Czechoslovak Mathematical Journal

Ladislav Bican; Pavel Jambor; Tomáš Kepka; Petr Němec

Hereditary and cohereditary preradicals

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 2, 192–206

Persistent URL: http://dml.cz/dmlcz/101390

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

HEREDITARY AND COHEREDITARY PRERADICALS

LADISLAV BICAN, PAVEL JAMBOR, TOMÁŠ KEPKA, PETR NĚMEC, Praha

(Received April 4, 1974)

In the last years, several authors have studied torsion theories and related idempotent radicals. Great emphasis was laid on hereditary radicals (see e.g. [9], [10], [11], [13]). J. A. BEACHY [2] has defined the dual notion, which he has called a cotorsion radical. However, the situation seems to be more lucid within the framework of the general theory of preradicals. In this paper, we are going to study hereditary preradicals together with the related dual problems. The first section provides (without proofs) a brief summary of the results from the theory of preradicals which are needed in the following parts. The second part is devoted to the investigation of the basic properties of hereditary preradicals, showing e.g. that a hereditary preradical is uniquely determined by its values on injective modules. We also discover a one-to-one correspondence between two-sided ideals and hereditary preradicals with torsion modules closed under direct products. In the third part we examine the hereditary closure of a given preradical r, i.e. the least hereditary preradical containing r. The following two sections deal with dualizations of these concepts. We investigate cohereditary preradicals, the one-to-one correspondence between the cohereditary preradicals and two-sided ideals and the cohereditary core. The final section includes some examples.

1. PRELIMINARIES

R always stands for an associative ring with identity and R-Mod is the category of all unitary left R-modules. The injective hull of a module M is denoted by $E(M)$, the direct product (sum) is denoted by $\prod_{i\in I} M_i$ ($\coprod_{i\in I} M_i$, $M_1 \oplus M_2$). Let $M \in R$-Mod and let N be a submodule of M. N is called *essential in* M if $K \cap N \neq 0$ for every nonzero submodule K of M. In this case, M is said to be an envelope of N. Dually, a submodule N of a module M is called superfluous in M, if $K = M$ whenever $K + N = M$. Then M is said to be a cover of M/N. A ring R is called left perfect if every module has a projective cover (see [1] for the details).

Let $\mathscr{A}$ be a class of modules. We shall say that $\mathscr{A}$ is

- hereditary if $\mathscr{A}$ is closed under isomorphisms and submodules,
- cohereditary if $\mathscr{A}$ is closed under epimorphic images,
- stable if every module $A \in \mathscr{A}$ has an injective presentation $0 \to A \to Q \to K \to 0$ with $Q \in \mathscr{A}$,
- costable if every module $A \in \mathscr{A}$ has a projective presentation $0 \to L \to P \to A \to 0$ with $P \in \mathscr{A}$.

A preradical r for R-Mod is a subfunctor of identity, i.e. r assigns to each module M its submodule $r(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $r(M)$ into $r(N)$ by restriction. Obviously, $r(R)$ is a two-sided ideal. We shall denote by $\mathscr{T}_r(\mathscr{F}_r)$ the class of all modules M such that $r(M) = M(r(M) = 0)$. Modules from $\mathscr{T}_r(\mathscr{F}_r)$ are called r-torsion (r-torsionfree). A preradical r is said to be idempotent (a radical) if for all $M \in R$-Mod, $r(r(M)) = r(M)\,(r(M/r(M)) = 0)$.

Let r, s be preradicals. We shall say that $r \subseteq s$ if $r(M) \subseteq s(M)$ for all $M \in R$-Mod. Obviously, if $r \subseteq s$ then $\mathscr{T}_r \subseteq \mathscr{T}_s$ and $\mathscr{F}_s \subseteq \mathscr{F}_r$. Conversely, if r is idempotent and $\mathscr{T}_r \subseteq \mathscr{T}_s$ then $r \subseteq s$, and if r is a radical and $\mathscr{F}_s \subseteq \mathscr{F}_r$ then $r \subseteq s$.

Let r be a preradical. Then $\mathscr{T}_r$ is a cohereditary class closed under direct sums and $\mathscr{F}_r$ is a hereditary class closed under direct products. If r is idempotent (a radical) then $\mathscr{F}_r(\mathscr{T}_r)$ is closed under extensions. For every $M \in R$-Mod, put $\bar{r}(M) = \sum K$, where K runs through all submodules K of M with $r(K) = K$, and $\tilde{r}(M) = \bigcap L$, L running through all submodules L of M such that $r(M/L) = 0$. Then $\bar{r}$ is an idempotent preradical, $\tilde{r}$ is a radical, $\mathscr{T}_r = \mathscr{T}_{\bar{r}}$, $\mathscr{F}_r = \mathscr{F}_{\tilde{r}}$, and if s is an idempotent preradical with $s \subseteq r$ (a radical with $r \subseteq s$) then $s \subseteq \bar{r}$ ($\tilde{r} \subseteq s$). Thus $\bar{r}$ is called the idempotent core and $\tilde{r}$ the radical closure of r.

