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UNIVERSALITY PROPERTY OF FREE GROUPOID EXTENSIONS OF HALFGROUPOIDS AND ITS GEOMETRICAL MEANING

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In his work [1], G. E. BATES presented a theory of free net extensipns of halfnets and interpreted it algebraically as the theory of loop extensions of halfloops (for the notion of a hafloop, cf. also [2], p. 15). In [3] and [4], using suitable algebraic modifications, T. EVANS and W. PEREMANS suggested to generalize the algebraic part of Bates' theory to a theory of free groupoid extensions of halfgroupoids. These suggestions were worked out by R. H. BRUCK in [2], pp. 1–8.

In the present article we deduce the known theorem on universality property of free groupoid extensions of halfgroupoids by a modification of a procedure of Pickert from [5], pp. 15–16. As we hope this concept achieves the result more quickly than the investigations in [2], pp, 2–6. Afterwards we outline a geometric counterpart of this theorem in terms of generalized nets and halftnets.

1. ALGEBRAIC PART

A *binary halfoperation* $\cdot$ over a set $G \neq \emptyset$ is defined as a mapping of a non-void set Dom $\cdot \subseteq G \times G$ into G. If particularly Dom $\cdot = G \times G$ then $\cdot$ is said to be a *binary operation* over G.

By a *halfgroupoid* we mean a couple $(G, \cdot)$ where G is a non-void set and $\cdot$ a binary halfoperation over G. When $\cdot$ is a binary operation, we get a *groupoid*.

Let $(G, \cdot), (G', .')$ be halfgroupoids. Then a mapping (a surjective mapping) $\theta : G \to G'$ is said to be a homomorphism of $(G, \cdot)$ into (onto) $(G', \cdot')$ if $(x, y) \in$ $\in$ Dom $\cdot \Rightarrow (x^\theta, y^\theta) \in$ Dom $\cdot$ & $(x \cdot y)^\theta = x^\theta \cdot' y^\theta$.

We shall use two important special cases of a homomorphism θ. Firstly, if θ is bijective and $(x, y) \in$ Dom $\cdot \Leftrightarrow (x^\theta, y^\theta) \in$ Dom $\cdot'$ for all $x, y \in G$ then it is called *isomorphism*[1]). Secondly, if $G \subseteq G'$ and $x^\theta = x\ \forall x \in G$ (i.e. $\theta = \mathrm{id}_G$), then we write $(G, \cdot) \subseteq (G', \cdot')$ and call $(G, \cdot)$ a *subhalfgroupoid* of $(G', \cdot')$.

[1]) In this case θ^{-1} is necessarily a homomorphism of $(G', .')$ onto $(G, .)$ which can be easily verified.

Let be given a non-void set $\mathfrak{G}$ of halfgroupoids $\mathbf{G} = (G, \cdot)$ or a family $(\mathbf{G}_\iota)_{\iota \in I}$, $I \neq \emptyset$ of halfgroupoids $\mathbf{G}_\iota = (G_\iota, \cdot_\iota)$ [2]) such that for any $(G, \cdot), (G, \cdot') \in \mathfrak{G}$ it holds $(x, y) \in \operatorname{Dom} \cdot \cap \operatorname{Dom} \cdot' \Rightarrow x \cdot y = x \cdot' y$ or $(x, y) \in \operatorname{Dom} \cdot_\alpha \cap \operatorname{Dom} \cdot_\beta \Rightarrow x \cdot_\alpha y = x \cdot_\beta y \; \forall \alpha, \beta \in I$. Then we speak of a *compatible* set or family of halfgroupoids.

