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ARCHIMEDEAN CLASSES IN AN ORDERED SEMIGROUP I

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INTRODUCTION

The archimedean equivalence on an ordered semigroup has been defined by the author [6], [7] and B. PONDĚLÍČEK [3]. But the difficulty occurs because of the fact that the archimedean equivalence is not necessarily a congruence relation, that is, the set product of two archimedean classes is not always contained in a single archimedean class.

The behavior of set products of archimedean classes has been studied by the author [9] for nonnegatively ordered semigroups. In the present paper we generalise the theory for general ordered semigroups. As in the previous paper, we define the operation $*$ between archimedean classes so that the set of archimedean classes forms an ordered idempotent semigroup. Then we show that the set products of archimedean classes are determined in some extent in terms of the operation $*$. These final results are given in § 6.

1. PRELIMINARIES

We use the terminology and notation in CLIFFORD and PRESTON [2] freely. By an *ordered semigroup* we mean a semigroup S with a simple order $\leqq$ which satisfies

$$\text{for} \quad x, y, z \in S\,, \quad x \leqq y \quad \text{implies} \quad xz \leqq yz \quad \text{and} \quad zx \leqq zy\,.$$

Let S be an ordered semigroup. An element x of S is called *positive* [*negative*; *nonnegative*; *nonpositive*] if $x < x^2$ [$x^2 < x$; $x \leqq x^2$; $x^2 \leqq x$]. The number of distinct natural powers of an element x of S is called the *order* of x. Thus x is an element of finite order m if and only if there is a natural number n such that $x^n = x^{n+1}$ and m is the least in the set of natural numbers n satisfying this condition. An ordered semigroup S is called *nonnegatively* [*nonpositively*] *ordered* if every element of S is nonnegative [nonpositive].

The *archimedean equivalence* $\mathscr{A}$ on an ordered semigroup S is defined by:

for $x, y \in S$, $x \mathscr{A} y$ if and only if there exist natural numbers p, q, r and s such that $x^p \leqq y^q$ and $y^r \leqq x^s$.

In our papers [6] and [7], we gave the following results:

Lemma 1.1. *The archimedean equivalence* $\mathscr{A}$ *on an ordered semigroup* S *is an equivalence relation on* S.

Lemma 1.2. *For elements* x *and* y *in an ordered semigroup* S, *the following conditions are equivalent*:

(1) $x \mathscr{A} y$;
(2) *there exist natural numbers* p, q *and* r *such that* $x^p \leqq y^q \leqq x^r$;
(3) *there exist natural numbers* p, q *and* r *such that* $y^p \leqq x^q \leqq y^r$.

An equivalence class of an ordered semigroup S modulo the archimedean equivalence $\mathscr{A}$ is called an *archimedean class*.

Lemma 1.3. *Each archimedean class of an ordered semigroup* S *is a convex subsemigroup of* S.

Lemma 1.4. *Each archimedean class of an ordered semigroup* S *has at most one idempotent. For an archimedean class* C *of* S, *the following conditions are equivalent*:

(1) C *contains an idempotent*;
(2) *the set of all nonnegative elements of* C *is nonempty and has the greatest element*;
(3) *the set of all nonpositive elements of* C *is nonempty and has the least element*;
(4) C *has the zero element*;
(5) *every element of* C *is an element of finite order*;
(6) C *contains an element of finite order*;
(7) C *contains at least one nonnegative element and at least one nonpositive element.*

Moreover, for an element e *of* C, *the following conditions are equivalent*;

(8) e *is an idempotent of* C;
(9) e *is the greatest nonnegative element of* C;
(10) e *is the least nonpositive element of* C;
(11) e *ist the zero element of* C.

If an archimedean class C of an ordered semigroup S satisfies one of the equivalent conditions (1)–(7) in Lemma 1.4, then C is called *periodic*. An archimedean class C

of S is called *torsion-free* if it is not periodic. If C is a torsion-free archimedean class, then either every element of C is positive or every element of C is negative. In the former case, C is called a *positive torsion-free* archimedean class and, in the latter case, C is called a *negative torsion-free* archimedean class of S.

Also in [4] and [5], we gave the following results:

Lemma 1.5. *An idempotent semigroup E is a semilattice of rectangular bands. Every rectangular band which is a constituent of the decomposition is a $\mathscr{D}$-class of E.*

Lemma 1.6. *The set of all idempotents of an ordered semigroup S is a subsemigroup of S, if it is nonvoid.*

Lemma 1.7. *In an ordered idempotent semigroup E, each $\mathscr{D}$-class consists of either only one $\mathscr{L}$-class or only one $\mathscr{R}$-class.*

Let S be an ordered semigroup. By Lemma 1.5, the set E of all idempotents of S is a subsemigroup of S, if it is nonvoid. The $\mathscr{D}$-equivalence in the semigroup E is denoted by *$\mathscr{D}_E$-equivalence.* Let D be a $\mathscr{D}$-class of an ordered idempotent semigroup E. By Lemma 1.7, D consists of either only one $\mathscr{L}$-class or only one $\mathscr{R}$-class in E. In the former case D is called of *L-type* and in the latter case D is called of *R-type*.

Let P be a simply ordered set. An element b of P is said to *lie between* two elements a and c of P if either $a \leqq b \leqq c$ or $c \leqq b \leqq a$.

Let L be a semilattice with respect to the order $\leqq$. L is called a *tree semilattice*, if for every $a \in L$, the set $\{x; x \leqq a\}$ forms a simply ordered subset of L.

Finally we give a result from [9].

Lemma 1.8. *Let A and B be archimedean classes in a nonnegatively ordered semigroup S such that $A \leqq B$. Suppose that the set product $BA[AB]$ is not contained in a single archimedean class. Then*

(1) *B is a periodic archimedean class with idempotent, say e, $B \setminus \{e\} \neq \square$ and every element of $B \setminus \{e\}$ is of order 2;*

(2) *the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type [R-type] and there exists an idempotent f of S such that $f < e$, $f \mathscr{D}_E e$ and f and e are consecutive in $e\mathscr{D}_E$;*

(3) *there exists a periodic archimedean class C with idempotent, say g, which satisfies the following conditions:*

(a) $g \mathscr{D}_E e$ *and* $g < f$;
(b) $A \leqq C$;
(c) $AC, CA \subseteq C$;
(d) $Bg = \{f, e\}$ $[gB = \{f, e\}]$;

(4) $BA \subseteq \{f\} \cup B$ $[AB \subseteq \{f\} \cup B]$.

In the rest of this paper, we denote always by S an ordered semigroup, by E the set of all idempotents of S and by $\mathscr{C}$ the set of all archimedean classes in S.

We define the *binary relation* γ on $\mathscr{C}$ by:

for $A, B \in \mathscr{C}$, $A \gamma B$ if and only if either $AB \cap A \neq \square$ or $BA \cap A \neq \square$.

Lemma 2.1. *Let $A, B \in \mathscr{C}$ such that $A \gamma B$. If A is positive [negative] torsion-free, then $B \leqq A[A \leqq B]$.*

Proof. Suppose A is positive torsion-free. Since $A \gamma B$, there exist $a \in A$ and $b \in B$ such that either $ab \in A$ or $ba \in A$. Since A is positive torsion-free, there exists a natural number n such that either $ab \leqq a^n < a^{n+1}$ or $ba \leqq a^n < a^{n+1}$. Hence we have $b < a^n$ and so $B \leqq A$.

Lemma 2.2. *Let $A, B \in \mathscr{C}$ such that A is torsion-free and let $a \in A$ and $b \in B$. Then $ab \in A$ if and only if $ba \in A$.*

Proof. We consider only the case when A is positive torsion-free. First suppose that $ab \in A$. Then there exists a natural number n such that $ab \leqq a^n < a^{n+1}$. Hence $b < a^n$ and so $ba \leqq a^{n+1}$. Also, since $a^2, ab \in A$, there exists a natural number m such that $a^2 \leqq (ab)^m$. Hence $a^2 < a^3 \leqq (ab)^m a = a(ba)^m$ and so $a < (ba)^m$. Thus we have $a \mathscr{A} ba$ and so $ba \in A$. Similarly we can prove that $ba \in A$ implies $ab \in A$.