Let $\mathscr{A}$ be an arbitrary class of R-modules. For every $M \in R$-Mod, we define $r_{\mathscr{A}}(M) = \sum \operatorname{Im} f$, $f \in \operatorname{Hom}(A, M)$, $A \in \mathscr{A}$ and $r^{\mathscr{A}}(M) = \bigcap \operatorname{Ker} f$, $f \in \operatorname{Hom}(M, A)$, $A \in \mathscr{A}$. Then $r_{\mathscr{A}}$ is an idempotent preradical, $\mathscr{A} \subseteq \mathscr{T}_{r_{\mathscr{A}}}$ and $\mathscr{T}_{r_{\mathscr{A}}}$ consists of all epimorphic images of direct sums of modules from $\mathscr{A}$. If $\mathscr{A}$ is hereditary then $\mathscr{F}_{r_{\mathscr{A}}}$ is stable, and if $\mathscr{A}$ is a cohereditary class closed under extensions and direct sums then $r_{\mathscr{A}}$ is an idempotent radical. Further, $r^{\mathscr{A}}$ is a radical, $\mathscr{A} \subseteq \mathscr{F}_{r^{\mathscr{A}}}$ and $\mathscr{F}_{r^{\mathscr{A}}}$ consists of all submodules of direct products of modules from $\mathscr{A}$. If $\mathscr{A}$ is a hereditary class closed under extensions and direct products then $r^{\mathscr{A}}$ is an idempotent radical. If r is a preradical then $\bar{r} = r_{\mathscr{T}_r}$ and $\tilde{r} = r^{\mathscr{F}_r}$.

The assertions listed above are presented without proofs, since a detailed study of these and related problems will appear in [6].

An i-subpreradical (p-subpreradical) s is an application that assigns to every injective (projective) module M its submodule $s(M)$ in such a way that every homomorphism of M into an injective (projective) module N induces a homomorphism of $s(M)$ into $s(N)$ by restriction.

A preradical r is called hereditary (cohereditary) if for every $M \in R$-Mod and every submodule $N \subseteq M$, $r(N) = N \cap r(M)$ $(r(M/N) = (r(M) + N)/N)$. Let I be a two-sided ideal of a ring R. We shall say that I satisfies the condition (a) if $x \in Ix$ for every $x \in I$.

Finally, we shall need the following simple assertion: *If I is a two-sided ideal then there exists a largest idempotent two-sided ideal contained in I.*

2. HEREDITARY PRERADICALS

2.1. Proposition. *The following conditions on a preradical r for R-Mod are equivalent:*

(i) *r is left exact as a functor,*
(ii) *r is hereditary,*
(iii) *r is idempotent and $\mathscr{T}_r$ is hereditary.*

Proof. (i) implies (ii). If N is a submodule of a module M then, by (i), the sequence $0 \to r(N) \to r(M) \to r(M/N)$ is exact, so that $r(N) = N \cap r(M)$.

(ii) implies (iii). We have $r(r(M)) = r(M) \cap r(M)$ by (ii) and r is idempotent. If $M \in \mathscr{T}_r$ and N is a submodule of M then $r(N) = N \cap r(M) = N \cap M = N$. Thus $N \in \mathscr{T}_r$ and $\mathscr{T}_r$ is hereditary.

(iii) implies (i). Let $0 \to A \to^f B \to^g C \to 0$ be an exact sequence. We are going to show that the sequence $0 \to r(A) \to^{\bar{f}} r(B) \to^{\bar{g}} r(C)$, where $\bar{f}$ and $\bar{g}$ are restrictions of f and g, is exact. The exactness in $r(A)$ is clear as well as the inclusion $\operatorname{Im} \bar{f} \subseteq \operatorname{Ker} \bar{g}$. Let $x \in \operatorname{Ker} \bar{g}$ be an arbitrary element. Then $x \in \operatorname{Im} f \cap r(B) = r(B) = r(\operatorname{Im} f \cap r(B))$ by (iii) and $r(\operatorname{Im} f \cap r(B)) \subseteq r(\operatorname{Im} f) = \operatorname{Im} \bar{f}$ as desired.

2.2. Proposition. *Let r be a preradical for R-Mod. Then*

(i) *if r is hereditary then $\mathscr{F}_r$ is stable and closed under envelopes,*
(ii) *if r is a radical and $\mathscr{F}_r$ is stable then r is a hereditary radical.*

Proof. (i) For $F \in \mathscr{F}_r$ we have $0 = r(F) = F \cap r(E(F))$ and $r(E(F)) = 0$, F being essential in $E(F)$.

(ii) We shall show that r satisfies the condition (iii) of Proposition 2.1. Consider the commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & \\
0 \arrow[r] & r(M)/r(r(M)) \arrow[r] \arrow[d, dashed, "f"'] & M/r(r(M)) \arrow[dl, "h"] \\
& Q &
\end{tikzcd}$$

with exact row and column, where Q is an injective module from $\mathscr{F}_r$, the existence of which follows from the stability of $\mathscr{F}_r$. Thus $f(r(M)/r(r(M))) = h(r(M)/r(r(M))) \subseteq$ $\subseteq h(r(M/r(r(M)))) \subseteq r(Q) = 0$, showing $r(M) = r(r(M))$. Now let N be a submodule of a module $M \in \mathscr{T}_r$. The stability of $\mathscr{F}_r$ yields the existence of the following commutative diagram with exact row and column where Q is an injective module from $\mathscr{F}_r$:

$$
\begin{tikzcd}
 & 0 \arrow[d] & \\
0 \arrow[r] & N/r(N) \arrow[r] \arrow[d, "f"'] & M/r(N) \arrow[dl, "h"] \\
 & Q &
\end{tikzcd}
$$

Now $M/r(N) \in \mathscr{T}_r$, $Q \in \mathscr{F}_r$ yields $h = 0$ so that $f = 0$, $N = r(N)$ which completes the proof.

2.3. Corollary. *The radical closure of a hereditary preradical is a hereditary radical.*

Proof. Since $\mathscr{F}_r = \mathscr{F}_{\bar{r}}$, it suffices to use Proposition 2.2.

2.4. Proposition. *Let s be a hereditary preradical, $\mathscr{A}$ a representative set of s-torsion cyclic modules and $M = \coprod_{A \in \mathscr{A}} A$. Then $s = r_{\mathscr{A}} = r_{\{M\}}$.*

Proof. Obviously, $\mathscr{T}_{r_{\mathscr{A}}} \subseteq \mathscr{T}_s$ and $\mathscr{T}_{r_{\{M\}}} \subseteq \mathscr{T}_s$. On the other hand, every module $T \in \mathscr{T}_s$ is an epimorphic image of $\coprod_{x \in T} Rx$ and consequently of a direct power of M. Hence the assertion follows.