If $\mathfrak{G}$ or $(\mathfrak{G}_\iota)_{\iota \in I}$ is a compatible set or family of halfgroupoids then we define its *union* $\bigcup_{\mathbf{G} \in \mathfrak{G}} \mathbf{G}$ or $\bigcup_{\iota \in I} \mathbf{G}_\iota$ as a halfgroupoid $(\bigcup_{\mathbf{G} \in \mathfrak{G}} G, \bigcup_{(G,\cdot) \in \mathfrak{G}} \cdot)$ or, respectively, $(\bigcup_{\iota \in I} G_\iota, \bigcup_{\iota \in I} \cdot_\iota)$ where $\operatorname{Dom} \bigcup_{(G,\cdot)\in\mathfrak{G}} \cdot = \bigcup_{(G,\cdot)\in\mathfrak{G}} \operatorname{Dom} \cdot$ and $x(\bigcup_{(G,\cdot)\in\mathfrak{G}} \cdot)\, y = x \cdot y$ with $(x, y) \in \operatorname{Dom} \cdot$ for some $(G, \cdot) \in \mathfrak{G}$ or, respectively, $\operatorname{Dom} \bigcup_{\iota\in I} \cdot_\iota = \bigcup_{\iota \in I} \operatorname{Dom} \cdot_\iota$, $x(\bigcup_{\iota\in I} \cdot_\iota)\, y = x \cdot y$ with $(x, y) \in \operatorname{Dom} \cdot_\iota$ for some $\iota \in I$. If $\mathfrak{G}$ or $(\mathbf{G}_\iota)_{\iota \in I}$ is a compatible set or family of halfgroupoids with $\bigcap_{\mathbf{G}\in\mathfrak{G}} G \neq \emptyset$ or $\bigcap_{\iota\in I} G_\iota \neq \emptyset$ then we define its *intersection* $\bigcap_{\mathbf{G}\in\mathfrak{G}} \mathbf{G}$ or $\bigcap_{\iota\in I} G_\iota$ as a halfgroupoid $(\bigcap_{(G,\cdot)\in\mathfrak{G}} G, \bigcap_{(G,\cdot)\in\mathfrak{G}} \cdot)$ or $(\bigcap_{\iota\in I} G_\iota, \bigcap_{\iota\in I} \cdot_\iota)$ where $\operatorname{Dom} \bigcap_{(G,\cdot)\in\mathfrak{G}} \cdot = \bigcap_{(G,\cdot)\in\mathfrak{G}} \operatorname{Dom} \cdot$, $x(\bigcap_{(G,\cdot)\in\mathfrak{G}} \cdot)\, y = x \cdot y$ independently of $(G, \cdot) \in \mathfrak{G}$ or, respectively, $\operatorname{Dom} \bigcap_{\iota\in I} \cdot_\iota = \bigcap_{\iota\in I} \operatorname{Dom} \cdot_\iota$, $x(\bigcap_{\iota\in I} \cdot_\iota)\, y = x \cdot y$ independently of $\iota \in I$.