Lemma 2.3. *Let $A, B \in \mathscr{C}$ such that A is torsion-free and let $a \in A$ and $b \in B$. Then the following conditions are equivalent:*

(1) $ab \in A$;
(2) $a^m b^m \in A$ *for every natural number* m;
(3) $a^m b^m \in A$ *for some natural number* m;
(4) $ba \in A$;
(5) $b^m a^m \in A$ *for every natural number* m;
(6) $b^m a^m \in A$ *for some natural number* m.

Proof. We consider only the case when A is positive torsion-free and $ab \leqq ba$.

(1) $\Rightarrow$ (2). Suppose $ab \in A$. By Lemma 2.1, we have $B \leqq A$ and so there exists a natural number n such that $b \leqq a^n$. Hence

$$(ab)^m \leqq b^m a^m \leqq a^{mn} a^m = a^{mn+m}$$

with $(ab)^m, a^{mn+m} \in A$. Hence $b^m a^m \in A$ and so, by Lemma 2.2, $a^m b^m \in A$.

(2) $\Rightarrow$ (3). Evident.

(3) $\Rightarrow$ (1). Suppose $a^m b^m \in A$ for some natural number m. Then, by Lemma 2.2, $b^m a^m \in A$. Also we have $a^m b^m \leqq (ab)^m \leqq b^m a^m$. Hence $(ab)^m \in A$ and so $ab \in A$.

By Lemma 2.2, (1) $\Leftrightarrow$ (4), (2) $\Leftrightarrow$ (5) and (3) $\Leftrightarrow$ (6).

Theorem 2.4. *Let A, $B \in \mathscr{C}$ such that A is torsion-free. Then the following conditions are equivalent*:

(1) $A \gamma B$;
(2) $AB \cap A \neq \square$;
(3) *for every $a \in A$ and $b \in B$, there exists a natural number n such that $a^n b \in A$*;
(4) $BA \cap A \neq \square$;
(5) *for every $a \in A$ and $b \in B$, there exists a natural number n such that $ba^n \in A$.*

Proof. By Lemma 2.2, (2) $\Leftrightarrow$ (4) and so the conditions (1), (2) and (4) are mutually equivalent. Also, by Lemma 2.2, (3) $\Leftrightarrow$ (5) and clearly (3) $\Rightarrow$ (2). Finally we assume the condition (2). Then there exist $a' \in A$ and $b' \in B$ such that $a'b' \in A$. Let $a \in A$ and $b \in B$. We consider only the case when A is positive torsion-free. Then, by Lemma 2.1, $B \leqq A$ and so there exists a natural number m such that $b \leqq a^m$. Since $b, b' \in B$, there exist natural numbers p and q such that $b'^p \leqq b^q$. Also, since $a, a' \in A$, there exists a natural number r such that $a' \leqq a^r$. We take a natural number n such that $pr < nq$. Then

$$a'^p b'^p \leqq a^{pr} b^q \leqq a^{pr} a^{mq} = a^{pr+mq} .$$

Here $a^{pr+mq} \in A$ and, by Lemma 2.3, $a'^p b'^p \in A$. Hence $a^{pr} b^q \in A$ and so

$$a^{nq} b^q = a^{nq-pr} a^{pr} b^q \in A .$$

Therefore, again by Lemma 2.3, we have $a^n b \in A$. This proves (2) $\Rightarrow$ (3).

Lemma 2.5. *Let a be an element of S and let e be an idempotent of S. Then $ea^m e = (eae)^m$ for every natural number m.*

Proof. First suppose $ea \leqq ae$. Then

$$(eae)^m = e(ae)^m \leqq e(a^m e^m) = ea^m e = (e^m a^m)\, e \leqq (ea)^m\, e = (eae)^m$$

and so $(eae)^m = ea^m e$. In the case when $ae \leqq ea$, we can prove the same conclusion in a similar way.

Lemma 2.6. *Suppose that $a = ab^m[a = b^m a]$ for some $a, b \in S$ and some natural number m. Then $a = ab[a = ba]$.*

Proof. By way of contradiction, we assume that $a \neq ab$. Then $a < ab$ or $ab < a$. If $a < ab$, then

$$a < ab \leqq ab^2 \leqq \ldots \leqq ab^m ,$$

which is a contradiction. In a similar way, we can prove that $ab < a$ implies a contradiction.

Theorem 2.7. *Let $A, B \in \mathscr{C}$ such that A is periodic with idempotent e. Then the following conditions are equivalent*:

(1) $A \gamma B$;

(2) $eb = e$ *or* $be = e$ *for some* $b \in B$;

(3) *if the* $\mathscr{D}_E$*-class* $e\mathscr{D}_E$ *is of L-type, then* $eb = e$ *for all* $b \in B$ *and, if* $e\mathscr{D}_E$ *is of R-type, then* $be = e$ *for all* $b \in B$;

(4) $ebe = e$ *for some* $b \in B$;

(5) $ebe = e$ *for all* $b \in B$.

Proof. (1) $\Rightarrow$ (2). Suppose $A \gamma B$. Then either $AB \cap A \neq \square$ or $BA \cap A \neq \square$. First we suppose $AB \cap A \neq \square$. Then there exist $a \in A$ and $b \in B$ such that $ab \in A$. Also there exists a natural number m such that $a^m = e$. Hence $eb = a^{2m}b = a^{2m-1}ab \in A$. But, by Lemma 1.4, e is the zero element of A. Hence $e = e(eb) = eb$. When $BA \cap A \neq \square$, we can prove in a similar way that there exists $b \in B$ such that $be = e$.

(2) $\Rightarrow$ (4). Clear.

(4) $\Rightarrow$ (5). Suppose $ebe = e$ for some $b \in B$. Let x be an arbitrary element of B. Then there exist natural numbers r, s and t such that $b^r \leqq x^s \leqq b^t$. Then, by Lemma 2.5,

$$e = (ebe)^r = eb^re \leqq ex^se \leqq eb^te = (ebe)^t = e$$

and so, again by Lemma 2.5, $e = ex^se = e(exe)^s$. Hence, by Lemma 2.6, we have $e = e(exe) = exe$.

(5) $\Rightarrow$ (3). Suppose $ebe = e$ for all $b \in B$. First suppose $e\mathscr{D}_E$ is of L-type. Then both eb and e are idempotents. Also $e(eb) = eb$ and $(eb)e = e$. Hence eb and e are $\mathscr{R}$-equivalent in the semigroup E. But, since $e\mathscr{D}_E$ is of L-type, we have $e = eb$. In the case when $e\mathscr{D}_E$ is of R-type, we can prove in a similar way that $e = be$ for all $b \in B$.

(3) $\Rightarrow$ (1). Clear.

Theorem 2.8. *γ is a quasi-order on $\mathscr{C}$.*

Proof. The reflexivity of γ is clear. In order to prove the transitivity, we suppose that $A, B, C \in \mathscr{C}$, $A \gamma B$ and $B \gamma C$.

First suppose that A is torsion-free. Since $B \gamma C$, either $BC \cap B \neq \square$ or $CB \cap B \neq \square$. First suppose $BC \cap B \neq \square$. Then there exist $b \in B$ and $c \in C$ such that $bc \in B$. Since $A \gamma B$, it follows from Theorem 2.4 that there exist natural numbers m and n such that a^mbc, $a^nb \in A$. Then $a^{m+n}b = a^m(a^nb) \in A$ and $(a^{m+n}b)c = a^n(a^mbc) \in A$. Hence $A \gamma C$. In the case when $CB \cap B \neq \square$, we can prove in a similar way that $A \gamma C$.

Next suppose A is periodic with idempotent e and B is torsion-free. First suppose that the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type. Since $B \gamma C$, it follows from Theorem 2.4 that

$BC \cap B \neq \square$. Hence there exist $b \in B$ and $c \in C$ such that $bc \in B$. Hence, by Theorem 2.7, we have $ebc = e = eb$. Hence $ec = ebc = e$. Hence, again by Theorem 2.7, we have $A \gamma C$. In the case when $e\mathscr{D}_E$ is of R-type, we can prove in a similar way that $A \gamma C$.