The following result is well-known (see e.g. J. P. Jans [10]) but we include it here for the sake of completeness.

2.5. Proposition. *Let s be a hereditary preradical. Then s is a radical iff there exists an s-torsionfree injective module Q such that $s = r^{\{Q\}}$.*

Proof. It is easy to see that for an injective module Q the preradical $r^{\{Q\}}$ is a hereditary radical. Conversely, let $\mathscr{A}$ be a representative set of cyclic s-torsionfree modules and $Q = \prod_{A \in \mathscr{A}} E(A)$. Obviously, $\mathscr{T}_s \subseteq \mathscr{T}_t$ where $t = r^{\{Q\}}$. Let $T \in \mathscr{T}_t \dot{-} \mathscr{T}_s$ be a module. Without loss of generality we can assume that $T \in \mathscr{F}_s \cap \mathscr{T}_t$ (take $T/s(T)$ instead of T, if necessary). If T is non-zero, then it contains a non-zero cyclic submodule K isomorphic to some $A \in \mathscr{A}$. But then $K \in \mathscr{F}_t \cap \mathscr{T}_t = 0$ yields a contradiction.

2.6. Proposition. *Let s be an i-subpreradical. Let M be a module and Q an injective module containing M. Then $r(M) = M \cap s(Q)$ does not depend on the particular choice of Q and r is a hereditary preradical.*

Proof. Let M, N be modules, Q, S injective modules, $M \subseteq Q$, $N \subseteq S$ and $f \in \operatorname{Hom}(M, N)$. Then there exists $g \in \operatorname{Hom}(Q, S)$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ Q \arrow[r, "g"] & S \end{tikzcd}$$

commutes. From this it easily follows that $r(M)$ does not depend on the choice of Q and r is a preradical. Finally, for a submodule N of $M, N \subseteq M \subseteq Q$, we have $r(N) = N \cap s(Q) = N \cap M \cap s(Q) = N \cap r(M)$ and r is hereditary.

2.7. Theorem. *There is a one-to-one correspondence between hereditary preradicals and i-subpreradicals.*

Proof. By Proposition 2.6.

2.8. Proposition. *Let r be a hereditary preradical. Then the following statements are equivalent:*

(i) *r is a radical,*
(ii) *$r(Q/r(Q)) = 0$, for every injective module Q.*

Proof. (i) implies (ii). Obvious.

(ii) implies (i). For a module M, let us consider the exact sequence

$$0 \to r(E(M))/r(M) \to E(M)/r(M) \to E(M)/r(E(M)) \to 0 .$$

By the hypothesis, $r(E(M)/r(E(M))) = 0$ so that $r(E(M)/r(M)) = r(E(M))/r(M)$ and consequently, $r(M/r(M)) = M/r(M) \cap r(E(M)/r(M)) = (M \cap r(E(M)))/r(M) = 0$.

2.9. Lemma. *Let r be a hereditary preradical and $I = \bigcap K$, K running through all left ideals with $r(R/K) = R/K$. Then I is a two-sided ideal.*

Proof. For $a \in R$ and $R/K \in \mathscr{T}_r$ we have $R/(K : a) \in \mathscr{T}_r$. Thus $I \subseteq (K : a)$, $Ia \subseteq K$ and consequently $Ia \subseteq I$.

2.10. Proposition. *Let r be a hereditary preradical such that $\mathscr{T}_r$ is closed under direct products. Put $I = \bigcap K$, where K runs through all left ideals such that $R/K \in \mathscr{T}_r$. Then*

(i) *$R/I \in \mathscr{T}_r$ and I is a two-sided ideal,*
(ii) *$T \in \mathscr{T}_r$ iff $IT = 0$,*

(iii) $r(M) = \{m \in M, \text{Im} = 0\}$, *for every* $M \in R$-Mod,

(iv) r *is a radical iff* $I = I^2$.

Proof. (i) There is a natural monomorphism $f : R/I \to \prod_K R/K$ which shows $R/I \in \mathscr{T}_r$. I is two-sided by Lemma 2.9.

(ii) If $T \in \mathscr{T}_r$ then $Rt \cong R/(0:t) \in \mathscr{T}_r$, for every $t \in T$. Hence $I \subseteq (0:t)$, $It = 0$ and consequently $IT = 0$. Conversely, if $IT = 0$ then $I \subseteq (0:t)$, for every $t \in T$, and $Rt \cong R/(0:t) \in \mathscr{T}_r$ yields $T \in \mathscr{T}_r$.

(iii) Since r is idempotent, this follows from (ii).

(iv) If $I = I^2$ and $m + r(M) \in r(M/r(M))$ is arbitrary, then $Im \subseteq r(M)$ and $Im = I^2 m = 0$ by (ii). Thus $m \in r(M)$ by (iii) and $r(M/r(M)) = 0$. Conversely, $I(I/I^2) = 0$ yields $I/I^2 \in \mathscr{T}_r$ and hence $R/I^2 \in \mathscr{T}_r$, since $R/I \in \mathscr{T}_r$ and $\mathscr{T}_r$ is closed under extensions. Thus $I = I^2$.

2.11. Proposition. *Let* $I \subseteq R$ *be a two-sided ideal. For every module* $M \in R$-Mod *put* $r(M) = \{m \in M, \text{Im} = 0\}$. *Then*

(i) r *is a hereditary preradical,*

(ii) $T \in \mathscr{T}_r$ *iff* $IT = 0$,

(iii) $\mathscr{T}_r$ *is closed under direct products,*

(iv) $r(R) = (0:I)_r = \{a \in R, Ia = 0\}$,

(v) $I = \bigcap K$, *where* K *runs through all left ideals such that* $R/K \in \mathscr{T}_r$,

(vi) r *is a radical iff* $I = I^2$.