Let $(G, \cdot)$ be a groupoid and $(G^*, \cdot^*)$ its subhalfgroupoid. Denote by $\mathfrak{G}$ the set of just all the groupoids $X, \circ)$ of the form $(G^*, \cdot^*) \subseteq (X, \circ) \subseteq (G, \cdot)$. Then $\bigcap_{(X,\circ)\in\mathfrak{G}} (X, \circ)$ is a groupoid belonging also to $\mathfrak{G}$. It will be denoted by $\mathbf{G}((G, \cdot), (G^*, \cdot^*))$ and said to be *generated* by $(G^*, \cdot^*)$ in $(G, \cdot)$. [3]) We give now its recursive construction: First, let $(G_1, \cdot_1) := (G^*, \cdot^*)$. Further assume that a halfgroupoid $(G_i, \cdot_i)$ is given for some $i \in \{1, 2, \ldots\}$ so that $(G^*, \cdot^*) \subseteq (G_i, \cdot_i) \subseteq (G, \cdot)$. Then put $G_{i+1} = G_i \cup \{x \cdot y \mid x, y \in G_i\}$, $\operatorname{Dom} \cdot_{i+1} = G_i \times G_i$, $x \cdot_{i+1} y = x \cdot y \; \forall x, y \in G_i$. Then $(G_{i+1}, \cdot_{i+1})$ is a halfgroupoid satisfying $(G^*, \cdot^*) \subseteq (G_{i+1}, \cdot_{i+1}) \subseteq (G, \cdot)$. Thus, by induction, a sequence $((G_i, \cdot_i))_{i=1}^\infty$ is defined. It is compatible and its union $\bigcap_{i=1}^{\infty} (G_i, \cdot_i)$ is the groupoid $\mathbf{G}((G, \cdot), (G^*, \cdot^*))$ as can be verified briefly. The preceding construction can be somewhat modified: Let $(G'_1, \cdot'_1) := (G^*, \cdot^*)$ and $\gamma_1 := \mathrm{id}_{G^*}$. Further, let be given a halfgroupoid $(G'_i, \cdot'_i)$ and let there exists for some $i \in \{1, 2, \ldots\}$ an isomorphism γ_i of $(G_i, \cdot_i)$ onto $(G'_i, \cdot'_i)$ fixing G^* element-wise. Then determine another halfgroupoid $(G'_{i+1}, \cdot'_{i+1})$, $(G'_i, \cdot'_i) \subseteq (G'_{i+1}, +'_{i+1})$ in such a manner that for a decomposition $\mathscr{D}_i$ on the set $G'_i \times G'_i \setminus \operatorname{Dom} \cdot'_i$ (described as follows) it is $G'_{i+1} = G'_i \cup \mathscr{D}_i$, $\operatorname{Dom} \cdot'_{i+1} = G'_i \times G'_i$ and $x_{i+1} \cdot' y = x \cdot'_i y$ for all $(x, y) \in \operatorname{Dom} \cdot'_i$, while $(x, y) \in x \cdot'_{i+1} y \in \mathscr{D}_i$ for all $(x, y) \in (G'_i \times G'_i) \setminus \operatorname{Dom} \cdot'_i$ such that $x^{\gamma_i^{-1}} \cdot y^{\gamma_i^{-1}}$ is equal to the same element of G. Now define the mapping $\gamma_{i+1} : G_{i+1} \to G'_{i+1}$ which prolongs γ_i and associates for every $x, y \in G_i$ to $x \cdot y$ the element $x \cdot'_{i+1} y$. This γ_{i+1} is an isomorphism of $(G_{i+1}, \cdot_{i+1})$ onto $(G'_{i+1}, \cdot'_{i+1})$. Then, by

[2]) This notation will be used frequently in the sequel.

[3]) In the following we adopt G. Pickert's methodical point of view (used in [5], pp. 12—26, for the explanation of the theory of free planar extensions of incidence structures).

induction, a compatible sequence $((G'_i, \cdot'_i))_{i=1}^{\infty}$ is defined and its union, $\bigcup_{i=1}^{\infty} (G'_i, \cdot'_i)$, is a groupoid which is isomorphic to $\bigcup_{i=1}^{\infty} (G_i, \cdot_i)$ under the isomorphism which prolongs all γ_i's.

This latter construction gives rise to a general recursion scheme which leads to all groupoids generated by a given halfgroupoid $G^*, \cdot^*$ with respect to all possible groupoids $(G, \cdot)$ such that $(G^*, \cdot^*) \subseteq (G, \cdot)$. Let $(G^*, \cdot^*)$ be a given halfgroupoid. Firstly put $(G_{(1)}, \cdot_{(1)}) := (G^*, \cdot^*)$. Secondly suppose we have a halfgroupoid $(G_{(i)}, \cdot_{(i)})$ for some $i \in \{1, 2, \ldots\}$. Then choose a decomposition $\mathscr{D}_{(i)}$ of the set $G_{(i)} \times G_{(i)} \setminus \mathrm{Dom} \cdot_{(i)}$ onto mutually disjoint nonvoid subsets and define $G_{(i+1)} := G_{(i)} \cup \mathscr{D}_{(i)}$, $\mathrm{Dom} \cdot_{(i+1)} = G_{(i)} \times G_{(i)}$, $x \cdot_{(i+1)} y = x \cdot_{(i)} y$ for all $(x, y) \in \mathrm{Dom} \cdot_{(i)}$ and $(x, y) \in x \cdot_{(i+1)} y \in \mathscr{D}_{(i)}$ for all $(x, y) \in (G_{(i)} \times G_{(i)}) \setminus \mathrm{Dom} \cdot_{(i)}$.

