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NONSINGULAR SEMILATTICES AND SEMIGROUPS

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1. INTRODUCTION

For a ring R, the condition that every large right ideal is dense (R nonsingular) implies that $Q(R)$, the maximal ring of quotients of R, is a regular ring, self-injective as a Q-module, and the R-injective hull of R. If S is a nonsingular semigroup, its maximal quotient semigroup $Q(S)$ need not be regular (see [5]), but C. V. HINKLE, JR. in [2] showed that $Q(S)$ is the injective hull of S and $Q(S)$ is self-injective as a Q-system. Hinkle [3] has also shown that if S is a semilattice E of groups, then $Q(S)$ is a semilattice $Q(E)$ of groups.

These considerations lead to the investigation of nonsingular semilattices and nonsingular semigroups that are semilattices of groups. We characterize nonsingular semilattices as disjunctive semilattices and point out an alternate description of $Q(S)$ for these semilattices. We then give a description of nonsingular semigroups that are semilattices of groups and simplify this description in two special cases.

2. NONSINGULAR SEMILATTICES

Throughout this paper all semilattices and semigroups will have a zero. If S is a semigroup in which every one-sided ideal is two-sided, we call an ideal D *dense* if and only if $x \neq y$ $(x, y \in S)$ implies that there exists $d \in D$ such that $xd \neq yd$. A nonzero ideal L is *large* if and only if L has a nonzero intersection with any other nonzero ideal of S. Clearly every dense ideal is large. If S is a semigroup in which every large ideal is dense, S is called *nonsingular*. This terminology corresponds to standard usage in ring theory.

Recall that a semilattice S (with 0) is *disjunctive* if, whenever $x \nleqq y$ $(x, y \in S)$, there exists $u \in S$ with $0 < u \leqq x$ and $u \wedge y = 0$. It is easy to show that this is equivalent to the requirement that every interval $[0, x] = \{z \in S : 0 \leqq z \leqq x\}$ be semicomplemented, i.e., for each $y \in [0, x)$ there exists $y' \in (0, x]$ with $y \wedge y' = 0$.

Theorem 1. *A semilattice S with 0 is nonsingular if and only if it is disjunctive.*

Proof. Assume S is nonsingular and $x \nleqq y$ $(x, y \in S)$. Set $I = \{a \in S : a \wedge z = 0$ or $a \leqq z\}$ where $z = x \wedge y < x$. Now I is a large ideal, for if J is a nonzero ideal and $b \in J \setminus \{0\}$ then $b \wedge z$ is a nonzero element of $I \cap J$ unless $b \wedge z = 0$, in which case $b \in I \cap J$. Hence I is dense and there exists $d \in I$ with $z \wedge d < x \wedge d$. We must have $d \wedge z = 0$, for otherwise $d \in I$ would give $d \leqq z$ and then $d = z \wedge d < x \wedge d \leqq d$. Letting $u = x \wedge d$ we have $0 < u \leqq x$ and $u \wedge y = d \wedge z = 0$.

Suppose conversely that S is disjunctive, that I is a large ideal and that $x \neq y$. We may also assume $x \nleqq y$, thus getting $u \in S$ with $0 < u \leqq x$ and $u \wedge y = 0$. If we now take d to be a nonzero element of $I \cap [0, u]$, we have $x \wedge d = d \neq 0 = y \wedge d$.

We remark at this point that if S is a disjunctive semilattice then $Q(S)$, being the injective hull of S (in the category of S-systems), is by the main theorem of [4] isomorphic to $I_D(S)$, the lattice of all D-ideals of S.

3. NONSINGULAR SEMIGROUPS THAT ARE SEMILATTICES OF GROUPS

For this section we let S be a semilattice Y of groups G_α $(\alpha \in Y)$ where Y is order isomorphic to $E(S)$, the idempotents of S. We assume that the reader is familiar with Clifford's result concerning S (Theorem 4.11 of [1]), and we let $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ $(\beta \leqq \alpha)$ denote the linking homomorphisms. Recall that the idempotents of S are central and every one-sided ideal is two-sided while being itself a union of the groups it contains. We let e_α denote the identity of G_α.

