf{N} \setminus (\{k_i \mid i \in I\} \cup \{1\})$. Define $f \in P$ by

$$nf = \begin{cases} (\bigvee_J nr_j) + 1 & \text{if} \quad n = q \\ 1 & \text{if} \quad n \neq q . \end{cases}$$

Since $q \neq 1$, $f \in (*(-\infty, 1, 1])\, p_1^{-1}$; since $k_i f = 1$ for all $i \in I$, $f \notin \bigcup_I^* [a_i, b_i, \infty)$; since $qs_j < qf$ for all $j \in J$, $f \notin \bigcup_J^* (-\infty, r_j, s_j]$. This contradicts (2), and thus $h \in (\bigcup_I^* [a_i, b_i, \infty)) \cup (\bigcup_J^* (-\infty, r_j, s_j])$. Therefore, by definition of $\mathscr{G}^*(P)$, $(*(-\infty, 1, 1])\, p_1^{-1}$ is not closed with respect to $\mathscr{G}^*(P)$, i.e., $\mathscr{G}^*(P)$ does not equal $\mathscr{P}^*$, and thus, by Theorem 4.2, $\mathscr{G}^*(P)$ is strictly contained in $\mathscr{P}^*$. By Proposition 3.3, $\mathscr{G}^*(\mathbf{N}) = \mathscr{I}(\mathbf{N}) = \mathscr{Y}(\mathbf{N}) = c\mathscr{P}(\mathbf{N})$. Thus, since $\mathbf{N}$ is a lattice, by Corollary I on p. 246 of [18], $c\mathscr{P}(P) = \mathscr{P}^*$, and, as remarked on p. 1007 of [1], $\mathscr{Y}(P) \supseteq \mathscr{P}^*$. We conclude that $\mathscr{G}^*(P) \not\supseteq \mathscr{Y}(P)$ and $\mathscr{G}^*(P) \not\supseteq c\mathscr{P}(P)$. Then by Proposition 3.2, $\mathscr{G}^*(P)$ contains none of $\mathscr{S}(P)$, $\mathscr{L}(P)$, or $\mathscr{T}\mathscr{L}(P)$.

Example 5.3. Let $P = |\prod_{\mathbf{R}}| \mathbf{R}$. Let $\mathscr{P}^*$ be the product of the $\mathscr{G}^*(\mathbf{R}) = \mathscr{I}(\mathbf{R}) = \mathscr{N}(\mathbf{R})$. By Corollary 4.3, $\mathscr{G}^*(P) = \mathscr{P}^*$. By Theorem 8 of [1], $\mathscr{N}(P) \supseteq \mathscr{P}^*$, but as noted on p. 1013 of [1], Exercise J on p. 240 of [14] and Theorem 3 of [3] imply that $\mathscr{N}(P) \neq \mathscr{P}^*$. Therefore, $\mathscr{N}(P) \nsubseteq \mathscr{G}^*(P)$.

Let $(P, \leqq)$ be a partially ordered set. We say that $x, y \in P$ are *incomparable* if $x \ngeq y$ and $x \nleq y$. Then $(P, \leqq)$ is *trivially ordered* if every pair of disjoint elements is incomparable. For a set X, the *cofinite topology* on X, denoted by $\mathscr{C}(X)$, has for open sets precisely the subsets of X with finite complements. Clearly, $\mathscr{C}(X)$ is the minimal T_1 topology on X.

Proposition 5.4. *If $(P, \leqq)$ is trivially ordered, then*

$$\mathscr{C}(P) = \mathscr{I}(P) = \mathscr{G}^*(P) = \mathscr{G}(P) = \mathscr{Y}(P) = c\mathscr{P}(P) ,$$

and $\mathscr{T}\mathscr{L}(P)$, $\mathscr{L}(P)$, $\mathscr{N}(P)$, $\mathscr{S}(P)$, $\mathscr{T}d(P)$, and $\mathscr{T}g(P)$ are all discrete.

Proof. It is easy to see that the **L**-closed prime ideals and dual ideals, the final and initial segments, and the generalized final and initial segments and star-segments are precisely the singletons. Furthermore, since $u(\square) = b(\square) = P$, it is also clear that the singletons are the only (proper) **F**-ideals and dual **F**-ideals and, as such, are completely irreducible. For the second part of the Proposition, it is straightforward to check that any subset of P is closed with respect to $\mathcal{T}\mathcal{L}(P)$, $\mathcal{N}(P)$, and $\mathcal{S}(P)$ or open with respect to $\mathcal{L}(P)$, $\mathcal{T}d(P)$, and $\mathcal{T}g(P)$.

Corollary 5.5. *If* $(P, \leqq)$ *is an infinite trivially ordered set, then* $\mathcal{I}(P)$, $\mathcal{G}^*(P)$, $\mathcal{G}(P)$, $\mathcal{Y}(P)$, *and* $c\mathcal{P}(P)$ *are all* T_1 *but not Hausdorff.*

Example 5.6. Let $\boldsymbol{Z}$ be trivially ordered by $\preccurlyeq$. Let $(P, \preccurlyeq) = (\boldsymbol{Z}, \preccurlyeq) \mid \times \mid (\boldsymbol{Z}, \preccurlyeq)$, and let $\mathcal{P}$ be the product of $\mathcal{G}(\boldsymbol{Z})$ with itself. We will show that $\mathcal{G}(P) \neq \mathcal{P}$. Clearly $(P, \preccurlyeq)$ is trivially ordered, and thus by Proposition 5.4, $\mathcal{G}(P) = \mathcal{C}(P)$. To see that $\mathcal{G}(P) \neq \mathcal{P}$ consider

$$\{(m, n) \in P \mid n \neq 0\} = (\boldsymbol{Z} \setminus \{0\})\, p_2^{-1} \in \mathcal{P}\,.$$

Clearly $\{(m, 0) \mid m \in \boldsymbol{Z}\}$ is infinite and equals $P \setminus \{(m, n) \mid n \neq 0\}$. Thus, $\{(m, n) \mid n \neq 0\} \notin \mathcal{C}(P) = \mathcal{G}(P)$, and hence $\mathcal{G}(P) \neq \mathcal{P}$.

Furthermore, we note that since $\mathcal{C}(P)$ is minimal, $\mathcal{Y}(P) = c\mathcal{P}(P) = \mathcal{C}(P) \subseteq \mathcal{P}$, but by the argument above, $\mathcal{Y}(P) \neq \mathcal{P} \neq c\mathcal{P}(P)$. Thus neither the remark on p. 1007 of [1] that the ideal topology of a cardinal product of lattices contains the product of the ideal topologies, nor Corollary I on p. 246 of [18] which says that the **CP**-ideal topology is preserved by cardinal products of lattices, holds for arbitrary partially ordered sets.

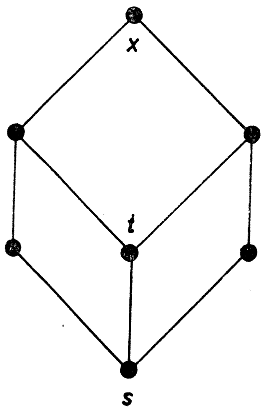

Fig. 1.

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