Thus by induction, a compatible sequence $((G_{(i)}, \cdot_{(i)}))_{i=1}^{\infty}$ (called *generating chain*) is defined and for its union $\bigcup_{i=1}^{\infty} (G_{(i)}, \cdot_{(i)})$, it results $\bigcup_{i=1}^{\infty} (G_{(i)}, \cdot_{(i)}) = \mathbf{G}(\bigcup_{i=1}^{\infty} (G_{(i)}, \cdot_{(i)}), (G^*, \cdot^*))$. The "freest" case occurs if each $\mathscr{D}_{(i)}$ is trivial (consists only of one-element blocks). Then we shall have in the above construction for all $i \in \{1, 2, \ldots\}$: $x \cdot_{(i+1)} y = (x, y)$ (we drop $\{(x, y)\}$) for all $(x, y) \in (G_{(i)} \times G_{(i)}) \setminus \mathrm{Dom} \cdot_{(i)}$ and the corresponding $\bigcup_{i=1}^{\infty} (G_{(i)}, \cdot_{(i)})$ will be called the *free groupoid extension* of $(G^*, \cdot^*)$ and denoted by $(G^{*f}, \cdot^{*f})$.

Theorem 1. *Let $(G^*, \cdot^*)$ be a halfgroupoid and $(G, \cdot)$ a groupoid such that $(G, \cdot) = \mathbf{G}((G, \cdot), (G^*, \cdot^*))$. Then there exists an isomorphism of $(G, \cdot)$ onto $(G^{*f}, \cdot^{*f})$ leaving each element of G^* fixed, if and only if to every groupoid $(G', \cdot') = \mathbf{G}((G', \cdot'), (G^*, \cdot^*))$ there exists a homomorphism of $(G, \cdot)$ onto $(G', \cdot')$ leaving each element of G^* fixed.*

Proof. 1. Necessity: We have to show that there exists a homomorphism of $(G^{*f}, \cdot^{*f})$ onto $(G', \cdot')$, leaving each element of G^* fixed. We shall construct such a homomorphism inductively using generating chains $((G_i^*, \cdot_i^*))_{i=1}^{\infty}$, $((G_{i'}, \cdot_{i'}))_{i=1}^{\infty}$ of $(G^{*f}, \cdot^{*f})$ or of $\mathbf{G}((G', \cdot'), (G^*, \cdot^*))$, respectively. First put $\theta_1^{(G',\cdot')} := \mathrm{id}_{G^*}$. This is obviously a homomorphism of $(G_{1*}, \cdot_{1*})$ onto $(G_{1'}, \cdot_{1'})$ leaving G^* element-wise fixed. Further assume that for some $i \in \{1, 2, \ldots\}$ a homomorphism $\theta_i^{(G_{i'},\cdot')}$ of $(G_{i*}, \cdot_{i*})$ onto $(G_{i'}, \cdot_{i'})$ is given leaving G^* element-wise fixed. We prolong $\theta_i^{(G',\cdot')}$ onto a mapping $\theta_{i+1}^{(G',\cdot')} : G_{(i+1)*} \to G_{(i+1)'}$ as follows: For all $(x, y) \in G_{i*} \times G_{i*} \setminus \mathrm{Dom} \cdot_{i*}$ define $(x, y)^{\theta_{i+1}^{(G',\cdot')}} := x^{\theta_i^{(G',\cdot')}} \cdot' y^{\theta_i^{(G',\cdot')}}$. By induction, we get a sequence $((\theta_i^{(G',\cdot')}))_{i=1}^{\infty}$ and it may be easily verified that there is just one mapping $\theta^{(G',\cdot')} : G^{*f} \to G'$ prolonging all $\theta_i^{(G',\cdot')}$. This mapping $\theta^{(G',\cdot')}$ is then easily shown to be a homomorphism of $(G^{*f}, \cdot^{*f})$ onto $(G', \cdot')$ keeping G^* element-wise fixed.