Proof. $r(M)$ is a submodule of M since I is two-sided. For every $f \in \text{Hom}(M, N)$ we have $I f(r(M)) = f(I\, r(M)) = 0$ so that $f(r(M)) \subseteq r(N)$ and (i) holds. The assertions (iii), (iv), (v) are obvious and (ii), (vi) follow from 2.10.

2.12. Theorem. *There is a one-to-one correspondence between the two-sided ideals of a ring* R *and the hereditary preradicals* r *for* R-Mod *with* $\mathscr{T}_r$ *closed under direct products. This correspondence induces a one-to-one correspondence between the idempotent two-sided ideals and the hereditary radicals* r *with* $\mathscr{T}_r$ *closed under direct products.*

Proof follows immediately from Propositions 2.10 and 2.11.

3. HEREDITARY CLOSURE

3.1. Proposition. *Let* r *be a preradical. For all* $M \in R$-Mod, *put* $h(r)(M) = M \cap r(E(M))$. *Further, let* $\mathscr{A}$ *be the class of all modules* M *such that* $M \subseteq r(E(M))$ *and let* $\mathscr{B}$ *be the class of all modules that are isomorphic to a submodule of a module from* $\mathscr{T}_r$. *Then*

(i) $h(r)$ *is a hereditary preradical and* $r \subseteq h(r)$,

(ii) *if s is a hereditary preradical with $r \subseteq s$, then $h(r) \subseteq s$, i.e. $h(r)$ is the least hereditary preradical containing r,*

(iii) $h(r) = r_{\mathscr{A}}$ *and* $\mathscr{A} = \mathscr{T}_{h(r)}$,

(iv) $F \in \mathscr{F}_{h(r)}$ *iff* $E(F) \in \mathscr{F}_r$,

(v) $h(\bar{r}) = r_{\mathscr{B}}$ *and* $\mathscr{B} = \mathscr{T}_{h(\bar{r})}$,

(vi) *if $Q/r(Q)$ is injective and $r(Q/r(Q)) = 0$ for every injective module Q then $h(r)$ is a hereditary radical.*

The preradical $h(r)$ is called *the hereditary closure of r.*

Proof. (i) follows from Proposition 2.6.

(ii) If s is hereditary, $r \subseteq s$ and $M \in R$-Mod, then $h(r)(M) = M \cap r(E(M)) \subseteq \subseteq M \cap s(E(M)) = s(M)$.

(iii) It is clear that $\mathscr{A} = \mathscr{T}_{h(r)}$ and hence $h(r) = \overline{h(r)} = r_{\mathscr{T}_{h(r)}} = r_{\mathscr{A}}$.

(iv) Obvious.

(v) The proof is similar to that of (iii).

(vi) If Q is injective then $h(r)(Q/h(r)(Q)) = h(r)(Q/r(Q)) = Q/r(Q) \cap r(Q/r(Q)) = = 0$ and it suffices to use Proposition 2.8.

3.2. Proposition. *Let r be a preradical and let $\mathscr{D}$ be the class of all modules M such that $E(M) \in \mathscr{F}_r$. Then*

(i) $\widetilde{h(r)}$ *is a hereditary radical and* $r \subseteq h(\bar{r}) \subseteq \widetilde{h(r)}$,

(ii) *if s is a hereditary radical and $r \subseteq s$ then $\widetilde{h(r)} \subseteq s$, i.e. $\widetilde{h(r)}$ is the least hereditary radical containing r,*

(iii) $\widetilde{h(r)} = r^{\mathscr{D}}$ *and* $\mathscr{F}_{\widetilde{h(r)}} = \mathscr{D}$,

(iv) *if for every injective module Q, $Q/r(Q)$ is injective and $r(Q/r(Q)) = 0$, then $h(r) = h(\bar{r}) = \widetilde{h(r)}$.*

Proof follows easily from Proposition 3.1 and Corollary 2.3.

3.3. Proposition. *Let s be a preradical such that $\mathscr{T}_s$ is closed under extensions. Denote by $\mathscr{C}$ the class of all $T \in \mathscr{T}_s$ such that every submodule of T belongs to $\mathscr{T}_s$. Then*

(i) *$\mathscr{C}$ is a hereditary cohereditary class closed under extensions and direct sums,*

(ii) *$r_{\mathscr{C}}$ is a hereditary radical and* $r_{\mathscr{C}} \subseteq \bar{s} = \tilde{\bar{s}} \subseteq s$,

(iii) *if t is a hereditary preradical and $t \subseteq s$ then $t \subseteq r_{\mathscr{C}}$.*

Proof. The class $\mathscr{C}$ is clearly hereditary, so that $\mathscr{F}_{r_{\mathscr{C}}} = \mathscr{F}_{\bar{r}_{\mathscr{C}}}$ is stable and $\bar{r}_{\mathscr{C}}$ is hereditary. Now we are going to show $\mathscr{C} = \mathscr{T}_{\bar{r}_{\mathscr{C}}}$. The inclusion $\mathscr{C} \subseteq \mathscr{T}_{\bar{r}_{\mathscr{C}}}$ is obvious.

For $T \in \mathcal{T}_{\tilde{r}_{\mathscr{C}}} \dotminus \mathscr{C}$ there is a submodule X of T such that $X \notin \mathcal{T}_s$. Thus $X \in \mathcal{T}_{\tilde{r}_{\mathscr{C}}}$, $\bar{s}(X) \subsetneqq X$ and $r_{\mathscr{C}}(X/\bar{s}(X)) \neq 0$ since $\tilde{r}_{\mathscr{C}}(X/\bar{s}(X)) = X/\bar{s}(X) \neq 0$. So X contains a submodule Y, $\bar{s}(X) \subset Y \subseteq X$, $\bar{s}(X) \neq Y$ with $Y/\bar{s}(X) \in \mathscr{C} \subseteq \mathcal{T}_s$. Hence $Y \in \mathcal{T}_s$, $Y \subseteq \bar{s}(X)$ which contradicts the choice of Y.