Finally suppose that A is periodic with idempotent e and B is periodic with idempotent f. Let $c \in C$. Then, by Theorem 2.7, we have $fcf = f$. Hence $fc \in E$ and, since $(fc) f(fc) = fc$ and $f(fc) f = f$, we have $fc \mathscr{D}_E f$. Since $A \gamma B$, it follows from Theorem 2.7, we have either $ef = e$ or $fe = e$. First we suppose $ef = e$. Then $ec = efc = e(fc) \in E$. By Lemma 1.5, $E/\mathscr{D}_E$ forms a semilattice and in this semilattice,

$$\begin{aligned}(ec) \mathscr{D}_E &= (e(fc)) \mathscr{D}_E = (e\mathscr{D}_E) \wedge ((fc) \mathscr{D}_E) \\ &= (e\mathscr{D}_E) \wedge (f\mathscr{D}_E) = (ef) \mathscr{D}_E = e\mathscr{D}_E .\end{aligned}$$

Since $e\mathscr{D}_E$ forms a rectangular band in E, we have $e = e(ec) e = ece$. Hence $A \gamma C$. In the case when $fe = e$, we can prove in a similar way that $A \gamma C$.

3. THE RELATION δ ON $\mathscr{C}$

We define the *binary relation* δ on $\mathscr{C}$ by:

$$\text{for} \quad A, B \in \mathscr{C}, \quad A \,\delta\, B \quad \text{if and only if} \quad A \gamma B \quad \text{and} \quad B \gamma A .$$

Since γ is a quasi-order on $\mathscr{C}$, the following theorem is a consequence of a well-known result about quasi-ordered set (cf. [1] p. 21).

Theorem 3.1. (1) *δ is an equivalence relation on $\mathscr{C}$;*

(2) *the quotient set $\mathscr{C}/\delta$ is a partially ordered set if, for $\mathscr{D}_1, \mathscr{D}_2 \in \mathscr{C}/\delta$, $\mathscr{D}_1 \leqq \mathscr{D}_2$ is defined to mean $A \gamma B$ for some $A \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$.*

(3) *if $\mathscr{D}_1 \leqq \mathscr{D}_2$ for $\mathscr{D}_1, \mathscr{D}_2 \in \mathscr{C}/\delta$, then $A \gamma B$ for all $A \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$.*

Theorem 3.2. *Suppose $A, B \in \mathscr{C}$ and $A \,\delta\, B$. Then*

(1) *A is torsion-free if and only if B is torsion-free;*

(2) *A is periodic if and only if B is periodic.*

Proof. (1) Suppose A is torsion-free. By way of contradiction, we assume B is periodic with idempotent f. Let $a \in A$. Then, since $B \gamma A$, it follows from Theorem 2.7 that $af = f$ or $fa = f$. First suppose $af = f$. Since $A \gamma B$, it follows from Theorem 2.4 that $a^n f \in A$ for some natural number n. Hence

$$f = af = \dots = a^n f \in A \cap B$$

and so $A = B$, which is a contradiction. In the case when $fa = f$, we obtain a contradiction in a similar way. This proves that if A is torsion-free, then B is torsion-free.

By symmetry, if B is torsion-free, then A is torsion-free. (2) is an immediate consequence of (1).

Let $\mathscr{D}$ be a δ-class in $\mathscr{C}$. By Theorem 3.2, either all archimedean classes contained in $\mathscr{D}$ are periodic or all archimedean classes contained in $\mathscr{D}$ are torsion-free. $\mathscr{D}$ is called a periodic δ-class in the former case and is called a torsion-free δ-class in the latter case.

Theorem 3.3. *Let $A, B \in \mathscr{C}$ such that A is periodic with idempotent e and B is periodic with idempotent f. Then $A \delta B$ if and only if $e \mathscr{D}_E f$.*

Proof. It follows from Theorem 2.7 that $A \delta B$ is equivalent to $efe = e$ and $fef = f$, which is equivalent to $e \mathscr{D}_E f$.

Lemma 3.4. *Let $A, B \in \mathscr{C}$ such that both A and B are torsion-free, $A \delta B$ and $A < B$. Then A is negative torsion-free and B is positive torsion-free.*

Proof. Let $a \in A$ and $b \in B$. Then, by Theorem 2.4, there exist natural numbers m and n such that $a^m b \in A$ and $ab^n \in B$. Hence, by Lemma 2.3, $a^{mn}b^n \in A$ and $a^m b^{mn} \in B$. Since $A < B$, we have $a^{mn}b^n < ab^n$ and $a^m b < a^m b^{mn}$. Hence $a^{mn} < a$ and $b < b^{mn}$. Therefore $mn > 1$, a is negative and b is positive.

Theorem 3.5. *A torsion-free δ-class $\mathscr{D}$ contains at most two elements of $\mathscr{C}$. If $\mathscr{D}$ contains exactly two elements of $\mathscr{C}$, then the lesser element of $\mathscr{D}$ is a negative torsion-free archimedean class of S and the greater element of $\mathscr{D}$ is a positive torsion-free archimedean class of S.*

Proof. By way of contradiction, we assume that $\mathscr{D}$ contains three elements A, B and C such that $A < B < C$. Then, by Lemma 3.4, B is positive torsion-free and is negative torsion-free at the same time, which is absurd. The second assertion is only the restatement of Lemma 3.4.

4. THE OPERATION ON $\mathscr{C}/\delta$

The negation of the relation $A \gamma B$ and $A \delta B$ are denoted by A non γ B and A non δ B, respectively.

Lemma 4.1. *Let $A, B \in \mathscr{C}$ such that A non γ B, B non γA and $A \leqq B$. Then the product set AB in S consists of only one idempotent, say e, BA consists of only one idempotent, say f, such that $e \mathscr{D}_E f$. Let C and D be archimedean classes of S containing e and f, respectively. Then $A < C < B$, $A < D < B$, $C \delta D$, $C \gamma A$, $C \gamma B$, $D \gamma A$ and $D \gamma B$.*

Proof. Let $a \in A$ and $b \in B$. Then, since $A \leqq B$ and A non γ B, we have $a < a^2 b$ and $a < ba^2$. But, if $ab \leqq ba$, then $a^2 b \leqq aba$ and, if $ba \leqq ab$, then $ba^2 \leqq aba$.

Hence we have $a < aba$. Similarly it follows from $A \leqq B$ and B non γ A that $b^2a < b$, $ab^2 < b$ and $bab < b$. Hence

$$ab \leqq (aba)\, b = (ab)^2 = a(bab) \leqq ab\,, \quad ba \leqq b(aba) = (ba)^2 = (bab)\, a \leqq ba\,,$$

and so ab and ba are idempotents. We put $ab = e$ and $ba = f$. Also we have

$$ab \leqq (a^2b)\, b = a^2b^2 = a(ab^2) \leqq ab\,, \quad ba \leqq b(ba^2) = b^2a^2 = (b^2a)\, a \leqq ba$$

and so $ab = a^2b^2$ and $ba = b^2a^2$. Hence

$$efe = ab^2a^2b = abab = ab = e\,, \quad fef = ba^2b^2a = baba = ba = f\,.$$

Hence $e\, \mathscr{D}_E\, f$. Now we suppose that the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type. Replacing a by a^2 and b by b^2, we see that a^2b, ba^2, ab^2 and b^2a are idempotents. Also we have

$$(ab)(ab^2) = (ab)^2\, b = (ab)\, b = ab^2\,, \quad (ab^2)(a^2b) = a(b^2a^2)\, b = a(ba)\, b = ab$$

and so $e = ab\, \mathscr{R}\, ab^2$ in the semigroup E. Hence $ab = ab^2$ and, similarly we have $ba = ba^2$. Also

$$ab = a^2b^2 = a(ab^2) = a(ab) = a^2b\,, \quad ba = b^2a^2 = b(ba^2) = b(ba) = b^2a\,.$$

In the case when $e\mathscr{D}_E$ is of R-type, we can prove

$$ab = ab^2 = a^2b\,, \quad ba = ba^2 = b^2a$$

in a similar way. Thus we have

$$ab = a^ib^j\,, \quad ba = b^ia^j$$

for every natural numbers i and j. Now we take an arbitrary element $a'b' \in AB$ with $a' \in A$ and $b' \in B$. Then there exist natural numbers p, q, r, s, t and u such that $a^p \leqq a'^q \leqq a^r$ and $b^s \leqq b'^t \leqq b^u$. Hence

$$ab = a^pb^s \leqq a'^qb'^t \leqq a^rb^u = ab$$

and so $a'b' = a'^qb'^t = ab = e$. This proves that AB consists of only one idempotent e and similarly BA consists of only one idempotent f. By Theorem 3.3, we have $C\, \delta\, D$. Since $a^2 \leqq ab = e \leqq b^2$, we have $A \leqq C \leqq B$. But, since A non γ B and B non γA, we have $ab \notin A$ and $ab \notin B$. Hence $A < C < B$. Since $ab = a^2b \in C \cap AC$ and $ab = ab^2 \in C \cap CB$, we have $C\, \gamma\, A$ and $C\, \gamma\, B$. Similarly we can prove that $A <$ $< D < B$, $D\, \gamma\, A$ and $D\, \gamma\, B$.