2. Sufficiency: Assume that for all $(G', \cdot')$ there exists a homomorphism $\varphi_{(G',\cdot')}$

of $(G, \cdot)$ onto $(G', \cdot')$ leaving G^* element-wise fixed even though we exploit only $\varphi := \varphi_{(G^{*f}, \cdot^{*f})}$ and its restrictions $\varphi_i := \varphi|_{G_i}$ $\forall i \in \{1, 2, \ldots\}$. By part 1 there exists also a homomorphism $\theta^{(G,\cdot)}$ of $(G^{*f}, \cdot^{*f})$ onto $(G, \cdot)$ which will be helpful for our next reasoning. We shall prove by induction that $\varphi_i \theta_i^{(G,\cdot)} = \mathrm{id}_{G_i}$ $\forall i \in \{1, 2, \ldots\}$: [4]) Obviously $\varphi_1 = \theta_1^{(G,\cdot)} = \mathrm{id}_{G^*}$ so that $\varphi_1 \theta_1^{(G,\cdot)} = \mathrm{id}_{G_1}$. Thus suppose that $\varphi_i \theta_i^{(G,\cdot)} = \mathrm{id}_{G_i}$ holds for some $i \in \{1, 2, \ldots\}$. Now turn to elements from $G_{i+1} \setminus G_i$. We know that every element of $G_{i+1} \setminus G_i$ is of the form $z = x \cdot y$ for some $(x, y) \in (G_i \times G_i) \setminus \mathrm{Dom} \cdot_i$. Then $z^\varphi = (x \cdot y)^\varphi = x^\varphi \cdot^{*f} y^\varphi$ and consequently $z^{\varphi\theta^{(G,\cdot)}} = (x^\varphi \cdot^{*f} y^\varphi)^{\theta^{(G,\cdot)}} = x \cdot y = z$. Thus $\varphi_{i+1} \theta_{i+1}^{(G',\cdot)} = \mathrm{id}_{G_{i+1}}$ and the proof of $\varphi_i \theta_i^{(G,\cdot)} = \mathrm{id}_{G_i}$ $\forall i \in \{1, 2, \ldots\}$ is complete. This fact implies also $\varphi\theta^{(G,\cdot)} = \mathrm{id}_G$. Thus φ and $\theta^{(G,\cdot)}$ are bijective and both leave G^* element-wise fixed. □

§ 2. GEOMETRIC PART

By a (*generalized*) *halfnet* we shall mean a quadruplet $\mathcal{N} = (\mathcal{P}, \mathcal{L}, \mathrm{I}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3))$ [5]) where $\mathcal{P}$, $\mathcal{L}$ are non-void sets, I a binary relation from $\mathcal{P}$ to $\mathcal{L}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ disjoint non-void subsets with $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$, $\#\mathcal{L}_1 = \#\mathcal{L}_2 = \#\mathcal{L}_3$ such that

(i) for every $P \in \mathcal{P}$ and every $i \in \{1, 2, 3\}$ there is just one $l_i \in \mathcal{L}_i$ with $P\mathrm{I}l_i$,

(ii) for all $l_1 \in \mathcal{L}_1$, $l_2 \in \mathcal{L}_2$ there is at most one $P \in \mathcal{P}$ with $P\mathrm{I}l_1, l_2$,

and

(iii) for all $l \in \mathcal{L}_3$ there is at least one $P \in \mathcal{P}$ with $P\mathrm{I}l$.

If, moreover,

(iv) for all $l_1 \in \mathcal{L}_1$, $l_2 \in \mathcal{L}_2$ there is at least one $P \in \mathcal{P}$ with $P\mathrm{I}l_1, l_2$,

then $\mathcal{N}$ is called a (*generalized*) *net.* We shall equip a halfnet $\mathcal{N}$ with a triple $(\sigma_1, \sigma_2, \sigma_3)$ of mappings $\sigma_1 : \mathcal{L}_1 \to S$, $\sigma_2 : \mathcal{L}_2 \to S$, $\sigma_3 : \mathcal{L}_3 \to S$ where σ_1, σ_2 are bijections and σ_3 is an injection. Then we say that $\mathcal{N}$ has *binding* $(\sigma_1, \sigma_2, \sigma_3)$.