Theorem 2. *S is nonsingular if and only if*

(i) *$E(S)$ is nonsingular*

and

(ii) *if L is a large ideal and $e_\alpha \in E(S) \setminus E(L)$, then*

$$\bigcap\{\operatorname{Ker} \phi_{\alpha,\beta} : e_\beta \in E(L),\ e_\beta < e_\alpha\} = \{e_\alpha\}\,.$$

Proof. Assume S is nonsingular and let F be a large ideal of $E(S)$ Then $L = \bigcup\{G_\beta : e_\beta \in F\}$ is easily seen to be a large ideal of S and is therefore dense. If $e_\alpha \neq e_\beta$ $(e_\alpha, e_\beta \in E(S))$ then there exists $x \in L$ such that $e_\alpha x \neq e_\beta x$, and if $x \in G_\gamma$ we have $e_\alpha e_\gamma \neq e_\beta e_\gamma$ with $e_\gamma \in F$ giving F dense in $E(S)$. Now let L be a large ideal of S and $e_\alpha \in E(S) \setminus E(L)$. Suppose the condition is not satisfied so that there exists $x \in G_\alpha$ with $x \neq e_\alpha$ and $\phi_{\alpha,\beta}(x) = e_\beta$ for all $e_\beta \in E(L)$ with $e_\beta < e_\alpha$. Let $d \in L$ be arbitrary where $d \in G_\gamma$. Then $e_\alpha e_\gamma < e_\alpha$ and $e_\alpha e_\gamma \in E(L)$. Therefore $\phi_{\alpha,\alpha\gamma}(x) = e_{\alpha\gamma}$ and we have $xd = \phi_{\alpha,\alpha\gamma}(x)\, \phi_{\gamma,\alpha\gamma}(d) = e_{\alpha\gamma}\phi_{\gamma,\alpha\gamma}(d) = \phi_{\alpha,\alpha\gamma}(e_\alpha)\, \phi_{\alpha,\alpha\gamma}(d) = e_\alpha d$, a contradiction to the fact that L is dense.

For the converse, let L be a large ideal of S. Then $E(L)$ is a large ideal of $E(S)$ so that $E(L)$ is dense in $E(S)$. Let $x, y \in S$ with $x \neq y$. Assume $x \in G_\alpha$ and $y \in G_\beta$ with $\alpha \neq \beta$. Since $E(L)$ is dense, there exists $e_\gamma \in E(L)$ such that $e_\alpha e_\gamma \neq e_\beta e_\gamma$ and thus $G_{\alpha\gamma} \cap G_{\beta\gamma} = \emptyset$. Since $xe_\gamma \in G_{\alpha\gamma}$ and $ye_\gamma \in G_{\beta\gamma}$, we have $xe_\gamma \neq ye_\gamma$. Now suppose $x, y \in G_\alpha$. If $e_\alpha \in E(L)$ we are done, so assume $e_\alpha \notin E(L)$. Then there must exist $e_\beta \in E(L)$ such that $e_\beta < e_\alpha$ and $\phi_{\alpha,\beta}(x) \neq \phi_{\alpha,\beta}(y)$, for otherwise $xy^{-1} \neq e_\alpha$ violates (ii). Now we have $xe_\beta = \phi_{\alpha,\beta}(x) \neq \phi_{\alpha,\beta}(y) = ye_\beta$ and the proof is complete.

Two special kinds of semilattices of groups are given in the following definitions. S is said to have *trivial multiplication* if and only if each $\phi_{\alpha,\beta}$ with $\beta < \alpha$ is the trivial homomorphism. S is *0-proper* if and only if $\phi_{\alpha,\beta}$ is one-to-one whenever $\beta \neq 0$.

Corollary. *Assume S is 0-proper. Then S is nonsingular if and only if $E(S)$ is nonsingular.*

Corollary. *Assume S has trivial multiplication. Then S is nonsingular if and only if $E(S)$ is nonsingular and $|G_\alpha| > 1$ implies that e_α is an atom of $E(S)$.*

Proof. Let S be nonsingular, $|G_\alpha| > 1$, and suppose e_α is not an atom. Then there exists $e_\beta \in E(S)$ with $0 < e_\beta < e_\alpha$. Then $I = \{e \in E(S) : ee_\beta = 0 \text{ or } e \leqq e_\beta\}$ is large in $E(S)$ with $e_\alpha \notin I$, and hence $L = \bigcup\{G_\gamma : e_\gamma \in I\}$ violates (ii) of the theorem. The converse follows from the theorem and the observation that a large ideal contains all atoms.

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