Corollary 4.2. *Let $A, B \in \mathscr{C}$. Then there exists $X \in \mathscr{C}$ such that X lies between A and B, $X\, \gamma\, A$ and $X\, \gamma\, B$.*

Proof. Without loss of generality, we assume that $A \leqq B$. If $A \gamma B$, then A satisfies the condition for X and, if $B \gamma A$, then B satisfies the condition for X. If A non γB and B non γA, then either one of C and D given in Lemma 4.1 satisfies the condition for X.

Lemma 4.3. *Let $A, B, C, D \in \mathscr{C}$ such that $C \gamma A$, $C \gamma B$ and $A \leqq D \leqq B$. Then $C \gamma D$.*

Proof. Let $d \in D$. Then there exist $a \in A$ and $b \in B$ such that $a \leqq d \leqq b$. First we suppose that C is torsion-free. Then, by Theorem 2.4, there exist natural numbers p and q such that $ac^p \in C$ and $bc^q \in C$. We put $r = \max\{p, q\}$. Then $ac^r \leqq dc^r \leqq bc^r$ with $ac^r, bc^r \in C$. Hence $dc^r \in C$ and so we have $C \gamma D$. Next we suppose that C is periodic with idempotent g. Then, by Theorem 2.7, we have

$$g = gag \leqq gdg \leqq gbg = g$$

and so $gdg = g$. Hence we have $C \gamma D$.

We define the operation $\wedge$ on $\mathscr{C}/\delta$ by:

$$\text{for} \quad \mathscr{D}_1, \mathscr{D}_2, \mathscr{D} \in \mathscr{C}/\delta\,, \quad \mathscr{D}_1 \wedge \mathscr{D}_2 = \mathscr{D} \quad \text{if and only if there exist}$$
$$A \in \mathscr{D}_1\,, \quad B \in \mathscr{D}_2 \quad \text{and} \quad C \in \mathscr{D} \quad \text{such that } C \text{ lies between } A \text{ and } B\,,$$
$$C \gamma A \quad \text{and} \quad C \gamma B\,.$$

Theorem 4.4. *$\wedge$ is a binary operation on $\mathscr{C}/\delta$.*

Proof. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be arbitrary elements of $\mathscr{C}/\delta$. We take $A \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$ arbitrarily. Then, by Corollary 4.2, there exists $C \in \mathscr{C}$ such that C lies between A and B, $C \gamma A$ and $C \gamma B$. Then the δ-class $C\delta$ is $\mathscr{D}_1 \wedge \mathscr{D}_2$, by definition. Next we show that $\mathscr{D}_1 \wedge \mathscr{D}_2$ is determined uniquely irrespective of the choice of A, B and C in the definition. In fact, suppose that A, $A_1 \in \mathscr{D}_1$, B, $B_1 \in \mathscr{D}_2$, $C \gamma A$, $C \gamma B$, $C_1 \gamma A_1$, $C_1 \gamma B_1$, C lies between A and B and C_1 lies between A_1 and B_1. Then $C \gamma A \delta A_1$ and $C \gamma B \delta B_1$ and so, by Lemma 4.3, we have $C \gamma C_1$. Similarly we have $C_1 \gamma C$. Hence $C\delta = C_1\delta$.

In the proof of Theorem 4.4, we have shown

Corollary 4.5. *For every $A \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$, there exists $C \in \mathscr{D}_1 \wedge \mathscr{D}_2$ such that $C \gamma A$, $C \gamma B$ and C lies between A and B.*

Theorem 4.6. *$\mathscr{C}/\delta$ is a commutative idempotent semigroup with respect to the operation $\wedge$.*

Proof. First we show the associativity of the operation $\wedge$. Let $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \in \mathscr{C}/\delta$. We take $A \in \mathscr{D}_1$, $B \in \mathscr{D}_2$ and $C \in \mathscr{D}_3$. Then, by Corollary 4.5, there exists $D \in \mathscr{D}_1 \wedge \mathscr{D}_2$ such that D lies between A and B, $D \gamma A$ and $D \gamma B$. Further there exists

$F \in (\mathscr{D}_1 \wedge \mathscr{D}_2) \wedge \mathscr{D}_3$ such that F lies between D and C, $F \gamma D$ and $F \gamma C$. Also there exist $G \in \mathscr{D}_2 \wedge \mathscr{D}_3$ and $H \in \mathscr{D}_1 \wedge (\mathscr{D}_2 \wedge \mathscr{D}_3)$ such that G lies between B and C, H lies between A and G, $G \gamma B$, $G \gamma C$, $H \gamma A$ and $H \gamma G$. Since $H \gamma A$ and $H \gamma G \gamma B$, it follows from Lemma 4.3 that $H \gamma D$. Moreover, since $H \gamma G \gamma C$, it follows again from Lemma 4.3 that $H \gamma F$. Similarly we can prove that $F \gamma H$. Hence

$$(\mathscr{D}_1 \wedge \mathscr{D}_2) \wedge \mathscr{D}_3 = F\delta = H\delta = \mathscr{D}_1 \wedge (\mathscr{D}_2 \wedge \mathscr{D}_3).$$

The commutativity and the idempotency of the operation $\wedge$ follows immediately from the definition.

By Theorem 4.6, with respect to the relation $\lesssim$ on $\mathscr{C}/\delta$ defined by

$$\mathscr{D}_1 \lesssim \mathscr{D}_2 \quad \text{if and only if} \quad \mathscr{D}_1 \wedge \mathscr{D}_2 = \mathscr{D}_1,$$

$(\mathscr{C}/\delta, \lesssim)$ is a meet semilattice and $\mathscr{D}_1 \wedge \mathscr{D}_2$ is the greatest lower bound of $\mathscr{D}_1$ and $\mathscr{D}_2$ (cf. [1] p. 10).

Lemma 4.7. *For $\mathscr{D}_1, \mathscr{D}_2 \in \mathscr{C}/\delta$, the following conditions are equivalent:*

(1) $\mathscr{D}_1 \lesssim \mathscr{D}_2$;

(2) *there exist $A \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$ such that $A \gamma B$;*

(3) *$A \gamma B$ for every $A \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$.*

Proof. First suppose that the condition (1) holds. Then $\mathscr{D}_1 \wedge \mathscr{D}_2 = \mathscr{D}_1$ and so there exist $A, C \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$ such that C lies between A and B, $C \gamma A$ and $C \gamma B$. Hence $A \delta C \gamma B$ and so $A \gamma B$. Next suppose that the condition (2) holds. Let $A_1 \in \mathscr{D}_1$ and $B_1 \in \mathscr{D}_2$. Then, since $A_1 \delta A \gamma B \delta B_1$, we have $A_1 \gamma B_1$. It is clear that (3) implies (1).

Theorem 4.8. *The semilattice $(\mathscr{C}/\delta, \lesssim)$ is a tree semilattice.*

Proof. Suppose $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D} \in \mathscr{C}/\delta$ such that $\mathscr{D}_1 \lesssim \mathscr{D}$ and $\mathscr{D}_2 \lesssim \mathscr{D}$. Let $A \in \mathscr{D}_1$, $B \in \mathscr{D}_2$ and $C \in \mathscr{D}$. Then, by Lemma 4.7, we have $A \gamma C$ and $B \gamma C$. Without loss of generality, we assume $A \leqq B$. By Corollary 4.5, there exists $D \in \mathscr{D}_1 \wedge \mathscr{D}_2$ such that $A \leqq D \leqq B$, $D \gamma A$ and $D \gamma B$. First suppose $C \leqq D$. Then, since $C \leqq D \leqq B$ with $B \gamma C$ and $B \gamma B$, it follows from Lemma 4.3 that $B \gamma D$. Hence $B \delta D$ and so

$$\mathscr{D}_1 \wedge \mathscr{D}_2 = D\delta = B\delta = \mathscr{D}_2.$$

Therefore $\mathscr{D}_2 \lesssim \mathscr{D}_1$. In the case when $D \leqq C$, we can prove $\mathscr{D}_1 \lesssim \mathscr{D}_2$ in a similar way.