If $\mathcal{N}, \mathcal{N}'$ are halfnets then define a *homomorphism* of $\mathcal{N}$ into (onto) $\mathcal{N}'$ as a couple (π, λ) of mappings (surjective mappings) $\pi : \mathcal{P} \to \mathcal{P}'$, $\lambda : \mathcal{L} \to \mathcal{L}'$ such that $P\mathrm{I}l \Rightarrow P^\pi \mathrm{I}' l^\lambda$ and for each $i \in \{1, 2, 3\}$, $l \in \mathcal{L}_i \Rightarrow l^\lambda \in \mathcal{L}'_i$.

If, moreover, π, λ are bijections and (π^{-1}, λ^{-1}) is a homomorphism of $\mathcal{N}'$ onto $\mathcal{N}$, then (π, λ) is called *isomorphism.* If, on the other hand, $\mathcal{P} \subseteq \mathcal{P}'$, $\mathcal{L} \subseteq \mathcal{L}'$ and $\pi = \mathrm{id}_{\mathcal{P}}$, $\lambda = \mathrm{id}_{\mathcal{L}}$, then $\mathcal{N}$ is said to be *subhalfnet* of $\mathcal{N}'$ (notation $\mathcal{N} \subseteq \mathcal{N}'$).

If $\mathcal{N}, \mathcal{N}'$ are halfnets with bindings $(\sigma_1, \sigma_2, \sigma_3)$, $(\sigma'_1, \sigma'_2, \sigma'_3)$ then a homomorphism (π, λ) of $\mathcal{N}$ into $\mathcal{N}'$ is called *bound* if the mappings $\hat\sigma_1 : l^{\sigma_1} \mapsto (l^\lambda)^{\sigma_1'}$ $\forall l \in \mathcal{L}_1$, $\hat\sigma_2 : l^{\sigma_2} \mapsto (l^\lambda)^{\sigma_2'}$ $\forall l \in \mathcal{L}_2$ are equal and the mapping $\hat\sigma_3 : l^{\sigma_3} \mapsto (l^\lambda)^{\sigma_3'}$ $\forall l \in \mathcal{L}_3$ is the restriction of $\hat\sigma_1 = \hat\sigma_2$. If $\mathcal{N} \subseteq \mathcal{N}'$ and $S \subseteq S'$, $\sigma_1 = \sigma'_1|_{\mathcal{L}_1}$, $\sigma_2 = \sigma'_2|_{\mathcal{L}_2}$, $\sigma_3 = \sigma'_3|_{\mathcal{L}_3}$ then $\mathcal{N}$ is called a *bound* subhalfnet of $\mathcal{N}'$ (notation $\mathcal{N} \leqq \mathcal{N}'$).

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[4]) $\varphi_i \theta_i^{(G,\cdot)} = \mathrm{id}_{G_i}$ implies that φ_i, $\theta_i^{(G,\cdot)}$ (as surjective mappings) are injective too.

[5]) For a halfnet $\mathcal{N}$ we shall use frequently the notation $(\mathcal{P}, \mathcal{L}, \mathrm{I}, (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3))$; similarly if $\mathcal{N}$ has an index; e.g. $\mathcal{N}' =: (\mathcal{P}', \mathcal{L}', \mathrm{I}', (\mathcal{L}'_1, \mathcal{L}'_2, \mathcal{L}'_3))$, and so on. This convention will be applied also to the notation of the corresponding bindings (defined in the sequel).

Let $\mathbf{G} = (G, \cdot)$ be a halfgroupoid. Choose disjoint sets G_1, G_2, G_3 such that $\# G = \# G_1 = \# G_2$, $\# G_3 = \#\{x \cdot y \mid (x, y) \in \mathrm{Dom} \cdot\}$ and bijections $\gamma_1 : G_1 \to G$, $\gamma_2 : G_2 \to G$, $\gamma_3 : G_3 \to \{x \cdot y \mid (x, y) \in \mathrm{Dom} \cdot\}$. Further define a binary relation $\mathrm{I}_{\mathbf{G},(\gamma_1,\gamma_2,\gamma_3)} =: \mathrm{I}$ from $\mathrm{Dom} \cdot$ to $G_1 \cup G_2 \cup G_3$ by means of $(x, y)\, \mathrm{I} g$ if and only if either $x^{\gamma_1^{-1}} = g \in G_1$ or $y^{\gamma_2^{-1}} = g \in G_2$ or $(x \cdot y)^{\gamma_3^{-1}} = g \in G_3$. Then $\mathfrak{N}(\mathbf{G}, (\gamma_1, \gamma_2, \gamma_3)) = (\mathrm{Dom} \cdot, G_1 \cup G_2 \cup G_3, \mathrm{I}_{\mathbf{G},(\gamma_1,\gamma_2,\gamma_3)}, (G_1, G_2, G_3))$ is a bound halfnet with binding $(\gamma_1, \gamma_2, \gamma_3)$; it will be called a halfnet *over* $\mathbf{G}$ corresponding to an admissible triple $(\gamma_1, \gamma_2, \gamma_3)$.