Now (i) is obvious and (ii) follows from $\mathcal{T}_{\tilde{r}_{\mathscr{C}}} = \mathscr{C} = \mathcal{T}_{r_{\mathscr{C}}}$. If $t \subseteq s$ then $\mathcal{T}_t \subseteq \mathcal{T}_s$ and $\mathcal{T}_t \subseteq \mathscr{C} = \mathcal{T}_{r_{\mathscr{C}}}$ since t is hereditary. Thus $t = r_{\mathcal{T}_t} \subseteq r_{\mathscr{C}}$ and the proof is complete.

4. COHEREDITARY PRERADICALS

4.1. Proposition. *The following statements are equivalent for a preradical r:*

(i) *r is cohereditary,*
(ii) *r preserves epimorphisms as a functor,*
(iii) *the functor $M \to M/r(M)$ is right exact,*
(iv) *r is a radical and $\mathscr{F}_r$ is cohereditary.*

Proof. (i) implies (ii). For the canonical projection $p : M \to M/N$ we have $p(r(M)) = (r(M) + N)/N = r(M/N)$ by (i).

(ii) implies (iii). For an exact sequence $0 \to A \to^i B \to^p C \to 0$ we get the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \downarrow & & \downarrow & & \downarrow \\ A/r(A) & \xrightarrow{f} & B/r(B) & \xrightarrow{g} & C/r(C) \end{array}$$

where $\operatorname{Im} f = (i(A) + r(B))/r(B)$ and $\operatorname{Ker} g = p^{-1}(r(C))/r(C)$. It is clear that g is an epimorphism and $\operatorname{Im} f \subseteq \operatorname{Ker} g$. Finally, $p(r(B)) = r(C)$ by (ii) so that $p^{-1}(r(C)) = r(B) + i(A)$.

(iii) implies (iv). The sequence $r(M)/r(r(M)) \to^i M/r(M) \to M/r(M)/r(M/r(M)) \to 0$ is exact by (iii) so that $r(M/r(M)) = 0$, since $\operatorname{Im} i = 0$.

Further, if $M \in \mathscr{F}_r$, $N \subseteq M$, then $r(N) = r(M) = 0$ and (iii) yields the exact sequence $N \to M \to M/N/r(M/N) \to 0$. Thus $r(M/N) = 0$.

(iv) implies (i). If N is a submodule of M then $M/(N + r(M)) \in \mathscr{F}_r$ and consequently $r(M/N) \subseteq (N + r(M))/N$.

4.2. Proposition. *Let r be a preradical. Then*

(i) *if r is cohereditary then $\mathcal{T}_r$ is closed under covers,*
(ii) *if r is cohereditary and R is left perfect then $\mathcal{T}_r$ is costable,*
(iii) *if r is cohereditary, $M \in \mathcal{T}_r$ and $0 \to K \to^f P \to^g M \to 0$ is an arbitrary projective presentation of M then $P = r(P) + f(K)$,*
(iv) *if r is idempotent cohereditary and R is left hereditary then $\mathcal{T}_r$ is costable.*

Proof. (i) If N is superfluous in M and $M/N \in \mathscr{T}_r$ then $M/N = r(M/N) = (r(M) + N)/N$ yields $r(M) = M$.

(ii) follows immediately from (i).

(iii) By Proposition 4.1, g induces an epimorphism $r(P) \to r(M) = M$. Hence for every $x \in P$ there is $y \in r(P)$ with $g(x) = g(y)$ and $x - y \in \operatorname{Ker} g = \operatorname{Im} f$.

(iv) Let $0 \to K \to P \to^g M \to 0$ be a projective presentation of a module $M \in \mathscr{T}_r$. By Proposition 4.1, g induces an epimorphism $r(P) \to M$ and $r(P) \in \mathscr{T}_r$ is projective by the hypothesis.

4.3. Proposition. *Let r be an idempotent preradical and let each $T \in \mathscr{T}_r$ have a projective presentation $0 \to K \to^f P \to T \to 0$ such that $P = r(P) + f(K)$. Then r is an idempotent cohereditary radical.*

Proof. Let B be a submodule of a module A. By the hypothesis, $r(A/B)$ has a projective presentation $0 \to K \to^f P \to^g r(A/B) \to 0$ with $P = f(A) + r(P)$. Then g induces the following commutative diagram

$$\begin{tikzcd} & & & P \arrow[d, "g"] \arrow[dl, "h"'] & \\ 0 \arrow[r] & B \arrow[r] & A \arrow[r] & A/B \arrow[r] & 0 \end{tikzcd}$$

which yields $r(A/B) = \operatorname{Im} g = g(r(P)) \subseteq (r(A) + B)/B$. Now it is easy to see that $r(M/r(M)) = 0$ for every $M \in R$-Mod, and $r(F/A) = 0$ for every submodule A of a module $F \in \mathscr{F}_r$, and it suffices to use Proposition 4.1.

4.4. Corollary. *Let R be a left perfect ring. Then the idempotent core of a cohereditary radical is an idempotent cohereditary radical.*

Proof. It suffices to use Proposition 4.2 (ii) and Proposition 4.3.

4.5. Proposition. *Let R be a left hereditary ring. Then the hereditary closure of a cohereditary radical r is a hereditary cohereditary radical.*

Proof. If B is a submodule of A then the factor-module $E(A)/B$ is injective by the hypothesis and $h(r)(A/B) = A/B \cap r(E(A)/B) = A/B \cap (r(E(A)) + B)/B = (A \cap r(E(A)) + B)/B = (h(r)(A) + B)/B$.