5. THE OPERATION $*$ ON $\mathscr{C}$

Lemma 5.1. *Let $A, C \in \mathscr{C}$ such that C is positive [negative] torsion-free, $C \gamma A$ and $C \leqq A[A \leqq C]$. Then $C = A$.*

Proof. Let $a \in A$ and $c \in C$. Then, by Theorem 2.4, there exists a natural number n such that $ac^n \in C$. Since C is positive torsion-free, there exists a natural number m such that $ac^n < c^m$. Hence $ac^n < c^m < c^{m+n}$ and so $a < c^m$. Hence $A \leqq C$ and so $A = C$.

Lemma 5.2. *Let $A, C \in \mathscr{C}$ such that C is periodic with idempotent g, $C \gamma A$, A non γ C and $A \leqq C[C \leqq A]$. Then, for $a \in A$, $h = aga$ is an idempotent which is determined irrespective of the choice of $a \in A$. Also if $g\mathscr{D}_E$ is of L-type, then $h = ag$ and if $g\ \mathscr{D}_E$ is of R-type, then $h = ga$. Moreover the archimedean class D of S which contains the idempotent h is the least [greatest] element in the set*

$$\{X \in \mathscr{C};\ A \leqq X \text{ and } X\,\delta\,C\}\ [\{X \in \mathscr{C};\ X \leqq A \text{ and } X\,\delta\,C\}]\,.$$

Proof. We consider only the case when $g\mathscr{D}_E$ is of L-type. Let $a \in A$. Then, by Theorem 2.7, $ga = g$ and so $aga = ag$. Also $agag = ag$ and so $h = aga = ag$ is an idempotent. Moreover $hg = agg = ag = h$ and $gh = gag = g$. Hence $h\,\mathscr{D}_E\,g$ and so, by Theorem 3.3, the archimedean class D containing the element h is δ-equivalent to C. Moreover, since A non γ C and $A \leqq C$, we have $a < g$ and so $a^2 \leqq ag = h$. Hence $A \leqq D$. Since $a^2 \in A$, it follows from a similar argument that $h_1 = a^2g$ is an idempotent and the archimedean class D_1 containing the idempotent h_1 satisfies the condition that $D_1\,\delta\,C$ and $A \leqq D_1$. Since A non γ C, we have A non γ D and A non γ D_1. Hence $a^2 < h = ag$ and $a < h_1 = a^2g$. Therefore $h_1 = a^2g \leqq (ag)\,g = h = ag \leqq (a^2g)\,g = h_1$. Hence $ag = h = h_1 = a^2g$ and so $h = ag = a^ng$ for every natural number n. Let a_1 be an arbitrary element of A. Then similarly we have $a_1g = a_1ga_1$ and $a_1g = a_1^ng$ for every natural number n. Since $a, a_1 \in A$, there exist natural numbers p, q, r and s such that $a^p \leqq a_1^q$ and $a_1^r \leqq a^s$. Hence

$$ag = a^pg \leqq a_1^qg = a_1g = a_1^rg \leqq a^sg = ag\,.$$

Hence $aga = ag = a_1g = a_1ga_1$. Thus aga is determined irrespective of the choice of $a \in A$.

Now let $X \in \mathscr{C}$ such that $A \leqq X$ and $X\,\delta\,C$. Then X is a periodic archimedean class with idempotent, say k. Since A non γ C, we have A non γ X and so $A < X$. Hence $a < k$. Also, since $X\,\delta\,C$, we have $h\,\mathscr{D}_E\,k$. Hence $h = ag \leqq kg = k$ and so $D \leqq X$. Therefore D is the least element in the set $\{X \in \mathscr{C};\ A \leqq X \text{ and } X\,\delta\,C\}$.

Let $\mathscr{D}$ be a δ-class in $\mathscr{C}$. If $\mathscr{D}$ is torsion-free, then we assign for $\mathscr{D}$ in an arbitrary way either one of L-type and R-type. If $\mathscr{D}$ is periodic, then, by Theorem 3.3, the idempotents of all archimedean classes of S, which are elements in $\mathscr{D}$ lie in a $\mathscr{D}_E$-class. If the $\mathscr{D}_E$-class is of L-type, then we say that the δ-class $\mathscr{D}$ is of L-type and, if the $\mathscr{D}_E$-class is of R-type, then we say that $\mathscr{D}$ is of R-type. Now we define the operation $*$ on $\mathscr{C}$ in the following way.

Let $A, B \in \mathscr{C}$ such that $A \leqq B$. If the δ-class $A\delta \wedge B\delta$ is of L-type, then we define

$$A * B = \min\{X \in \mathscr{C};\ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta\}\,;$$
$$B * A = \max\{X \in \mathscr{C};\ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta\}\,.$$

If the δ-class $A\delta \wedge B\delta$ is of R-type, then we define

$$A * B = \max\{X \in \mathscr{C};\ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta\}\,;$$
$$B * A = \min\{X \in \mathscr{C};\ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta\}\,.$$

Theorem 5.3. *The operation $*$ is a binary operation on $\mathscr{C}$.*

Proof. Let $A, B \in \mathscr{C}$ such that $A \leqq B$. First we suppose that the δ-class $A\delta \wedge B\delta$ is of L-type. If $A \gamma B$, then $A\delta = A\delta \wedge B\delta$ and evidently

$$A = \min\{X \in \mathscr{C};\ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta\} = A * B.$$

Suppose A non γ B. By Corollary 4.5, the set

$$\{X \in \mathscr{C};\ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta\}$$

is nonvoid. First we suppose that $A\delta \wedge B\delta$ is a torsion-free δ-class. Let $X \in \mathscr{C}$ such that $A \leqq X \leqq B$ and $X \in A\delta \wedge B\delta$. Then X is a torsion-free archimedean class. Since $A\delta \wedge B\delta \preceq A\delta$ and $A\delta \wedge B\delta \preceq B\delta$, it follows from Lemma 4.7 that $X \gamma A$ and $X \gamma B$. If X were negative torsion-free, then, by Lemma 5.1, $A = X \gamma B$, which is a contradiction. Hence X is positive torsion-free. Therefore, again by Lemma 5.1, we have $X = B$. Thus the set $\{X \in \mathscr{C};\ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta\}$ consists of only one element B and so $B = A * B$. Next we suppose that $A\delta \wedge B\delta$ is a periodic δ-class. We take $C \in A\delta \wedge B\delta$ such that $A \leqq C \leqq B$. Then C is a periodic archimedean class. Then, by Lemma 5.2, there exists a δ-class D which is the least element of the set $\{X \in \mathscr{C};\ A \leqq X \text{ and } X \delta C\}$. It is clear that $D = A * B$. In a similar way, we can prove that $B * A$ is defined. The case when $A\delta \wedge B\delta$ is of R-type can be treated similarly.

From the proof of the preceding Theorem 5.3, we have

Corollary 5.4. *Let $A, B \in \mathscr{C}$ such that $A \leqq B, A \gamma B$ and $A\delta$ is of L-type [R-type]. Then $A * B = A[B * A = A]$.*

Corollary 5.5. *Let $A, B \in \mathscr{C}$ such that $A \leqq B$, A non γ B and $A\delta \wedge B\delta$ is a torsion-free δ-class of L-type [R-type]. Then $A * B = B[B * A = B]$.*

Lemma 5.6. *Let $A, B, C \in \mathscr{C}$ such that $A \leqq B \leqq C$. Then $A\delta \wedge C\delta \preceq B\delta$ in the semilattice $\mathscr{C}/\delta$.*

Proof. We have $A \leqq A * C \leqq C$ and $A * C \in A\delta \wedge C\delta$. Since $A\delta \wedge C\delta \preceq A\delta$ and $A\delta \wedge C\delta \preceq C\delta$, it follows from Lemma 4.7 that $A * C \gamma A$ and $A * C \gamma C$. Hence, by Lemma 4.3, we have $A * C \gamma B$ and, again by Lemma 4.7, we have $A\delta \wedge C\delta \preceq B\delta$.