Conversely, let $\mathcal{N}$ be a halfnet with some binding $(\sigma_1, \sigma_2, \sigma_3)$. Then define a halfgroupoid $\mathfrak{G}(\mathcal{N}, (\sigma_1, \sigma_2, \sigma_3)) = (S, \bullet)$ such that $\mathrm{Dom} \bullet = \{(l_1^{\sigma_1}, l_2^{\sigma_2}) \mid \exists P \in \mathcal{P},\ P\mathrm{I}l_1 \in \mathcal{L}_1,\ P\mathrm{I}l_2 \in \mathcal{L}_2\}$ and for any $(l_1^{\sigma_1}, l_2^{\sigma_2}) \in \mathrm{Dom} \bullet$, let $(l_1^{\sigma_1} \bullet l_2^{\sigma_2})^{\sigma_3^{-1}}$ be such a line of $\mathcal{L}_3$ which passes through the common point of l_1, l_2. $\mathfrak{G}(\mathcal{N}, (\sigma_1, \sigma_2, \sigma_3))$ is called *coordinatizing* halfgroupoid of $\mathcal{N}$.

Theorem 2. A. *Let* $\mathbf{G}$ *be a halfgroupoid,* $\mathfrak{N}(\mathbf{G}, (\gamma_1, \gamma_2, \gamma_3))$ *one of halfnets over* $\mathbf{G}$. *Then* $\mathfrak{G}(\mathfrak{N}(\mathbf{G}, (\gamma_1, \gamma_2, \gamma_3), (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3))$ *coincides with* $\mathbf{G}$.

B. *Let* $\mathcal{N}$ *be a halfnet with a binding* $(\sigma_1, \sigma_2, \sigma_3)$. *Then each* $\mathfrak{N}(\mathfrak{G}(\mathcal{N}, (\sigma_1, \sigma_2, \sigma_3)), (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3))$ *is boundly isomorphic to* $\mathcal{N}$.

Proof. A. Let be given a halfgroupoid $\mathbf{G} = (G, \cdot)$. Denote $\mathfrak{N}(\mathbf{G}, (\gamma_1, \gamma_2, \gamma_3))$ by $(\mathrm{Dom} \cdot, G_1 \cup G_2 \cup G_3, \mathrm{I}_{\mathbf{G},(\gamma_1,\gamma_2,\gamma_3)}, (G_1, G_2, G_3))$ as in the definition of a halfnet over $\mathbf{G}$. Finally put $\mathfrak{G}(\mathfrak{N}(\mathbf{G}, (\gamma_1, \gamma_2, \gamma_3)) =: (G, \odot)$. Then the mapping id_G expresses an isomorphism of $(G, \cdot)$ onto $(G, \odot)$ so that also the binary halfoperations $\cdot, \odot$ coincide.