4.6. Proposition. *Let s be a cohereditary radical and let $\mathscr{A}$ be a representative set of s-torsionfree cocyclic modules. Then $s = r_{\mathscr{A}} = r^{\{M\}}$ where $M = \prod_{A \in \mathscr{A}} A$.*

Proof. Obviously, $r_{\mathscr{A}} = r^{\{M\}}$ and $\mathscr{A} \subseteq \mathscr{F}_s$ implies that $\mathscr{F}_{r_{\mathscr{A}}} \subseteq \mathscr{F}_s$. Let $F \in \mathscr{F}_s$, $m \in F$ be arbitrary and let $C \in R$-Mod, $f \in \operatorname{Hom}(F, C)$ be such that C is cocyclic and $f(m) \neq 0$. Then $\operatorname{Im} f$ is an s-torsionfree cocyclic module. Thus every s-torsionfree module can be imbedded into a direct product of modules from $\mathscr{A}$, i.e. $\mathscr{F}_s \subseteq \mathscr{F}_r \mathscr{A}$.

4.7. Proposition. *Let R be a left perfect ring and s an idempotent cohereditary radical for R-Mod. Then there is an s-torsion projective module P such that $s = r_{\{P\}}$.*

Proof. Let $\mathscr{A}$ be a representative set of s-torsion cocyclic modules, let P be the direct sum of projective covers of modules from $\mathscr{A}$ and let $t = r_{\{P\}}$. It is easy to see that t is a radical and $\mathscr{F}_t$ is cohereditary. Further, $\mathscr{F}_s \subseteq \mathscr{F}_t$ since $\mathscr{T}_t \subseteq \mathscr{T}_s$. Suppose there is a module $F \in \mathscr{F}_t \dot{-} \mathscr{F}_s$. Then $B = s(F) \neq 0$, $B \in \mathscr{T}_s$, s being idempotent. Now B has a factormodule B/C isomorphic to an element from $\mathscr{A}$ and $\operatorname{Hom}(P, B/C) \neq 0$, a contradiction to the previous assumption.

4.8. Proposition. *Let r be a cohereditary radical for R-Mod and $r(R) = I$. Then*

(i) *$r(M) = IM$ for all $M \in R$-Mod,*

(ii) *$T \in \mathscr{T}_r$ iff $IT = T$,*

(iii) *$F \in \mathscr{F}_r$ iff $IF = 0$,*

(iv) *if I is finitely generated as a right ideal then $\mathscr{T}_r$ is closed under products,*

(v) *r is idempotent iff $I^2 = I$,*

(vi) *r is hereditary iff satisfies the condition* (a).

Proof. (i) For $M \in R$-Mod, let $0 \to K \to P \to^g M \to 0$ be a projective presentation of M. We have $r(P) = IP$, P being projective, and consequently $r(M) = g(r(P)) = g(IP) = I\, g(P) = IM$.

(ii) and (iii) follow immediately from (i).

(iv) Suppose $I = \sum_{k=1}^{n} a_k R$, $T_i \in \mathscr{T}_r$, $i \in I$ and $t = (t_i)_{i \in I} \in T = \prod_{i \in I} T_i$. Then $t_i = \sum_{j=1}^{m_i} b_{ij} t_{ij} = \sum_{k=1}^{n} a_k t'_{ik}$ and $t = \sum_{k=1}^{n} a_k t_k$, where $t_k = (t'_{ik})_{i \in I}$, which shows $T = IT$ and $T \in \mathscr{T}_r$.

(v) If r is idempotent then $I^2 = I \cdot r(R) = r(I) = r(r(R)) = I$. Conversely, $I^2 = I$ yields $r(r(M)) = I^2 M = IM = r(M)$.

(vi) Let I satisfy the condition (a). With respect to Proposition 2.2 it suffices to show that $\mathscr{F}_r$ is stable. If $I \cdot E(F) \neq 0$ for some $F \in \mathscr{F}_r$, then there exists $f \in E(F)$ and $x \in I$ with $xf \neq 0$ and therefore $0 \neq axf \in F$ for some $a \in R$. The condition (a) yields the existence of $k \in I$ such that $ax = kax$ and $axf = kaxf = 0$ implies a contradiction. Conversely, for $x \in I$ we have $Rx \subseteq I = r(R)$ and consequently $Rx = r(Rx) = IRx = Ix$.

4.9. Corollary. *Let r be a cohereditary radical for R-Mod and $r(r(R)) = r(R)$. Then r is idempotent.*

Proof. Obvious.

4.10. Proposition. *Let $I \subseteq R$ be a two-sided ideal. If we put $r(M) = IM$ for every module M then r is a cohereditary radical and $r(R) = I$.*

Proof. r is clearly a preradical and for every submodule N of a module M, $r(M/N) = I(M/N) = (IM + N)/N = (r(M) + N)/N$.

4.11. Theorem. *There is a one-to-one correspondence between the two-sided ideals of R and the cohereditary radicals for R-Mod. This correspondence induces*

- *a one-to-one correspondence between the idempotent two-sided ideals and the idempotent cohereditary radicals,*
- *a one-to-one correspondence between the two-sided ideals satisfying the condition (a) and the hereditary cohereditary radicals.*

Proof. By Propositions 4.8 and 4.10.

4.12. Proposition. *Let r be a preradical for R-Mod. The following statements are equivalent:*

(i) *r is exact as a functor,*

(ii) *r is right exact as a functor,*

(iii) *r is hereditary and cohereditary,*

(iv) *r is an idempotent radical, $\mathcal{T}_r$ is hereditary and $\mathcal{F}_r$ is cohereditary,*

(v) *the functor $M \to M/r(M)$ is exact,*

(vi) *the functor $M \to M/r(M)$ is left exact,*

(vii) *$r(R)$ satisfies the condition (a) and $r(M) = r(R) \cdot M$ for all $M \in R$-Mod.*

Proof. The conditions (i), (ii), (iii) and (iv) are equivalent by Propositions 2.1 and 4.1 while (iii), (vii) are equivalent by Propositions 4.8 and 4.10.