Lemma 5.7. *Let $A, B, C \in \mathscr{C}$ such that $A \leqq B$. Then $A * C \leqq B * C$ and $C * A \leqq C * B$.*

Proof. If $A \leqq C \leqq B$, then $A \leqq A * C \leqq C \leqq B * C \leqq B$ and $A \leqq C * A \leqq C \leqq C * B \leqq B$. Next suppose $C \leqq A \leqq B$. By way of contradiction we assume

$B * C < A * C$. Then we have $C \leqq B * C < A * C \leqq A \leqq B$ and so, by Lemma 5.6,

$$A\delta \wedge C\delta \preceqq (B * C)\,\delta = B\delta \wedge C\delta \preceqq (A * C)\,\delta = A\delta \wedge C\delta \,.$$

Hence $A\delta \wedge C\delta = B\delta \wedge C\delta$. First we assume $B\delta \wedge C\delta$ is of L-type. Then $C < < A * C \leqq B$ and $A * C \in A\delta \wedge C\delta = B\delta \wedge C\delta$ with $B * C < A * C$, which contradicts the definition of $B * C$. Next we suppose $B\delta \wedge C\delta$ is of R-type. Then $C \leqq \leqq B * C < A$ and $B * C \in B\delta \wedge C\delta = A\delta \wedge C\delta$ with $B * C < A * C$, which contradicts the definition of $A * C$. Thus we have $A * C \leqq B * C$. In a similar way we can prove that $C * A \leqq C * B$. The case when $A \leqq B \leqq C$ can be treated in a similar way.

Lemma 5.8. *The operation $*$ is idempotent: for every $A \in \mathscr{C}$, $A * A = A$.*

Proof. Clear from the definition of the operation $*$.

Lemma 5.9. *For $A, B \in \mathscr{C}$, $A * (A * B) = (A * B) * B = A * B$.*

Proof. In the proof we only consider the case when $A\delta \wedge B\delta$ is of L-type. We have $(A * B)\,\delta = A\delta \wedge B\delta \preceqq B\delta$ and so, by Lemma 4.7, we have $A * B \,\gamma\, B$. Hence, by Corollary 5.4, $(A * B) * B = A * B$. Suppose $A \leqq B$. Then $A \leqq A * (A * B) \leqq \leqq A * B \leqq B$. On the other hand, we have $(A * (A * B))\,\delta = A\delta \wedge B\delta$ and so $A * B \leqq A * (A * B)$ by the definition of $A * B$. Hence $A * (A * B) = A * B$. In the case when $B \leqq A$, we can prove $A * (A * B) = A * B$ in a similar way.

Lemma 5.10. *$\mathscr{C}$ is a semigroup with respect to the operation $*$.*

Proof. Let $A, B, C \in \mathscr{C}$. We show that $(A * B) * C = A * (B * C)$ by dividing into the following cases. In the proof we only consider the case when $A\delta \wedge B\delta \wedge C\delta$ is of L-type.

(a) The case when $A \leqq B \leqq C$:

By Lemmas 5.7, 5.8 and 5.9, we have

$$A * C = (A * A) * C \leqq (A * B) * C \leqq (A * C) * C = A * C\,;$$

$$A * C = A * (A * C) \leqq A * (B * C) \leqq A * (C * C) = A * C\,.$$

Hence $(A * B) * C = A * C = A * (B * C)$.

(b) The case when $A \leqq C \leqq B$:

By Lemma 5.6, we have $(A * B)\,\delta = A\delta \wedge B\delta \preceqq C\delta$ and so, by Lemma 4.7, $A * B \,\gamma\, C$ and also $A\delta \wedge B\delta = A\delta \wedge B\delta \wedge C\delta$ is of L-type. Hence, by Corollary 5.4, $(A * B) * C = A * B$. Also we have $A \leqq A * B \leqq B$ and so

$$A * B = A * (A * B) = A * ((A * B) * C) \leqq A * (B * C)\,.$$

On the other hand

$$A * (B * C) \leqq A * (B * B) = A * B .$$

Hence $(A * B) * C = A * B = A * (B * C)$.

(c) The case when $C \leqq A \leqq B * C \leqq B$:

We have $(A * B)\,\delta = A\delta \wedge B\delta \precsim (B * C)\,\delta = B\delta \wedge C\delta \precsim C\delta$ and so $A * B\,\gamma\,C$ and $A\delta \wedge B\delta = A\delta \wedge B\delta \wedge C\delta$ is of L-type. Hence $(A * B) * C = A * B$. By Lemma 4.7, we have $B * C\,\gamma\,B$. Also, by Lemma 5.6, we have $B\delta \wedge C\delta \precsim A\delta$ and so $B\delta \wedge C\delta = A\delta \wedge B\delta \wedge C\delta$ is of L-type. Hence $(B * C) * B = B * C$ and so

$$A * B = A * (A * B) \leqq A * ((B * C) * B) = A * (B * C) .$$

On the other hand

$$A * (B * C) \leqq A * (B * B) = A * B .$$

Hence $(A * B) * C = A * B = A * (B * C)$.

(d) The case when $C \leqq B * C \leqq A \leqq B$:

By Lemma 5.9, we have $B * C = (B * C) * C \leqq A * C \leqq B * C$ and so $B * C = A * C$. Hence $A * (B * C) = A * (A * C) = A * C$. Also, since $A \leqq A * B \leqq B$, we have $A * C \leqq (A * B) * C \leqq B * C = A * C$. Hence $(A * B) * C = A * C = A * (B * C)$.

Thus we have proved that $(A * B) * C = A * (B * C)$ in the case when $A \leqq B$. We can prove the same associative condition in a similar way in the case when $B \leqq A$.

Theorem 5.11. *The system $(\mathscr{C}, *, \leqq)$ is an ordered idempotent semigroup and the relation δ on $\mathscr{C}$ is equal to the $\mathscr{D}$-equivalence on the idempotent semigroup $\mathscr{C}$.*

Proof. It follows from Lemmas 5.7, 5.8 and 5.10 that $(\mathscr{C}, *, \leqq)$ is an ordered idempotent semigroup. Let $A, B \in \mathscr{C}$ such that $A\,\mathscr{D}\,B$ in the semigroup $\mathscr{C}$. Then, by Lemma 1.5, A and B belong to the same rectangular band in the semigroup $\mathscr{C}$ and so $A = A * B * A$ and $B = B * A * B$. Hence

$$A\delta = (A * B * A)\,\delta = A\delta \wedge B\delta = (B * A * B)\,\delta = B\delta$$

and so $A\,\delta\,B$. Conversely suppose $A, B \in \mathscr{C}$ such that $A\,\delta\,B$. First suppose $A\delta$ is of L-type. Then, by Corollary 5.4, we have $A * B * A = A * (B * A) = A$ and $B * A * B = B * (A * B) = B$ and so $A\,\mathscr{D}\,B$ in the semigroup $\mathscr{C}$. In the case when $A\delta$ is of R-type, we obtain the same conclusion in a similar way.