B. Now let $\mathcal{N}$ be a halfnet with a binding $(\sigma_1, \sigma_2, \sigma_3)$. Denote $\mathfrak{G}(\mathcal{N}, (\sigma_1, \sigma_2, \sigma_3))$ by $(S, \bullet)$ and choose some admissible triple $(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$. In this way three disjoint sets S_1, S_2, S_3 and three bijections $\hat{\sigma}_1 : S_1 \to S$, $\hat{\sigma}_2 : S_2 \to S$, $\hat{\sigma}_3 : S_3 \to \{x \bullet y \mid (x, y) \in \mathrm{Dom} \bullet\}$ are chosen. Finally we construct the halfnet $\mathcal{N}' = \mathfrak{N}((S, \bullet), (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3))$ with $\mathcal{P}' := \mathrm{Dom} \bullet$, $\mathcal{L}' := S_1 \cup S_2 \cup S_3$, $\mathrm{I}' := \mathrm{I}_{(S,\bullet),(\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3)}$, $\mathcal{L}'_1 := S_1$, $\mathcal{L}'_2 := S_2$, $\mathcal{L}'_3 := S_3$ and define mappings $\pi : \mathcal{P} \to \mathcal{P}'$, $\lambda : \mathcal{L} \to \mathcal{L}'$: For all $i \in \{1, 2, 3\}$, $l \in \mathcal{L}_i$ we put $l^\lambda = l^{\sigma_i \hat{\sigma}_i^{-1}}$. For all $P \in \mathcal{P}$ let P^π be the intersection point of lines $l_1^{\sigma_1 \hat{\sigma}_1^{-1}}, l_2^{\sigma_2 \hat{\sigma}_2^{-1}}$ where $P\mathrm{I}l_1 \in \mathcal{L}_1$, $P\mathrm{I}l_2 \in \mathcal{L}_2$. Then (π, λ) can be shown to be a bound isomorphism of $\mathcal{N}$ onto $\mathcal{N}'$. □

The above reasoning permits to formulate Theorem 1 in the terms of the theory of halfnets with bindings. The notion of a groupoid $\mathbf{G}(\mathbf{G}, \mathbf{G}^*)$ generated in a groupoid $\mathbf{G}$ by a given halfgroupoid $\mathbf{G}^* \subseteq \mathbf{G}$ corresponds to the notion of a bound net $\mathbf{N}(\mathcal{N}, \mathcal{N}')$ generated in a bound in a bound net $\mathcal{N}$ by a bound halfnet $\mathcal{N}^* \lesssim \mathcal{N}$.

The notion of a free groupoid extension $\mathbf{G}^{*\mathbf{f}}$ of a halfgroupoid corresponds to the notion of a free bound net $\mathcal{N}^{*\mathbf{f}}$ of a bound halfnet $\mathcal{N}^*$.

Theorem 1 can be then re-written in the following form: *Let* $\mathcal{N}^*$ *be a bound subhalfnet of a bound net* $\mathcal{N}$ *such that* $\mathcal{N} = \mathbf{N}(\mathcal{N}, \mathcal{N}')$. *Then there exists a bound isomorphism* (π, λ) *of* $\mathcal{N}$ *onto* $\mathcal{N}^{*\mathbf{f}}$ *with* $\pi|_{\mathcal{P}} = \mathrm{id}_{\mathcal{P}}$, $\lambda|_{\mathcal{L}} = \mathrm{id}_{\mathcal{L}}$ *if and only if to every* *bound net $\mathcal{N}'$ such that $\mathcal{N} \leqq \mathcal{N}' = \mathbf{N}(\mathcal{N}, \mathcal{N}')$ there exists a bound homomorphism (π', λ') of $\mathcal{N}$ onto $\mathcal{N}'$ such that* $\pi'|_{\mathscr{P}} = \mathrm{id}_{\mathscr{P}}$, $\lambda'|_{\mathscr{L}} = \mathrm{id}_{\mathscr{L}}$.

We do not give here the details.

References

[1] *G. E. Bates,* Free loops and nets and its generalizations, Amer. Journ. Math. *69* (1947), 499—550.

[2] *R. H. Bruck,* A survey of binary systems, Berlin—Göttingen—Heidelberg 1958.

[3] *T. Evans,* Homomorphisms of non-associative systems, Journ. London Math. Soc. *24* (1949), 254—260.

[4] *W. Peremans,* Abstract algebraic systems, Rapport ZW 1949—003 of Math. Centrum Amsterdam, 12 pp.; Math. Rev. *16*, 897.

[5] *K. Pickert,* Projektive Ebenen, Berlin—Göttingen—Heidelberg 1955.

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