(iii) implies (v). The functor $M \to M/r(M)$ is right exact by Proposition 4.1. A monomorphism $f: A \to B$ induces $g: A/r(A) \to B/r(B)$. If $x + r(A) \in \operatorname{Ker} g$ then $f(x) \in f(A) \cap r(B) = r(f(A)) = f(r(A))$. Thus $x \in r(A)$, since f is a monomorphism, and $\operatorname{Ker} g = 0$.

It remains to prove that (vi) implies (iii) since (vi) follows from (v) trivially. The inclusion $i: A \to B$ induces a monomorphism $A/r(A) \to B/r(B)$ which yields the inclusion $A \cap r(B) \subseteq r(A)$ and r is hereditary. If $0 \to A \to^f B \to^g C \to 0$ is an exact sequence then $0 \to A/r(A) \to B/r(B) \to C/r(C)$ is exact by the hypothesis and $g^{-1}(r(C)) = f(A) + r(B)$, which yields $g(r(B)) = r(C)$ and r is cohereditary by Proposition 4.1.

4.13. Proposition. *Let s be a p-subpreradical and let for every module $M \in R$-Mod, $0 \to K \to P \to^f M \to 0$ be a projective presentation of M. If we put $r(M) = f(s(P))$ then $r(M)$ does not depend on the particular choice of the projective presentation and r is a cohereditary radical.*

Proof. Let $M, N \in R$-Mod, $h \in \mathrm{Hom}(M, N)$ and let $0 \to K \to P \to^f M \to 0$, $0 \to L \to Q \to^g N \to 0$ be projective presentations of M and N, respectively. Then there exists $k \in \mathrm{Hom}(P, Q)$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & M \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} \\ Q & \xrightarrow{g} & N \end{array}$$

commutes. Hence the independence of $r(M)$ of the particular choice of the projective presentation and the fact that r is preradical immediately follow. Moreover, if h is an epimorphism then $r(N) = hf(s(P)) = h(r(M))$ and the proof is complete.

4.14. Theorem. *There is a one-to-one correspondence between the cohereditary radicals and the p-subpreradicals.*

Proof follows from Proposition 4.13.

5. COHEREDITARY CORE

5.1. Proposition. *Let r be a preradical and let $\mathscr{A}$ be the class of all modules M possessing a projective presentation $0 \to K \to^f P \to M \to 0$ with $r(P) \subseteq f(K)$. For every $M \in R$-Mod, put $ch(r)(M) = r(R) \cdot M$. Then*

(i) *$ch(r)$ is a cohereditary radical and $ch(r) \subseteq r$,*

(ii) *if s is a cohereditary radical and $s \subseteq r$ then $s \subseteq ch(r)$, i.e. $ch(r)$ is the largest cohereditary radical contained in r,*

(iii) *if $M \in R$-Mod and $0 \to K \to P \to^g M \to 0$ is its projective presentation then $ch(r)(M) = g(r(P))$,*

(iv) *$ch(r) = r_{\mathscr{A}}$ and $\mathscr{F}_{ch(r)} = \mathscr{A}$,*

(v) *$ch(r)$ is idempotent iff $r(R)$ is an idempotent two-sided ideal,*

(vi) *if $r(R)$ is a projective module and $r(r(R)) = r(R)$ then $ch(r)$ is an idempotent cohereditary radical,*

(vii) *if Ra is projective and r-torsion for all $a \in r(R)$, then $ch(r)$ is a hereditary cohereditary radical.*

The preradical $ch(r)$ is called *the cohereditary core of r.*

Proof. (i) follows from Proposition 4.10.

(ii) If $s \subseteq r$ is a cohereditary radical then Proposition 4.8 yields $s(M) = s(R) \cdot M \subseteq$ $\subseteq r(R) \cdot M = ch(r)(M)$.

(iii) For P projective we have $r(P) = r(R) \cdot P$. Then, by (i) and 4.1, $g(r(P)) =$ $= g(r(R) \cdot P) = g(ch(r)(P)) = ch(r)(M)$.

(iv) $\mathscr{A} = \mathscr{F}_{ch(r)}$ by (iii) and $ch(r) = \widetilde{ch(r)} = r^{\mathscr{F}_{ch(r)}} = r^{\mathscr{A}}$.

(v) $ch(r)(R) = r(R)$ by (iii) and it suffices to use Proposition 4.8.

(vi) By (iii) we have $ch(r)(ch(r)(R)) = ch(r)(r(R)) = r(r(R)) = r(R) = ch(r)(R)$ and it suffices to use Corollary 4.9.

(vii) For every $a \in r(R)$ we have $r(R)a = r(R) \cdot Ra = ch(r)(Ra) = r(Ra) = Ra$ by (iii) and Proposition 4.8 completes the proof.

5.2. Proposition. *Let R be a left hereditary ring and r a preradical for R-*Mod. *Then*

(i) $ch(\bar{r}) = (\overline{ch(r)})$ *is an idempotent cohereditary radical,*

(ii) $h(\tilde{r}) = \widetilde{h(r)}$,

(iii) $h(ch(r)) \subseteq ch(h(r))$ *are both hereditary cohereditary radicals.*

Proof. (i) Obviously $ch(\bar{r}) \subseteq ch(r) \subseteq r$ and Proposition 5.1 yields $ch(\bar{r}) \subseteq$ $\subseteq ch(\overline{ch(r)}) \subseteq ch(\bar{r})$.

(ii) follows immediately from Proposition 3.2.