6. THE CONNECTION OF SET PRODUCTS OF ARCHIMEDEAN CLASSES WITH THE OPERATION $*$

Theorem 6.1. *Let $A, B \in \mathscr{C}$ such that $A\delta \wedge B\delta$ is torsion-free and A non δ B. Then either $A \gamma B$ or $B \gamma A$. If $A \gamma B$, then*

$$AB, BA \subseteq A = A * B = B * A .$$

If $B \gamma A$, then

$$AB, BA \subseteq B = A * B = B * A .$$

Proof. We consider only the case when $A \leqq B$. By Corollary 4.5, there exists $C \in A\delta \wedge B\delta$ such that $C \gamma A$, $C \gamma B$ and $A \leqq C \leqq B$. Since $A\delta \wedge B\delta$ is torsion-free, C is positive torsion-free or negative torsion-free. By Lemma 5.1, if C is positive torsion-free, then $B = C \gamma A$ and, if C is negative torsion-free, then $A = C \gamma B$. Now suppose $A \gamma B$. Then, by Lemma 4.7, we have $A \in A\delta = A\delta \wedge B\delta$. Since $A \gamma B$ and A non δ B, we have B non γ A. In particular $A \neq B$ and so, by Lemma 5.1, A is negative torsion-free. By Corollaries 5.4 and 5.5, we have $A * B = B * A = A$. Let $a \in A$ and $b \in B$. Then, since B non γ A, we have $ab^4 < b^2$ and $b^4a < b^2$. If $ab \leqq ba$, then $b^2ab^2 \leqq b^4a < b^2$ and, if $ba \leqq ab$, then $b^2ab^2 \leqq ab^4 < b^2$. Hence always we have $(ab^2)^2 \leqq ab^2$ and so ab^2 is non-positive. Let D be the archimedean class containing the element ab^2 and let D_- be the set of all nonpositive elements of D. Then, since $a^3 \leqq ab^2$, we have $A \leqq D$. Since A is negative torsion-free, it follows from the dual of Lemma 1.8 that AD_- is contained in a single archimedean class. On the other hand, since $A \gamma B$ and $b^2 \in B$, there exists a natural number n such that $a^nb^2 \in A$. Hence $a^{n+1}b^2 = a(a^nb^2) = a^n(ab^2) \in A \cap AD_-$ and so $AD_- \subseteq A$. Hence $a^2b^2 = a(ab^2) \in AD_- \subseteq A$ and, by Lemma 2.3, we have $AB \subseteq A$. Similarly we have $BA \subseteq A$. In the case when $B \gamma A$, we can prove $AB, BA \subseteq B = A * B = B * A$ in a similar way.

Corollary 6.2. *Let $A, B \in \mathscr{C}$ such that $A\delta \wedge B\delta$ is torsion-free. Then the following conditions are equivalent to each other:*

(1) *$A \neq B$ and $A \delta B$;*
(2) *AB is not contained in a single archimedean class;*
(3) *BA is not contained in a single archimedean class.*

Proof. (2) $\Rightarrow$ (1). Clear by Theorem 6.1. (1) $\Rightarrow$ (2). Suppose $A \neq B$ and $A \delta B$. Then $A \gamma B$ and $B \gamma A$. Also $A\delta = B\delta = A\delta \wedge B\delta$ and so both A and B are torsion-free. We take $a \in A$ and $b \in B$. Then, by Theorem 2.4, there exist natural numbers m and n such that $a^mb \in A$ and $ab^n \in B$. Hence $AB \cap A \neq \square$ and $AB \cap B \neq \square$. In a similar way we can prove that (1) $\Leftrightarrow$ (3).

Lemma 6.3. *Let $A, B \in \mathscr{C}$ such that $A * B[B * A]$ is a periodic archimedean class with idempotent e and the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type [R-type]. Then $ae = e$ [$ea = e$] for every $a \in A$.*

Proof. In the proof we only consider the case when $A \leqq B$. By Lemma 4.7, we have $A * B \gamma A$ and $A * B \gamma B$. First suppose that $A \gamma A * B$. Then we have $A \gamma B$ and $A\delta = (A * B)\delta$ is a δ-class of L-type. Hence, by Corollary 5.4, we have $A = A * B$. But, by Lemma 1.4, e is the zero element of A and so $ae = e$. Next suppose A non γ $A * B$. Then, by Lemma 5.2, ae is the idempotent of $A * B$ and so $ae = e$.

Corollary 6.4. *Suppose that $A, B \in \mathscr{C}$ such that $A \leqq B$ and $A\delta \wedge B\delta$ is a periodic δ-class of L-type [R-type]. Let e and f be the idempotents in $A * B$ [$B * A$] and $B * A$ [$A * B$], respectively.*

(1) *If $A\delta \wedge B\delta \neq B\delta$, then $ab = e$ [$ba = e$] for every $a \in A$ such that $a \leqq e$ and for every $b \in B$;*

(2) *If $A\delta \wedge B\delta \neq A\delta$, then $ba = f$ [$ab = f$] for every $a \in A$ and for every $b \in B$ such that $f \leqq b$.*

Proof. (1) Let $a \in A$ such that $a \leqq e$ and let $b \in B$. By Lemma 6.3, we have $ae = e$ and, by Theorem 2.7, we have $eb = e$. Since $A\delta \wedge B\delta \neq B\delta$, we have $A * B < B$ and so $a \leqq e < b$. Hence

$$e = ae \leqq ab \leqq eb = e$$

and so $ab = e$. The assertion (2) can be proved in a similar way.

Theorem 6.5. *Suppose $A, B \in \mathscr{C}$ such that $A\delta \wedge B\delta$ is periodic and $A\delta \neq A\delta \wedge B\delta \neq B\delta$. Then AB consists of only one element which is the idempotent in $A * B$ and BA consists of only one element which is the idempotent in $B * A$.*

Proof. In the proof we only consider the case when $A \leqq B$ and $A\delta \wedge B\delta$ is of L-type. Since $A\delta \neq A\delta \wedge B\delta \neq B\delta$, we have $A < A * B$ and $B * A < B$. We denote by e and f the idempotents of $A * B$ and $B * A$, respectively. Let $a \in A$ and $b \in B$. Then, since $a < e$ and $f < b$, it follows from Corollary 6.4 that $ab = e$ and $ba = f$.

Let A be an archimedean class of S. We denote by A_+ and A_- the set of all nonnegative elements of A and the set of all nonpositive elements of A, respectively.

Theorem 6.6. *Suppose that $A, B \in \mathscr{C}$ such that $A \leqq B$ and $A\delta \wedge B\delta$ is a periodic δ-class of L-type [R-type].*

(1) *If $A\delta = A\delta \wedge B\delta \neq B\delta$ and AB [BA] is contained in a single archimedean class, then $AB \subseteq A_-$ [$BA \subseteq A_-$];*

(2) *If $A\delta \neq A\delta \wedge B\delta = B\delta$ and BA [AB] is contained in a single archimedean class, then $BA \subseteq B_+$ [$AB \subseteq B_+$].*

Proof. (1) Let $a \in A$ and $b \in B$. Denote by e the idempotent of the periodic archimedean class A. If $a \in A_+$, then $a \leqq e$ and so, by Corollary 6.4, we have $ab = e$. In particular, $eb = e \in A$. Since AB is contained in a single archimedean class, we have $AB \subseteq A$. Moreover if $a \in A_-$, then $e \leqq a$ and so $e = eb \leqq ab$. Hence $ab \in A_-$. Thus $AB \subseteq A_-$. (2) can be proved in a similar way.

Lemma 6.7. *Suppose that $A, B \in \mathscr{C}$ such that $A \leqq B$ and $A\delta \wedge B\delta$ is a periodic δ-class of L-type [R-type].*

(1) *Suppose $A\delta = A\delta \wedge B\delta$. Let e and g be the idempotents of A and $B * A$ $[A * B]$, respectively. Then the following conditions are equivalent:*

(a) *$AB\,[BA]$ is not contained in a single archimedean class;*

(b) *there exists an idempotent f of S such that $e < f < g$, $e\,\mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$ and also there exists $a \in A_- \setminus \{e\}$ such that $ag = f\,[ga = f]$.*

(2) *Suppose $A\delta \wedge B\delta = B\delta$. Let e and g be the idempotents of B and $A * B$ $[B * A]$, respectively. Then the following conditions are equivalent:*

(a) *$BA\,[AB]$ is not contained in a single archimedean class;*

(b) *there exists an idempotent f of S such that $g < f < e$, $e\,\mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$ and also there exists $b \in B_+ \setminus \{e\}$ such that $bg = f\,[gb = f]$.*

Proof. (1) First suppose that the condition (a) holds. Since $A\delta = A\delta \wedge B\delta \leqq B\delta$, we have $A\,\gamma\,B$. Hence, by Lemma 2.7,

$$eb = e \quad \text{for every} \quad b \in B. \tag{6.7.1}$$

By the condition (a), we have $A \neq B$ and so

$$A < B. \tag{6.7.2}$$

Let $a \in A_+$ and $b \in B$. Then, by (6.7.2), we have $a \leqq e < b$ and, by (6.7.1), $a^2 \leqq \leqq ab \leqq eb = e$. Hence

$$ab \in A \quad \text{for every} \quad a \in A_+ \quad \text{and} \quad b \in B. \tag{6.7.3}$$

By the condition (a) and (6.7.1),

$$\text{there exist} \quad x \in A \quad \text{and} \quad y \in B \quad \text{such that} \quad xy \notin A. \tag{6.7.4}$$

By (6.7.3) and (6.7.4),

$$x \in A_- \setminus \{e\}. \tag{6.7.5}$$

Put

$$z = \max\{g, y\}. \tag{6.7.6}$$

Let C be the archimedean class containing the element xy. Then $x^2 \leqq xy$ and so, by (6.7.4), we have $A < C$. Since $xy \leqq xz$, we have

$$xz \notin A. \tag{6.7.7}$$

We have $B * A \leqq B$. If $B * A = B$, then $g, y \in B$ and so $z \in B$. If $B * A < B$, then $g < y$ and so $z = y \in B$. Hence

$$z \in B. \tag{6.7.8}$$

Let i be a natural number and let D_i be the archimedean class containing the element $z^i e$. Then $e = x^i e \leqq z^i e \leqq z^{i+1}$ and so $A \leqq D_i \leqq B$. Also, by (6.7.1) and (6.7.8)

$$(z^i e)^2 = z^i e z^i e = z^i e, \quad (z^i e) e = z^i e, \quad e(z^i e) = (e z^i) e = e.$$

Hence D_i is a periodic archimedean class and, by Theorem 3.3, $D_i \in A\delta = A\delta \wedge B\delta$. Since $(B * A)\delta = A\delta \wedge B\delta = A\delta$ and since $A\delta \wedge B\delta$ is of L-type, $e \mathscr{L} g$ in the semigroup E. Hence $g = ge = g^i e \leqq z^i e$ and so $B * A \leqq D_i$. Therefore $D_i = B * A$ by the definition of $B * A$ and so

$$(6.7.9) \qquad z^i e = g \quad \text{for every natural number} \quad i.$$

By way of contradiction, we assume that $zx \leqq xz$. Since x is an element of a periodic archimedean class with idempotent e, there exists a natural number n such that $x^n = e$. Hence, by (6.7.1),

$$x^{2n} \leqq (xz)^n \leqq x^n z^n = e z^n = e$$

and so $xz \in A$, which contradicts (6.7.7). Hence

$$(6.7.10) \qquad xz < zx.$$

Since $A \leqq B * A$, we have $e \leqq g$. If $e = g$ were true, then, by (6.7.1), (6.7.10) and (6.7.9),

$$e = ez^n = x^n z^n \leqq (xz)^n \leqq z^n x^n = z^n e = g = e$$

and so $xz \in A$, contradicting (6.7.7). Hence

$$(6.7.11) \qquad e < g.$$

Put

$$(6.7.12) \qquad f = xg.$$

Since $B * A \, \gamma \, A$, it follows from Theorem 2.7 that $gx = g$ and so

$$(6.7.13) \qquad f \in E.$$

Since $e \mathscr{L} g$ in the semigroup E, we have $fe = xge = xg = f$. Since e is the zero element of A, we have $ef = exg = eg = e$. Hence

$$(6.7.14) \qquad e \mathscr{D}_E f.$$

By (6.7.5) and (6.7.12), $e = eg \leqq xg = f$. But if $e = f$ were true, then, by (6.7.10), (6.7.9), (6.7.12), (6.7.8) and (6.7.1),

$$e \leqq (xz)^{n+1} = x(zx) z \leqq xz^n x^n z = xz^n ez = xgz = fz = ez = e$$

and so $xz \in A$, contradicting (6.7.7). Hence $e < f$. Also, by (6.7.11), we have $x^n =$

$= e < g = g^n$ and so $x < g$. Hence $f = xg \leqq g^2 = g$. But, if $f = g$ were true, then $g = f = xg$ and so $g = x^n g = eg = e$, contradicting (6.7.11). Hence

$$(6.7.15) \qquad e < f < g\,.$$

By way of contradiction, we assume there exists $h \in E$ such that $e < h < f$ and $e\,\mathscr{D}_E\,h$. Then, since $A\delta$ is of L-type, $g\,\mathscr{L}\,h$ in the semigroup E. Also, since $e < h$, we have $x < h$ and so $f = xg \leqq hg = h$, which is a contradiction. Hence

$$(6.7.16) \qquad e \text{ and } f \text{ are consecutive in } e\mathscr{D}_E\,.$$

Thus we have the condition (b).

Conversely suppose that the condition (b) holds. Let $b \in B$. By Theorem 2.7, we have $eb = e \in A$. On the other hand, by the condition (b), there exists $a \in A$ such that $ag = f$. Since $g \in B * A$ and $B * A \leqq B$, there exists $b' \in B$ such that $g \leqq b'$. Then $f = ag \leqq ab'$ and so $ab' \notin A$. Thus we have the condition (a).

(2) can be proved in a similar way.

Theorem 6.8. *Suppose that $A, B \in \mathscr{C}$ such that $A \leqq B$ and $A\delta \wedge B\delta$ is a periodic δ-class of L-type [R-type].*

(1) *Suppose $A\delta = A\delta \wedge B\delta \neq B\delta$ and AB [BA] is not contained in a single archimedean class. Let e and g be idempotents of A and $B * A$ [$A * B$], respectively. Then there exists an idempotent f of S such that $e < f < g$, $e\,\mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$. Also $AB \subseteq A_- \cup \{f\}$ [$BA \subseteq A_- \cup \{f\}$].*

(2) *Suppose $A\delta \neq A\delta \wedge B\delta = B\delta$ and BA [AB] is not contained in a single archimedean class. Let e and g be idempotents of B and $A * B$ [$B * A$], respectively. Then there exists an idempotent f of S such that $g < f < e$, $e\,\mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$. Also $BA \subseteq B_+ \cup \{f\}$ [$AB \subseteq B_+ \cup \{f\}$].*

Proof. (1) Suppose $A\delta = A\delta \wedge B\delta \neq B\delta$ and AB is not contained in a single archimedean class. Then, by Lemma 6.7, there exists an idempotent f of S such that $e < f < g$, $e\,\mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$. Let $x \in A$ and $y \in B$. If $x \in A_+$, then, by Corollary 6.4, we have $xy = e \in A_-$. If $x \in A_-$ and $xy \in A$, then $e = ey \leqq$ $\leqq xy$ and so $xy \in A_-$. Finally suppose $xy \notin A$. Since $A\delta \wedge B\delta \neq B\delta$, we have $B * A < B$ and so $z = \max\{g, y\} = y$. Since $A\ \delta\ B * A\ \gamma\ B$, it follows from Theorem 2.7 that

$$(6.8.1) \qquad ey^i = e \quad \text{for every natural number} \quad i\,.$$

By way of contradiction, we assume that $xyx \leqq x$. Then, by (6.7.10), (6.8.1), (6.7.9) and (6.7.12), we have

$$x \geqq x(yx) \geqq x(yx)^{n+1} = xy(xy)^n x \geqq xyx^n y^n x = xyey^n x =$$
$$= xyex = xye = xg = f$$

and so $e = x^n \geqq f^n = f$, which is a contradiction. Hence

$$x < xyx\,. \tag{6.8.2}$$

By way of contradiction, we assume $y \leqq yxy$. Then, by (6.7.10), (6.8.1) and (6.7.9),

$$y \leqq (yx)\,y \leqq (yx)^n\,y \leqq y^n x^n y = y^n e y = y^n e = g\,.$$

Hence $B \leqq B * A$, which is a contradiction. Hence

$$yxy < y\,. \tag{6.8.3}$$

By (6.8.2) and (6.8.3), we have $xy \leqq (xy)^2 \leqq xy$ and so

$$xy \in E\,. \tag{6.8.4}$$

By (6.8.4),

$$xy = (xy)^{n+1} = xy(xy)^n \geqq xyx^n y^n = xyey^n = xye = xg = f\,,$$
$$xy = (xy)^{n+1} = x(yx)^n y \leqq xy^n x^n y = xy^n ey = xy^n e = xg = f\,.$$

Therefore

$$xy = f\,. \tag{6.8.5}$$

Hence we have $AB \subseteq A_- \cup \{f\}$.

(2) can be proved in a similar way.

Appendix. The behavior of set products AB of two archimedean classes A and B of S such that $A\delta = B\delta$ will be treated in other papers. A result in such a situation was given in [8].

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