(iii) $r \subseteq h(r)$ yields $ch(r) \subseteq ch(h(r))$. Now $ch(h(r))$ is a hereditary cohereditary radical by Proposition 5.1 (vii) and hence $h(ch(r)) \subseteq ch(h(r))$ by Proposition 3.1. Further, $ch(r) \subseteq ch(h(ch(r))) \subseteq h(ch(r))$ gives $h(ch(r)) = ch(h(ch(r)))$ by Proposition 3.1, since $ch(h(ch(r)))$ is hereditary. This shows that $h(ch(r))$ is a hereditary cohereditary radical.

5.3. Proposition. *Let r be a preradical for R-*Mod, *I the largest idempotent two-sided ideal contained in $r(R)$ and let s be the cohereditary radical defined by $s(M) = IM$ for all $M \in R$-*Mod. *Then*

(i) *s is an idempotent cohereditary radical and $s \subseteq ch(\bar{r})$, $s \subseteq ch(r) \subseteq r$,*

(ii) *if t is an idempotent cohereditary radical with $t \subseteq r$ then $t \subseteq s$, i.e. s is the largest idempotent cohereditary radical contained in r,*

(iii) *if R is left perfect then $s = \overline{ch(r)} = r_{\mathscr{A}}$ and $\mathscr{T}_s = \mathscr{A}$, where $\mathscr{A}$ is the class of all modules $T \in \mathscr{T}_r$ with a projective presentation $0 \to K \to^f P \to T \to 0$ such that $P = r(P) + f(K)$,*

(iv) *if R is left hereditary then* $s = ch(\bar{r})$,

(v) *if $r(R)$ is projective and $r(r(R)) = r(R)$ then* $s = ch(r) = ch(\bar{r}) = \overline{ch(r)}$.

Proof. (i) follows from Propositions 4.8 and 4.10.

(ii) $t(R)$ is an idempotent ideal of R contained in $r(R)$ by Proposition 4.8 and hence $t(R) \subseteq s(R)$, $t \subseteq s$.

(iii) $s = \overline{ch(r)}$ by Corollary 4.4 and (ii). It is easy to see that $\mathscr{A} = \mathscr{T}_{r_{\mathscr{A}}}$ and $r_{\mathscr{A}}$ is cohereditary. Thus $r_{\mathscr{A}} \subseteq s$, $\mathscr{A} = \mathscr{T}_{r_{\mathscr{A}}} \subseteq \mathscr{T}_s$ and the assertion easily follows.

(iv) follows from Proposition 5.2 (i).

(v) By (i), (ii) and Proposition 5.1 (vi) we have $s \subseteq ch(\bar{r}) \subseteq ch(r) = \overline{ch(r)} \subseteq s$.

6. EXAMPLES

In this final section we present some examples illustrating some of the above results. Let n be a positive integer. For every abelian group G we define

$$r(G) = nG\,, \qquad s(G) = \{g \in G,\ ng = 0\}\,,$$
$$t(G) = r(G) + s(G)\,, \quad v(G) = r(G) \cap s(G)\,,$$

Soc (G) as the sum of all minimal subgroups of G, $J(G)$ as the intersection of all maximal subgroups of G ($J(G) = G$ if there is no such subgroup). Then r, s, t, v, Soc, J are preradicals for the category of abelian groups. Moreover,

(i) r is a cohereditary radical and $\mathscr{T}_r$ is not costable,

(ii) s is a hereditary preradical with $\mathscr{T}_s$ closed under direct products,

(iii) Soc is a hereditary preradical,

(iv) J is a radical,

(v) t is neither idempotent nor a radical and $\mathscr{F}_t = 0$,

(vi) v is neither idempotent nor a radical and $\mathscr{T}_v = 0$,

(vii) $\bar{r}$ is not cohereditary,

(viii) $h(J) = \mathrm{id}$, $ch(J) = \mathrm{zer}$, where id (zer) is the identity (zero) functor,

(ix) $\bigcap K = 0$, where K runs through all ideals K of the ring of integers Z such that $Z/K \in \mathscr{T}_{\mathrm{Soc}}$.

References

[1] *H. Bass:* Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. *95* (1960), 466—488.

[2] *J. A. Beachy:* Cotorsion radicals and projective modules, Bull. Austral. Math. Soc. *5* (1971), 241—253.

[3] *L. Bican:* QF-3'modules and rings, Comment. Math. Univ. Carol. *14* (1973), 295—303.

[4] *L. Bican:* Corational extensions and pseudo-projective modules (to appear).

[5] *L. Bican, P. Jambor, T. Kepka, P. Němec:* On rings with trivial torsion parts, Bull. Austral. Math. Soc. *9* (1973), 275—290.

[6] *L. Bican, P. Jambor, T. Kepka, P. Němec:* Preradicals, Commemt. Math. Univ. Carol. *15* (1974), 75—83.

[7] *L. Bican, P. Jambor, T. Kepka, P. Němec:* Stable and costable preradicals Acta Univ. Carolinae Math. et Phys. *16*, 2 (1975), 63—69.

[8] *L. Bican, P. Jambor, T. Kepka, P. Němec:* Preradicals and change of rings, Comment. Math. Univ. Carol. *16* (1975), 201—217.

[9] *S. E. Dickson:* A torsion theory for abelian categories, Trans. Amer. Math. Soc. *121* (1966), 223—235.

[10] *J. P. Jans:* Some aspects of torsion, Pacif. J. Math. *15* (1965), 1249—1259.

[11] *A. P. Mišina, L. A. Skornjakov:* Abelevy gruppy i moduli, Nauka, Moskva, 1969.

[12] *R. A. Rubin:* Absolutely torsion-free rings, Bull. Amer. Math. Soc. *78* (1972), 854—856.

[13] *B. Stenström:* Rings and modules of quotients, Lecture Notes in Mathematics *237*, Springer Verlag, Berlin—Heidelberg—New York 1971.

Authors' address: 180 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).